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THE NUMBER OF CONFIGURATION ON RECTANGULAR ARRAYS

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ABSTRACT. In this paper, we derive the formular that the number of configuration on the rectangular arrays is counted by the composite shift operator method.

1. Introduction

We study the structure of arrays integers. This problem not only has own merits but also has intimate relation to dimer problems, Ising model, Potts model and Self avoiding walks.

The purpose of the present article is to give a exact composite degeneracice of $M \times N$ lattice system by utilizing composite shift operator matrice method.

2. One Line Arrays of Two Elements

We consider simple particles (which occupy a single lattice site) distributed on a $1 \times N$ lattice. The particles are assumed to interact with their nearest neighbors only.

We first define an $\alpha_0(N)$ -(see Fig.1) in which the site on the right-hand end, i.e., the Nth site, is 0 ; and $\alpha_1(N)$ -space to be a $1 \times N$ lattice in which the N-th site is 1.

$f_1(N, m_0, m_1, s)$ is the number of sequences of arranging $m_0 \, zero, m_1$ ones on an $\alpha_i(N)$-space in such a way as to creates changes.

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By inquiring about the state of occupation of the (N-1)th site (see Fig.2). We can write an exhaustive and mutually exclusive set.

$$f_0(N,m_0,m_1,s) = f_0(N-1,m_0-1,m_1,s) + f_1(N-1,m_0,m_1,s-1) \tag{1}$$

$$f_1(N,m_0,m_1,s) = f_0(N-1,m_0,m_1-1,s-1) + f_1(N-1,m_0,m_1-1,s) \tag{2}$$

Equations (1) and (2) may be represented succinctly as

$$\begin{pmatrix} RS & RSU \\ RTU & TR \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \tag{3}$$

where R,S,T and U are operators that decrease by one, the quantities N, m_0, m_1 and s respectively, i.e.,

(4)

$$(R^r S^s T^t U^u) f_i(N,m_0,m_1,s) = f_i(N-r, m_o-s, m_1-t, s-u)$$

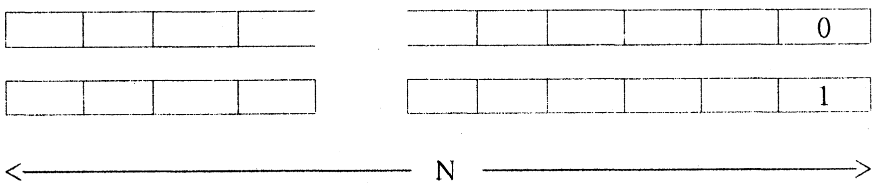

Fig. 1. This figure serves to define two exhaustive and mutually exclusive one mutually exclusive one dimensional lattice space.

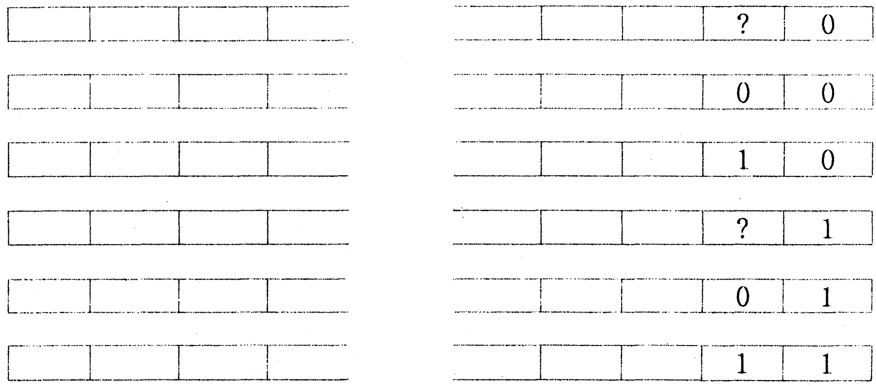

Fig. 2. The decomposition of f_0 and f_1 required to establish a recursion for f(Eq. (1),(2)) for a $1XN$ lattice.

We see from rewriting Eq.(3) as

$$R\begin{pmatrix} S & TU \\ SU & T \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{5a}$$

or as

$$Q_1 F = R^{-1} F$$

where

$$Q_1 = \begin{pmatrix} S & SU \\ TU & T \end{pmatrix}$$

or as

$$\begin{pmatrix} S - 1/R & SU \\ TU & T - 1/R \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \tag{5b}$$

that R^{-1} is an eigenvalue of the SOM. Thus $1/R = \eta_1^{-1} \equiv \lambda_1$ is the largest eigenvalue of Q_1 (expressd in the conjugate variables w, x and y) i.e.,

$$Q = \begin{pmatrix} w & wy \\ xy & x \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 1 & y \\ y & 1 \end{pmatrix} \tag{6}$$

It is of interest and will prove of value to note that Q_1 can be factored into a paricle matrix and a matrix that describes the nearest neighbor pair interaction.

The Perron-Frobenius theorem[3] applied to finite matrices of this type (where all the matrix elements are positive) guarantees a simple, nondegenerate, largest positive eigenvalue.

3. Extension to MXN lattices

We next construct a recursion for the composite degeneracy for a planar $2XN$ lattice space by first defining the spaces $\alpha_j(N) = 0,1,2,3$. The required relationship among the corresponding degeneracies $f_i(N, m_0, m_1, s, r)$ are

$$
\begin{aligned}
f_0(N,m_0,m_1,s,r) &= f_0(N-1,m_0-2,m_1,s,r)\\
&\quad + f_1(N-1,m_0-2,m_1,s-1,1)\\
&\quad + f_2(N-1,m_0-2,m_1,s-1,1)\\
&\quad + f_3(N-1,m_0-2,m_1,s-2,r)\\
f_1(N,m_0,m_1,s,r) &= f_0(N-1,m_0-1,m_1-1,s-1,r-1)\\
&\quad + f_1(N-1,m_0-1,m_1-1,s,r-1)\\
&\quad + f_2(N-1,m_0-1,m_1-1,s-2,r-1)\\
&\quad + f_3(N-1,m_0-1,m_1-1,s-1,r-1)\\
f_2(N,m_0,m_1,s,r) &= f_0(N-1,m_0-1,m_1-1,s-1,r-1)\\
&\quad + f_1(N-1,m_0-1,m_1-1,s-2,r-1)\\
&\quad + f_2(N-1,m_0-1,m_1-1,s,r-1)\\
&\quad + f_3(N-1,m_0-1,m_1-1,s-1,r-1)\\
f_3(N,m_0,m_1,s,r) &= f_0(N-1,m_0,m_1-2,s-2,r)\\
&\quad + f_1(N-1,m_0,m_1-2,s-1,r)\\
&\quad + f_2(N-1,m_0,m_1-2,s-1,r)\\
&\quad + f_3(N-1,m_0,m_1-2,s,r) \qquad (7)
\end{aligned}
$$

Those may written as

$$\text{(9)}\qquad \begin{pmatrix} w^2 & w^2y & w^2y & w^2y^2 \\ wxyz & wxz & wxy^2z & wxyz \\ wxyz & wxy^2z & wxz & wxyz \\ x^2y^2 & x^2y & x^2y & x^2 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \eta^{-1} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

or

$$Q_2F = \eta^{-1}F$$

From Eq.(9), we see again that $\eta^{-1} = \lambda$ is an eigenvalue of the SOM, Q_2. Again, Q_2 may be factored

$$\begin{aligned} Q_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} w^2 & w^2y & w^2y & w^2y^2 \\ wxyz & wx & wxy^2z & wxy \\ wxy & wxy^2 & wx & wxy \\ x^2y^2 & x^2y & x^2y & x^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} w & wy \\ xy & x \end{pmatrix} \otimes \begin{pmatrix} w & wy \\ xy & x \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \left(\begin{pmatrix} w & 0 \\ 0 & x \end{pmatrix} \otimes \begin{pmatrix} w & 0 \\ 0 & x \end{pmatrix} \right) \end{aligned}$$

$$\text{(10)}\qquad \left(\begin{pmatrix} 1 & y \\ y & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & y \\ y & 1 \end{pmatrix} \right) = P_2 Q_1^{[2]}$$

where [2] denotes the Kronecker product[1] of Q_1 with itself and

$$\text{(11)}\qquad P_2 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$Q_1^{[2]}$ in Eq.(10) can be interpreted as representing the nearest neighbor degeneracies of two independent $1XN$ lattices and P_2 describes the nearest neighor interaction between the two $1XN$ strips of which the $2XN$ space is composed.

We continue for a $3XN$ lattice by writing a $1XN$ space united with a $2XN$ space.

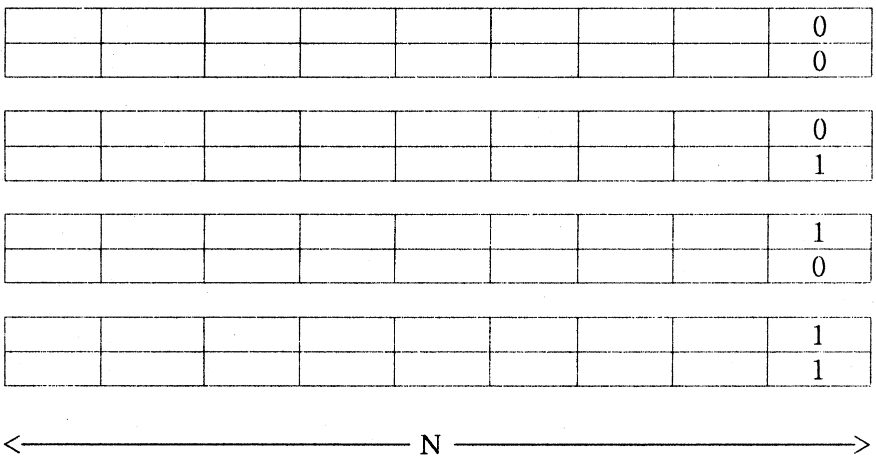

Fig. 3. The figure serves to define four exhaustive and mutually $2 \times N$ lattice space.

$$Q_3 = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && z &&&&& \\ &&& z &&& 0 & \\ &&&& z &&& \\ &&&&& z && \\ &&& 0 &&& 1 & \\ &&&&&&& 1 \end{pmatrix} Q_1 \otimes Q_2$$

$$= (P_2 \otimes I)(Q_1 \otimes P_2 Q_1^{[2]})$$

$$= (P_2 \otimes I)(I Q_1 \otimes P_2 Q_1^{[2]})$$

or

$$Q_3 = (P_2 \otimes I)(I \otimes P_2)Q_1^{[3]}$$
$$= P_3 Q_1^{[3]} \tag{12}$$

where I is the 2×2 identity matrix, where(because the matrices are commensurate) we have ussd^2 for any non-singular matrix G_j

$$(G_1 \otimes G_2)(G_3 \otimes G_4) = (G_1 G_3) \otimes (G_2 G_4) \tag{13}$$

and where the diagonal matrix P_3 is given by

$$P_3 \equiv (P_2 \otimes I)(I \otimes P_2) \tag{14}$$

Thus, Q_3 can be considered to be the result either of joining a $1XN$ lattice (represented by Q_1) with a $2XN$ lattice (represented by Q_2) or of joining three $1XN$ lattice(see Eq.(12)). To make the generalization manifest, we consider a $4XN$ lattice;

$$Q_4 = (P_2 \otimes I^{[2]})Q_1 \otimes Q_3$$
$$= (P_2 \otimes I^{[2]})Q_1 \otimes [(P_2 \otimes I)(I \otimes P_2^{[3]})]$$
$$= [P_2 \otimes I^{[2]}]\{I \otimes [(P_2 \otimes I)(I \otimes P_2)]Q_1^{[4]}\} \tag{15}$$

where we have used Eq.(13) ; finally

$$Q_4 = [P_2 \otimes I \otimes I][I \otimes P_2 \otimes I][I \otimes I \otimes P_2]Q_1^{[4]}$$
$$= P_4 Q_1^{[4]} \tag{16}$$

where the diagonal matrix

$$P_4 \equiv [P_2 \otimes I \otimes I][I \otimes P_2 \otimes I][I \otimes I \otimes P_2] \tag{17}$$

(note the similatrity with Eq.(14).)

The various factors in P_4 can be best interpreted by means of Fig. 5. In this figure, the sixteen possible N-th columns of a $4XN$ lattice are shown; next to each partition is the conjugate variable associated with change or not.

The sequence of columns has been chosen to be consistent with the decomposition enumerated in Fig. 2 and 3.

The factor, $I \otimes I \otimes P_2$ multiplying $Q_1^{[4]}$ is a diagonal matrix whoose elements describe the interaction between the top and the next-to-the-top $1XN$ lattice strips of which the $4XN$ lattice is composed.

The diagonal elements of this matrix are given in the order(from left to right) of operators shown between the top and the next-to-the-top sites of the sixteen columns shown in the figure, i.e.$1, z, z, 1, 1, z, z, 1, 1, z, z, 1, 1, z, z, 1$.

Similarly, $I \otimes P_2 \otimes I$ describes the interaction between the middle two rows. The diagonal of this matrix are given in the order(from left to right) of the operators shown between the middle two sites of the sixteen N-th columns shown in the figure, i.e.$1, 1, z, z, z, z, 1, 1, 1, 1, z, z, z, z, z, 1, 1$.

By analogy, it is seen that $P_2 \otimes I \otimes I$ is diagonal matrix whose non-zero elements are $1, 1, 1, 1, z, z, z, z, z, z, z, z, 1, 1, 1, 1$.

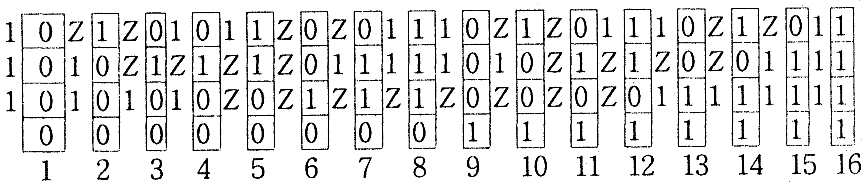

Fig. 5. The figure shows all 16 of the possible occupational arrangements for the N-th column a $4XN$ lattice. Besides each separation in a column i.e., besides each neighbor pair, is the activity associated

with it. The product of all three activities in our arrangement yields the associated element in the diagonal matrix. P_{M_p}(see Eqs.(17) and (18))

Thus, by considering a figure such as Fig.5, one can easily obtain the (diagonal) elements of P_4 by inspection, without carring out the indicated matrix multiplications. The (k,k)-th element of P_4 can be obtained by forming the product of 1 and z in the N-th column.

Thus for example, the diagonal elements of P_4 are $1, z, z^2, z, z^2, z^3, z^2, z, z, z^2, z^3, z^2, z, z^2, z, 1$. Utilizing Eqs. (12) and (16), we may generalize the foregoing results to be

$$P_{MP} = \prod_{K=1}^{M-1} \prod_{\substack{\otimes \\ J=1}}^{M-1} \gamma_{jk} \tag{18}$$

where $\prod$ denotes ordinary matrix multiplication, where $\prod_{\otimes}$ implies the Kronecker product and

$$\gamma_{jk} = \begin{cases} P_2 & j = k \\ I & j \neq k \end{cases} \tag{19}$$

Thus, for a planar MXN lattice

$$Q_{MP} = P_{MP} Q_1^{[M]} \tag{20}$$

In Eq.(20), $Q_1^{[M]}$ is the matrix describing the occupational and composite nearest neighbor degeneracise of M non-interacting $1 \times N$ lattice spaces. P_{MP} describes the nearest neighboring $1 \times N$ strips with neighboring $1 \times N$ strips.

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