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PREORDERINGS ON LOCAL GLOBAL RINGS

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ABSTRACT. Suppose A is a local global ring (with many units) and $T \subset A$ is a preordering. Let $a_i \in A^*, i = 1, 2, \cdots, n$ and $a \in (\sum_{i=1}^{l-1} a_i T) \cap A^*$. Then, for any integer $l, 1 < l \leq n$, there exist $x \in (\sum_{i=1}^{l-1} a_i T) \cap A^*$ and $y \in (\sum_{i=l}^{n} a_i T) \cap A^*$ such that $a = x + y$

1. Introduction

Let A be a ring, $f \in A[x_1, \cdots, x_n]$. We say that f has unit values if there exist $a_1, \cdots, a_n \in A$ such that $f(a_1, \cdots, a_n) \in A^*$. We say f has local unit values if for each maximal ideal $m \subset A$, f has local unit values as a polynomial over the local ring A_m. If every polynomial over A with local unit values has unit values, we call A a ring with many units (or a local global ring, in short LG-ring). We will show the Theorem 6 over the local global ring.

We need some words and notations.

Let $A^2 = \{a^2 \mid a \in A, A : LG - \text{ring}\}$ and denote by $\sum A^2$: the set of all finite sums of elements of A^2. The level of A is the smallest natural number s such that -1 is a sum of s-squares in A and if $-1 \notin A^2, A$ has finite level.

A subset $T \subset A$ is called a preordering on A if $T + T \subset T, T \cdot T \subset T, A^2 \subset T$ and $-1 \notin T$. Note that A has finite level iff $\sum A^2$ is a preordering, if A has infinite level then $T_0 = \sum A^2$ is the smallest preordering on A.

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An ordering on A is a subset $P \subset A$ such that $P + P \subset P, P \cdot P \subset P, P \cup -P = A$, and $P \cap -P$ is a prime ideal of A. Denote by X_A the set of all orderings on A. It is easy to cheek that every ordering on A is also a preordering.

2. Some Propositions and Main Result

Next some propositions are well known.

PROPOSITION 1. *A preordering T on A is an ordering iff it satisfies.*

$$xy \in -T \Rightarrow x \in T \text{ or } y \in T$$

PROPOSITION 2. *Let T be a preordering on A. Then*

$$xy \in -T \Rightarrow \text{ one of } T + xT \text{ and } T + yT \text{ is a preordering.}$$

PROOF. Note both $T + xT$ and $T + yT$ are closed under additon, multiplication and contain A^2.

Suppose $xy \in -T$ and neither $T + xT$ nor $T + yT$ is a preordering. Then, $-1 \in T + xT$ and $-1 \in T + yT$. Thus, there exist $s_1, t_1, s_2, t_2 \in T$ such that $-1 = s_1 + xt_1$, and $-1 = s_2 + xt_2$. Hence, $-xt_1 = 1 + s_1, -yt_2 = 1 + s_2$. By multiplying, $xyt_1t_2 = 1 + s$ for some $s \in T$. But then $-1 = s - xyt_1t_2 \in T$. This is a contradiction.

PROPOSITION 3. *Every maximal preordering is an ordering. Every preordering on A is contained in an ordering.* [1], [5]

PROPOSITION 4. *Let $P \in X_A$. Then the set of all ordering containing P forms a chain under inclusion.*
In particular, there is a unique maximal ordering containing P.

PROOF. Suppose $Q_1, Q_2 \in X_A$ such that $P \subset Q_1, P \in Q_2$ and $Q_1 \not\subset Q_2, Q_2 \not\subset Q_1$. Then there exist $x \in Q_1 \setminus Q_2$ and $y \in Q_2 \setminus Q_1$.

So, $-x \in Q_2$ and $-y \in Q_1$ and $x - y \in Q_1$. If $x - y \in P \subset Q_2$, then $x = (x-y)+y \in Q_2$, that is a contradiction. So $x - y \notin P$ and hence, $y - x \in P \subset Q_1$. Thus, $y = (y - x) + x \in Q_1$, a contradition too. Hence, one of $Q_1 \subset Q_2$ and $Q_2 \subset Q_1$ holds. If T is a preordering on A, let T^* denote $T \cap A^*$. Note this is a subgroup of A^* of exponent 2.

PROPOSITION 5. *Suppose A is a LG-ring. Let $T \subset A$ be a preordering and $P \subset A$ a maximal ordering. Then $T \subset P$ iff $T^* \subset P^*$.*

PROOF. The implication only if part is clear. Assume $T \not\subset P$. Choose $x \in T$. $x \notin P$. By the maximality of P, $-1 \in P + xP$ and hence, $-(1 + x) \in P + xP$.

Say $-(1 + x) = s + xt, s, t \in P$. Then $-1 = s + x(1 + t)$ so $-(1+t) = s(1+t) + x(1+t)^2$ and hence, $1 + x(1+t)^2 = s'$ for some $s' \in P$.

Let m be a maximal ideal of A. Then there exists $y \in A$ such that

$$1 + x(1+t)^2 + xy^2 \not\equiv 0 \pmod{m}$$

If $1 + x(1+t)^2 \equiv 0$ (mod m) take $y = 1$. Otherwise, take $y = 0$. Since A has many units, there exits $y \in A$ such that $a = 1 + x(1+t)^2 + xy^2 \in A^*$. Clearly $a \in T^*$ but $a = -s' + xy^2 \in -P$ so $a \notin P^*$.

The following main theorem is a special case of a transversality theorem in [1], but it is smart technique.

THEOREM 6. *Suppose A is a local global ring and $T \subset A$ is a preordering.*
Let $a_1 \cdots, a_n \in A^*$ and $a \in (a_1T + \cdots + a_nT) \cap A^*$. Then, for any integer $l, 1 < l \leq n$, there exist $x \in (a_1T + \cdots + a_{l-1}T) \cap A^*$ and $y \in (a_lT + \cdots + a_nT) \cap A^*$ such that $a = x + y$.

PROOF. Let us prove it by two steps.
First we show that we can write $a = x + y$, where $x \in (a_1T + \cdots + a_{l-1})T$ and $y \in (a_lT + \cdots + a_nT) \cap A^*$.

Step 1. Let $u \in (a_1T + \cdots + a_{l-1}T)$ and $v \in (a_lT + \cdots + a_nT)$ such that $a = u + v$. Since $2 \in A^*$, $T - T = A$ and hence, there exist $s, t \in T$ such that $a_l/a = s - t$.

Suppose $s \in A^*$. Then $a = (\frac{1}{s})(at + a_l) = (\frac{1}{s})ut + (\frac{1}{s})(a_l + vt)$. Let $x = (\frac{1}{s})ut \in (a_1T + \cdots + a_{l-1}T)$ and $y = (\frac{1}{s})(a_l + vt) \in (a_lT + \cdots + a_nT)$. Thus, it is sufficient to show that there exists $s', t' \in T$ such that $a_l/a = s' - t', s' \in A^*$ and $a_l + vt' \in A^*$.

Let m be a maximal ideal of A. Then there exists $\alpha, \beta \in A$ such that

$$s + \alpha^2 + \beta^2 \not\equiv 0 \pmod{m}$$

$$a_l + v(t + \alpha^2 + \beta^2) \not\equiv 0 \pmod{m}.$$

(Since $2 \in A^*$, $\mathrm{char}(A/m) \neq 2$ and hence, $\alpha^2 + \beta^2$ represents at least 3 distinct values in A/m so this is always possible) Since A has many units, there exist $\alpha, \beta \in A$ such that $s' = s + \alpha^2 + \beta^2 \in A^*$ and if $t' = t + \alpha^2 + \beta^2$ then $a_l + vt' \in A^*$. Clearly, $s', t' \in T$ and $a_l/a = s' - t'$.

Step 2. By the above, we can write $a = u + v$, where $u \in (a_1T + \cdots + a_{l-1}T) \cap A^*$ and $v \in (a_lT + \cdots + a_nT)$ and $a = u' + v'$, where $u' \in (a_1T + \cdots, +a_{l-1}T), v' \in (a_lT + \cdots, +a_nT) \cap A^*$. Let $\alpha, \beta \in A$. Then $\alpha^2 a = \alpha^2 u + \alpha^2 v, \beta^2 a = \beta^2 u' + \beta^2 v'$ and hence, $(\alpha^2 + \beta^2)a = (\alpha^2 u + \beta^2 u') + (\alpha^2 v + \beta^2 v')$

Suppose $\alpha^2 + \beta^2 \in A^*$. Let

$$x = \frac{(\alpha^2 u + \beta^2 u')}{\alpha^2 + \beta^2} \in (a_1T + \cdots + a_{l-1}T)$$

and

$$y = \frac{(\alpha^2 v + \beta^2 v')}{\alpha^2 + \beta^2} \in (a_lT + \cdots + a_nT).$$

Thus, it is sufficient to show that we can choose $\alpha, \beta \in A$ such that $\alpha^2 + \beta^2 \in A^*, \alpha^2 u + \beta^2 u' \in A^*$ and $\alpha^2 v + \beta^2 v' \in A^*$. Let m be a maximal ideal of A. Then there exists $\alpha, \beta \in A$ such that

$$\alpha^2 + \beta^2 \not\equiv 0 \pmod{m}$$
$$\alpha^2 u + \beta^2 u' \not\equiv 0 \pmod{m}$$
$$\alpha^2 v + \beta^2 v' \not\equiv 0 \pmod{m}.$$

If $u + u' \not\equiv 0 \pmod{m}, v + v' \not\equiv 0 \pmod{m}$ then just choose $\alpha = \beta = 1$. Ohterwise, we may assume $u + u' \equiv 0 \pmod{m}$. Then $u \equiv -u' \pmod{m}$ so we may take $\alpha = 0, \beta = 1$. Since A has many units, the result follows.

If T is preordering of A, denote by X_T the set of all maximal orderings of A containing T. Note that by proposition 4, if $P \in X_A$ then X_P is singleton set.

COROLLARY 7. *Suppose A is a local global ring and let $T \subset A$ be a preordering.*
Then $T^* = \cap_{p \in X_T} P^*$.

PROOF. Clearly $T^* \subset \cap_{p \in X_T} P^*$. Suppose $a \in \cap_{p \in X_T} P^*$. Then $-a \notin P$ for all $P \in X_T$ and hence, $T - aT$ is not a preordering as otherwise $-a \in T - aT \subset P$ for some $P \in X_T$.
Thus, $-1 \in T - aT$. By the Theorem6, there exits $s, t \in T^*$ such that $-1 = s - at$ and hence, $a = \frac{(1+s)}{t} \in T$. So $\cap_{p \in X_T} P^* \subset T^*$.

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