JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 8, June 1995

A NEW EQUILIBRIUM EXISTENCE VIA CONNECTEDNESS

WON KYU KIM

ABSTRACT. The purpose of this note is to give a new existence of fixed point using the connectedness, and next an equilibrium existence theorem for 1-person game is established as an application.

In 1950, Nash first proved the existence of equilibrium for games where the player's preferences are representable by continuous quasi-concave utilities and the strategy sets are simplexes. Next Debreu proved the existence of equilibrium for abstract economies. Recently, the existence of Nash equilibrium can be further generalized in more general settings by several authors, e.g. Tan-Yuan [4], Tarafdar [5] and the references therein. In the previous many results on the existence of equilibrium, the convexity assumption is very essential and the main proving tool is the maximal element technique. Still there have been a number of generalizations and applications of equilibrium existence theorem in generalized games.

In this note, we first give a new topological fixed point theorem using the connectedness and next we shall prove a new equilibrium existence theorem for non-compact non-convex 1-person game.

We first recall the following notations and definitions. Let A be a non-empty set. We shall denote by 2^A the family of all subsets of A. Let X, Y be non-empty topological spaces and $T : X \to 2^Y$ be a

This work was partially supported by KOSEF in 1995.
Received by the editors on June 30, 1995.
1991 *Mathematics subject classifications*: Primary 54C60, 90A12.

multimap. Then T is said to be *open* or have *open graph* (respectively, *closed* or *closed graph*) if the graph of T ($\mathrm{Gr}\, T = \{(x, T(x)) \in X \times Y \mid x \in X\}$) is open in $X \times Y$. We may call $T(x)$ the *upper section* of T, and $T^{-1}(y)(= \{x \in X \mid y \in T(x)\})$ the *lower section* of T. It is easy to check that if T has open graph, then the upper and lower sections of T are open ; however the converse is not true in general. A multimap $T : X \to 2^Y$ is said to be *closed at* x if for each net $(x_\alpha) \to x$, $y_\alpha \in T(x_\alpha)$ and $(y_\alpha) \to y$, then $y \in T(x)$. And T is *closed* on X if it is closed at every point of X. Note that if T is single-valued, then the closedness is equivalent to continuity as a function. A multimap $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, then there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$. It is easy to see that if T is upper semicontinuous and each $T(x)$ is non-empty closed, then T has closed graph; so T is closed (for the proof, see Proposition 11.9 in [1]).

We begin with the following :

LEMMA. *Let X be a non-empty connected subset of a Hausdorff topological space E and $T : X \to 2^X$ be a closed multimap such that $T(x)$ is non-empty for each $x \in X$. If $T^{-1}(y_o)$ is non-empty open in X for some $y_o \in X$, then T has a fixed point $y_o \in X$, i.e. $y_o \in T(y_o)$.*

PROOF. We first show that the lower section $T^{-1}(y_o)$ is closed. In fact, for every net $(x_\alpha)_{\alpha\in\Gamma} \subset T^{-1}(y_o)$ with $(x_\alpha) \to x$, we have $y_o \in T(x_\alpha)$ for each $\alpha \in \Gamma$ and $(x_\alpha) \to x$, so by the closedness of T at x, $y_o \in T(x)$. Hence $x \in T^{-1}(y_o)$, so $T^{-1}(y_o)$ is closed. By the assumption, $T^{-1}(y_o)$ is a non-empty open and closed subset of X. Therefore, by the connectedness of X, $T^{-1}(y_o) = X$. Hence we have $y_o \in T(x)$ for each $x \in X$ and hence $y_o \in T(y_o)$, which completes the proof.

We can generalize the above Lemma in more general settings, e.g. see [3]. Now we can give a simple example which is suitable for our Lemma:

EXAMPLE. Let $X = \{(x,y) \in R^2 \mid 0 \leq x,\ 0 \leq y \leq \frac{1}{x}\}$ be a connected set in R^2 and a multimap $T : X \to 2^X$ be defined as follows :

$$T(x,y) := \text{line segment from } (0,0) \text{ to } \frac{1}{2}(x,y) \quad \text{for each} \quad (x,y) \in X.$$

Then it is easy to see that the multimap T is closed at every $(x,y) \in X$ and note that $T^{-1}(0,0) = X$ is open. Therefore, by Lemma, T has a fixed point $(0,0)$ in X. Note that since X is not compact convex, many known fixed point theorems can not work for this multimap T.

Using Lemma, we shall prove the following new equilibrium existence theorem for a connected 1-person game.

THEOREM. *Let $\Gamma = (X, A, P)$ be an 1-person game such that*

(1) X is a non-empty connected subset of a Hausdorff topological space,

(2) the multimap $A : X \to 2^X$ is upper semicontinuous such that for each $x \in X$, $A(x)$ is non-empty closed in X,

(3) the multimap $P : X \to 2^X$ is upper semicontinuous such that $P(x)$ is closed in X for each $x \in X$,

(4) the set $C := \{x \in X : (A \cap P)(x) \neq \emptyset\}$ is closed,

(5) for some $y_o \in X$, $A^{-1}(y_o)$ and $A^{-1}(y_o) \cap P^{-1}(y_o)$ are non-empty open in X,

(6) for such $y_o \in X$, $y_o \notin P(y_o)$.

Then Γ has an equilibrium choice $y_o \in X$, i.e.

$y_o \in A(y_o)$ *and* $A(y_o) \cap P(y_o) = \emptyset$.

PROOF. We first define a multimap $\phi : X \to 2^X$ by

$$\phi(x) = \begin{cases} A(x), & \text{if} \quad x \notin \mathcal{C}, \\ A(x) \cap P(x), & \text{if} \quad x \in \mathcal{C}. \end{cases}$$

Then $\phi(x) \neq \emptyset$ for each $x \in X$. We shall show that ϕ is upper semicontinuous. Let V be any open subset of X containing $\phi(x)$. Then we let

$$\begin{aligned} U &:= \{x \in X : \phi(x) \subset V\} \\ &= \{x \in \mathcal{C} : \phi(x) \subset V\} \cup \{x \in X \setminus \mathcal{C} : \phi(x) \subset V\} \\ &= \{x \in \mathcal{C} : (A \cap P)(x) \subset V\} \cup \{x \in X \setminus \mathcal{C} : A(x) \subset V\} \\ &= \{x \in X : (A \cap P)(x) \subset V\} \cup \{x \in X \setminus \mathcal{C} : A(x) \subset V\}. \end{aligned}$$

Since $X \setminus \mathcal{C}$ is open, A is upper semicontinuous and $A \cap P$ is upper semicontinuous, U is open and hence ϕ is also upper semicontinuous. Since each $\phi(x)$ is non-empty closed, by Proposition 11.9 in [1], ϕ is closed.

Next we shall show that $\phi^{-1}(y_o)$ is an open subset of X. In fact, by the assumption (4), we have that

$$\begin{aligned} \phi^{-1}(y_o) &= \{x \in X : y_o \in \phi(x)\} \\ &= \{x \in \mathcal{C} : y_o \in \phi(x)\} \cup \{x \in X \setminus \mathcal{C} : y \in \phi(x)\} \\ &= [\mathcal{C} \cap (A \cap P)^{-1}(y_o)] \cup [(X \setminus \mathcal{C}) \cap A^{-1}(y_o)] \\ &= A^{-1}(y_o) \cap [P^{-1}(y_o) \cup ((X \setminus \mathcal{C}) \cap A^{-1}(y_o))] \\ &= [A^{-1}(y_o) \cap P^{-1}(y_o)] \cup [(X \setminus \mathcal{C}) \cap A^{-1}(y_o)] \end{aligned}$$

is non-empty open in X. Therefore, by Lemma, we have a point $y_o \in X$ such that $y_o \in \phi(y_o)$. If $y_o \in \mathcal{C}$, then $y_o \in \phi(y_o) = A(y_o) \cap P(y_o)$, which contradicts the assumption (6). Therefore, we have $y_o \notin \mathcal{C}$ and so $y_o \in \phi(y_o) = A(y_o)$, i.e. $y_o \in A(y_o)$ and $A(y_o) \cap P(y_o) = \emptyset$. This completes the proof.

REMARKS. (i) The set $\mathcal{C}$ is a proper (maybe empty) subset of X. In fact, if $\mathcal{C} = X$, then by applying Lemma to $A \cap P$, then $y_o \in (A \cap P)(y_o)$, which contradicts the assumption (6).
(ii) Our Theorem is quite different from the previous many equilibrium existence theorems, e.g. see [4,5]. In these results, the compactness and convexity assumptions are very essential. But in the above Theorem, we do not need any compact convex assumption, but we need the connectedness assumption.

Finally, it should be noted that by modifying the methods in [3,4], we can show that the case of m agents can be reduced to the 1-person game.

REFERENCES

1. K. C. Border, *Fixed Point Theorem with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge, 1985.
2. J. Dugundji and A. Granas, *Fixed Point Theory, Vol. 1*, PWN, Warsaw, 1982.
3. W. K. Kim and K. H. Lee, *New existence of equilibrium via connectedness*, submitted.
4. K.-K. Tan and X.-Z. Yuan, *Maximal elements and equilibria for $\mathcal{U}$-majorized preferences*, Bull. Austral. Math. Soc **49** (1994), 47 –54.
5. E. Tarafdar, *A fixed point theorem and equilibrium point of an abstract economy*, J. Math. Econom. **20** (1991), 211–218.

DEPARTMENT OF MATHEMATICS EDUCATION
CHUNGBUK NATIONAL UNIVERSITY
CHEONGJU 360-763, KOREA