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ON THE PETTIS INTEGRABILITY

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ABSTRACT. A function $f : \Omega \to X$ is called *intrinsically-separable valued* if there exists $E \in \Sigma$ with $\mu(E) = 0$ such that $f(\Omega - E)$ is a separable in X. For a given Dunford integrable function $f : \Omega \to X$ and a weakly compact operator T, we show that if f is intrinsically-separable valued, then f is Pettis integrable, and if there exists a sequence (f_n) of Dunford integrable and intrinsically-separable valued functions from Ω into X such that for each $x^* \in X^*$, $x^* f_n \to x^* f$ a.e., then f is Pettis integrable. We show that a function f is Pettis integrable if and only if for each $E \in \Sigma$, $F(E)$ is weak*-continuous on B_{X^*} if and only if for each $E \in \Sigma$, $M = \{x^* \in X^* : F(E)(x^*) = 0\}$ is weak*-closed.

1. Introduction

In [6], R. Huff investigates the Pettis integrability with respect to complete probability space. In this paper, we generalize the ideas put forth in [5] and [6].

Let us fix some terminology and notation. Let X be a real Banach space with continuous dual X^* and the closed unit ball of X^* will be denoted by B_{X^*}. Throughout, (Ω, Σ, μ) will denote a finite measure space.

A function $f : \Omega \to X$ is *Pettis integrable* if

(a) f is weakly μ-measurable and weakly μ-integrable, and
(b) for every $E \in \Sigma$, there exists an $x_E \in X$ such that

$$x^*(x_E) = \int_E x^* f \, d\mu$$

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for every $x^* \in X^*$

By the closed graph theorem, $T : X^* \to L^1(\mu)$ is bounded linear operator. Hence, if T^* denotes the adjoint of T, then we can define $T^* : L^\infty(\mu) \to X^{**}$ by

$$T_g^*(x^*) = \int_\Omega gT(x^*)\, d\mu = \int_\Omega gx^* f d\mu$$

for every $g \in L^\infty(\mu)$. In particular, define $F : \Sigma \to X^{**}$ by

$$F(E) = (D) - \int_E f\, d\mu$$

for each $E \in \Sigma$, and $F(E) = T^*_{\chi_E}$ is called the *Dunford integral* of f over E. In the case that $F(E) \in X$ for each $E \in \Sigma$, we write $F(E) = (P) - \int_E f\, d\mu$ and it is called the Pettis integral of f over E.

The function F is not necessarily countably additive. It can be shown that F is countably additive *if and only if* T is a weakly compact operator *if and only if* $\{x^* f : x^* \in B_{X^*}\}$ is uniformly integrable in $L^1(\mu)$.

2. Pettis Integrability

A weakly measurable function $f : \Omega \to X$ is *separable-like* if there exists a separable subspace D of X such that for every $x^* \in X^*$,

$$x^*|_D f = x^* f \quad \mu\text{-a.e.}.$$

In [6], R. Huff defined such function separable-like. This property of the function f is generalized in the following definition.

DEFINITION 2.1. A function $f : \Omega \to X$ is said to be *intrinsically-separable valued* if there exists $E \in \Sigma$ with $\mu(E) = 0$ such that $f(\Omega - E)$ is a separable subset in X.

In Particular, if f be a weakly measurable, then it is separable-like.

EXAMPLE 2.2. (a) Let f be a simple function. Then f is intrinsically-separable valued since f is strongly measurable function.

(b) There exists a function $f : \Omega \to X$ such that $x^*f = 0$ a.e., but f is not intrinsically-separable valued. [7, Example 9].

PROPOSITION 2.3[6, Corollary 4]. *Let f be Dunford integrable and T is weakly compact. If f is separable-like, then it is Pettis integrable.*

The next two Theorems give a sufficient condition for the Pettis integrable.

THEOREM 2.4. *Let $f : \Omega \to X$ be Dunford integrable* and *T be a weakly compact. If f is intrinsically-separable valued, then f is Pettis integrable.*

PROOF. Suppose that f is intrinsically-separable valued. Since f is weakly measurable, f is separable-like. By Proposition 2.3., f is Pettis integrable.

By [2, Theorem 8, p.55] and [5, Theorem 3], we obtain the following proposition.

PROPOSITION 2.5. *Let $f : \Omega \to X$. If there is a sequence (f_n) of Pettis integrable functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^*f_n = x^*f$ μ-a.e., then f is Pettis integrable.*

THEOREM 2.6. *Let f be Dunford integrable and T be a weakly compact. If there exists a sequence (f_n) of Dunford integrable and intrinsically-separable valued functions from Ω into X such that for each $x^* \in X^*$, $x^*f_n \to x^*f$* a.e., *then f is Pettis integrable.*

PROOF. By hypothesis and Theorem 2.4., (f_n) is a sequence of Pettis integrable functions from Ω into X such that for each $x^* \in X^*$, $x^*f_n \to x^*f$ a.e.. Then, by Proposition 2.5, f is Pettis integrable.

From the above two Theorems, we obtain the following Corollaries.

COROLLARY 2.7. *Let $f : \Omega \to X^{**}$ be Dunford integrable, X be a reflexive, and T be a weakly compact. If f is intrinsically-separable valued, then f is Pettis integrable.*

COROLLARY 2.8. *Let $f : \Omega \to X^{**}$ be Dunford integrable, X be a reflexive, and T be a weakly compact. If there exists a sequence (f_n) of Dunford integrable and intrinsically-separable valued functions from Ω into X^{**} such that for each $x^* \in X^*$, $x^* f_n \to x^* f$ a.e., then f is Pettis integrable.*

3. Properties of The Pettis Integrability

In a real Banach space X, if X is reflexive, then $X^{**} = X$. And so, we have the following theorems.

THEOREM 3.1. *Let $T : X^* \to L^1(\mu)$ be a weakly compact operator and X be a reflexive. Then F has a relatively weakly compact range.*

PROOF. Since T be a weakly compact, $F : \Sigma \to X^{**}$ is a countably additive vector measure., say $F : \Sigma \to X$. Then there exist a $\mu : \Sigma \to [0, \infty)$ be a countably additive measure such that $F \ll \mu$. Define $T^* : L^\infty(\mu) \to X$ by

$$T_g^* = \int g \, dF$$

for every $g \in L^\infty(\mu)$.

Since $x^* F \ll \mu$ for each $x^* \in X^*$, there exists an $f_{x^*} \in \mathcal{L}(\Omega, \Sigma, \mu, \mathbb{R})$ such that

$$x^* F(E) = \int_E f_{x^*} \, d\mu$$

for each $E \in \Sigma$.

Hence, for each $x^* \in X^*$, there exists an $dx^* F/d\mu = f_{x^*} \in L^1(\mu)$ is the Radon - Nikodym derivative of $x^* F$ with respect to μ such that

$$x^* T_g^* = \int g \, dx^* F = \int g \frac{dx^* F}{d\mu} \, d\mu = \int g f_{x^*} \, d\mu.$$

If (g_α) is a net in $L^\infty(\mu)$ converging weak* to g_0, then, for each $x^* \in X^*$,

$$\begin{aligned}\lim_\alpha x^* T^*_{g_\alpha} &= \lim_\alpha \int g_\alpha f_{x^*}\, d\mu \\ &= \int g_0 f_{x^*}\, d\mu \\ &= x^* T^*_{g_0},\end{aligned}$$

i.e., $(T^*_{g_\alpha})$ converges weakly to $T^*_{g_0}$. Hence T^* is a weak*-weak continuous.

Note that $\{g \in L^\infty(\mu) : \|g\|_\infty \leq 1\}$ is weak*-compact. Since T^* is a continuous, $T^*(\{g \in L^\infty(\mu) : \|g\|_\infty \leq 1\})$ is a compact in X. But $\{F(E) : E \in \Sigma\} = \{T^*_{\chi_E} : E \in \Sigma\} \subseteq \{T^*_g : \|g\|_\infty \leq 1\}$ and so $\overline{\{F(E) : E \in \Sigma\}} \subseteq T^*(\{g \in L^\infty(\mu) : \|g\|_\infty \leq 1\})$.

THEOREM 3.2. *Let $f : \Omega \to X$ be a measurable, T be a weakly compact, and X be a reflexive. Then f is Pettis integrable.*

PROOF. Since T is a weakly compact and X is a reflexive, F has a relatively weakly compact range by Theorem 3.1.. Therefore f is Pettis integrable by [2, Corollary 9, p. 56].

The following results give necessary and sufficient conditions for the Pettis integrable.

THEOREM 3.3. *Let $f : \Omega \to X$ be Dunford integrable. Then the following statements are equivalent :*

(a) *f is Pettis integrable*

(b) *For each $E \in \Sigma$, there exists $x_E \in X$ such that $F(E)(x^*) = x^*(x_E)$ for every $x^* \in X^*$, i.e., $F(E) \in X$.*

(c) *For each $E \in \Sigma$, $F(E)$ is weak*-continuous on B_{X^*}.*

(d) *For each $E \in \Sigma$, $M = \{x^* \in X^* : F(E)(x^*) = 0\}$ is weak*-closed.*

PROOF. (a) $\Longleftrightarrow$ (b). Since f is Pettis integrable, for each $E \in \Sigma$, we can define $F(E)(x^*) = T^*_{\chi_E}(x^*)$, for every $x^* \in X^*$. Since, for each $E \in \Sigma$,

$$T^*_{\chi_E}(x^*) = \int_E x^* f \, d\mu$$

is in X, there exists $x_E \in X$ such that $x^*(x_E) = \int_E x^* f \, d\mu$ for every $x^* \in X^*$.

It is trivial that (b) implies (c).

(c) $\Longrightarrow$ (d). Since $F(E)(M) = \{0\}$, for each $E \in \Sigma$, and $\{0\}$ is closed in $\mathbb{R}$, $F(E)^{-1}(\{0\}) = M$ is weak*-closed by hypothesis.

(d) $\Longrightarrow$ (b). Suppose that $M = \{x^* \in X^* : F(E)(x^*) = 0\}$, for each $E \in \Sigma$, is weak*-closed. We will show that, for each $E \in \Sigma$, there exists an element $x_E \in X$ such that $F(E)(x^*) = x^*(x_E)$ for every $x^* \in X^*$, and hence f is Pettis integrable. For each $E \in \Sigma$, define $F(E)(x^*) = \int_E x^* f \, d\mu$. We may assume that $M \neq X^*$. Otherwise we can take $x_E = 0$. Let $x_0^* \in X^*$ with $F(E)(x_0^*) = 1$. Then there exists an $x_E \in X$ such that $x_0^*(x_E) = 1$ and $x^*(x_E) = 0$ for every $x^* \in M$. Hence, for each $x^* \in X^*$, $x^* - F(E)(x^*)x_0^* = y^* \in X^*$ satisfies $F(E)(y^*) = 0$, *i.e.*, $y^* \in M$. Therefore $F(E)(x^*) = x^*(x_E)$.

As a Corollary of Theorem 3.3., we find the next Corollary.

COROLLARY 3.4. *Let f be Dunford integrable. Then the following statements are equivalent:*

(a) *f is Pettis integrable*

(b) *For each $E \in \Sigma$, $T^*_{\chi_E}$ is in X*

(c) *For each $E \in \Sigma$, $F(E)$ restricted on B_{X^*} is weak*-continuous at 0.*

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