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ON THE INTEGRAL AND INTEGRO-DIFFERENTIAL INEQUALITIES

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ABSTRACT. We complete the proofs of integral and integro-differential inequalities presented in Lakshmikantham, Leela and Raos' results.

1. Introduction

There are various types of differential inequalities, e.g., Gronwall type, Wendroff type, Bihari type inequalities. Theory of differential inequalities plays a prominent role in the study of qualitative and quantitative behavior of nonlinear differential systems.

In this paper we complete the proofs of integral and integro-differential inequalities appeared without proof in [3]. Our main reference is [2].

2. Main Results

First, the following scalar comparison result contains the central idea of the theory of inequalities.

THEOREM 1 [2, Theorem 1.5.2]. *Consider the saclar differential equation*

$$(1) \qquad \begin{cases} u' &= g(t,u) \\ u(t_0) &= u_0 \end{cases}$$

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where $g \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$. *Let* $r(t)$ *be the maximal solution of (1) existing on* $[t_0, \infty)$. *Suppose that* $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and*

$$Dm(t) \leq g(t, m(t)), \quad t \geq t_0.$$

where D *is any fixed Dini derivative. Then* $m(t_0) \leq u_0$ *implies*

$$m(t) \leq r(t), \quad t \geq t_0.$$

THEOREM 2 [2, Theorem 1.8.3]. *Let* $m \in C(\mathbb{R}^+, \mathbb{R}^+), f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ *and* $H \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$. *Suppose that* $g_0 \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ *satisfies* $g_0 \leq F$, *where*

$$F(t, u; t_0) = f(t, u) + \int_{t_0}^{t} K(t, s, u) ds$$

and

$$K(t, s, u) = H(t, s, \eta(s, t, u)),$$

where $\eta(s, T, v_0)$ *is the left maximal solution of*

$$(2) \qquad \begin{cases} u' &= g_0(t, u) \\ u(T) &= v_0 \end{cases}$$

existing on $[t_0, T]$. *Let the following hold:*

$$D^+ m(t) \leq f(t, m(t)) + \int_{t_0}^{t} H(t, s, m(s)) ds, \quad t \geq t_0.$$

Then $m(t_0) \leq u_0$ *implies*

$$m(t) \leq r(t), \quad t \geq t_0,$$

where $r(t)$ *is the maximal solution of*

$$(3) \qquad \begin{cases} u' &= F(t, u, t_0) \\ u(t_0) &= u_0 \end{cases}$$

existing on $[t_0, \infty)$.

Now, we obtain some results using the above fundamental results.

THEOREM 3. *Assume that* $f = g_0 = -c(u)$ *and* $H \geq 0$, *where* $c(u)$ *is a continuous, nondecreasing function in* u *and* $c(0) = 0$. *Then*

$$m(t) \leq r(t) \quad t \geq t_0.$$

where $r(t)$ *is the maximal solution of*

$$\begin{cases} u'(t) = & f(t, u(t)) + \int_{t_0}^{t} H\left(t, s, J^{-1}\left[J\left\{u(t) - (t-s)\right\}\right)\right]\right) ds \\ u(t_0) = & u_0 \end{cases}$$

existing on $[t_0, \infty)$, *whenever* $m(t_0) \leq u_0$. *Here*

$$J(u) = \int_{u_0}^{u} \frac{1}{c(s)} ds \quad \text{and} \quad J^{-1} \text{ is the inverse of } J.$$

PROOF. Notice that $c(u) > 0$ for $u > 0, J'(u) = \frac{1}{c(u)} > 0$ for $u > 0$ and

$$J(u) - J(u_0) = \int_{u_0}^{u} \frac{1}{c(s)} ds$$

is strictly nondecreasing. Thus J^{-1} exists and it is strictly nondecreasing. It is sufficient to show that

$$\eta(t, T, v_0) = J^{-1}\left[J(u(T)) - (t - T)\right]$$

is the maximal solution of

$$\begin{cases} u' & = -c(u) \\ u(T) & = v_0 \geq 0. \end{cases} \tag{4}$$

First, we must find the right maximal solution of

$$\begin{cases} u' & = c(u) \\ u(t_0) & = u_0. \end{cases} \tag{5}$$

Note that

$$J(u) - J(u_0) = \int_{t_0}^{t} \frac{u'}{c(u)} ds = \int_{t_0}^{t} ds$$
$$= \int_{u_0}^{u} \frac{1}{c(r)} dr = t - t_0.$$

Since $J(u)$ is strictly nondecreasing in u, we have

$$u(t) = J^{-1}\left[J(u_0) + t - t_0\right], \quad t \geq t_0.$$

Thus $v(t, t_0, u_0) = u(t)$ is the right maximal solution of (5).

Next, we find the left maximal solution of

$$((2)') \qquad \begin{cases} u' &= g_0(t,u) = -c(u) \\ u(t) &= v_0 \geq 0. \end{cases}$$

Note that $J(u) = J(u(T)) - (t-T)$ since $J(u) - J(u_0) = \int_T^t ds = T - t$. Since $J(u)$ is strictly nondecreasing in u, we have

$$u(t) = J^{-1}[J(u(T) - (t - T)], \quad t_0 \leq t \leq T.$$

Therefore $\eta(t, T, v_0) = J^{-1}[J(u(T) - (t - T)]$ is the right maximal solution of (2)'.

THEOREM 4. *Consider the special case* $f = g_0 = -\alpha u, \alpha > 0$ *and* $H(t, s, u) = H(t, s)u$ *with* $H(t, s) \geq 0$. *Then*

$$m(t) \leq m(t_0) \exp\left[\int_{t_0}^{t} B(s, t_0) ds\right], \quad t \geq t_0.$$

where $B(t, t_0) = -\alpha + \int_{t_0}^{t} H(t, s) e^{\alpha(t-s)} ds$.

PROOF. Note that the right maximal solution of

$$(6) \qquad \begin{cases} u' &= \alpha u \\ u(t_0) &= u_0 \end{cases}$$

is $u(t) = u_0 e^{\alpha(t-t_0)}, t \geq t_0$. Moreover,

$$\eta(t, T, v_0) = v_0 e^{\alpha(T-t_0)} = u(T) e^{\alpha(T-t)}$$

is a left maximal solution of

$$(7) \qquad \begin{cases} u' & = -\alpha u \\ u(T) & = v_0 \geq 0, \end{cases}$$

existing for $t_0 \leq t \leq T$. Thus we have

$$\begin{aligned} u' = F(t, u; t_0) &= f(t, u) + \int_{t_0}^{t} K(t, s, u) ds \\ &= -\alpha u + \int_{t_0}^{t} H(t, s, \eta(s, t, u)) ds \\ &= -\alpha u + \int_{t_0}^{t} H(t, s) e^{\alpha(t-s)} u(t) ds, \quad t_0 \leq s \leq t \\ &= B(t, t_0) u, \end{aligned}$$

where

$$B(t, t_0) = -\alpha + \int_{t_0}^{t} H(t, s) e^{\alpha(t-s)} ds.$$

Hence

$$u(t) = r(t) = u_0 \exp \left[\int_{t_0}^{t} B(s, t_0) ds \right], \quad t \geq t_0.$$

Consequently, we have

$$m(t) \leq m(t_0) \exp \left[\int_{t_0}^{t} B(s, t_0) ds \right], \quad t \geq t_0,$$

provided that $m(t_0) = u_0$.

THEOREM 5 [2, Theorem 1.8.1]. *Let* $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}), H \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}), H(t,s,u)$ *be nondecreasing in* u *for each* (t,s) *and for* $t \geq t_0$,

$$D^+m(t) \leq f(t, m(t)) + \int_{t_0}^{t} H(t,s,m(s))ds. \tag{8}$$

where $m \in C(\mathbb{R}^+, \mathbb{R}^+)$. *Suppose that* $r(t)$ *is the maximal solution of*

$$\begin{cases} u'(t) & = f(t,u(t)) + \int_{t_0}^{t} H(t,s,u(s))ds \\ v(t_0) & = u_0 \geq 0, \end{cases} \tag{9}$$

existing on $[t_0, \infty)$. *Then* $m(t_0) \leq u_0$ *implies* $m(t) \leq r(t), t \geq t_0$.

For the proof of the existence of the maximal solution for (9), see [4].

COROLLARY 6. *Assume that (8) hold with*

$$f = a(t)u, \quad H = H(t,s)u.$$

Then $m(t) \leq R(t,t_0)u(t_0), t \geq t_0$, *where* $R(t,s)$ *is the solution of*

$$\frac{\partial R(t,s)}{\partial s} + R(t,s)a(s) + \int_{s}^{t} R(t,\sigma)H(\sigma,s)d\sigma = 0, R(t,t) = I$$

on the interval $t_0 \leq s \leq t$.

PROOF. It is enough to show that $r(t) = R(t,t_0)u(t_0)$ is the solution of (9). Let $u(t)$ be any solution of (9). We prove that $u(t) = R(t,t_0)u(t_0)$. Put $v(s) = R(t,s)u(s)$ for $t_0 \leq s \leq t$. Then we obtain

$$\begin{aligned} v'(s) &= \frac{\partial R(t,s)}{\partial s}u(s) + R(t,s)u'(s) \\ &= \frac{\partial R(t,s)}{\partial s}u(s) + R(t,s)\left[a(s)u(s) + \int_{t_0}^{t} H(s,\sigma)u(\sigma)d\sigma\right] \\ &= \left[\frac{\partial R(t,s)}{\partial s} + R(t,s)a(s)\right]u(s) + R(t,s)\int_{t_0}^{s} H(s,\sigma)u(\sigma)d\sigma. \end{aligned}$$

By integrating from t_0 to t, we obtain

$$\begin{aligned}
&R(t,t)u(t) - R(t,t_0)u(t_0) \\
&= \int_{t_0}^{t} \left[\frac{\partial R(t,s)}{\partial s} + R(t,s)a(s)\right] u(s)ds \\
&\quad + \int_{t_0}^{t} R(t,s)\left[\int_{t_0}^{t} H(s,\sigma)u(\sigma)d\sigma\right] ds, \quad t_0 \le s \le t, \\
&= \int_{t_0}^{t} \left[\frac{\partial R(t,s)}{\partial s} + R(t,s)a(s) + \int_{s}^{t} R(t,\sigma)H(\sigma,s)d\sigma\right] u(s)ds \\
&= \int_{t_0}^{t} 0 \cdot u(s)ds = 0.
\end{aligned}$$

Thus $R(t,t)u(t) = R(t,t_0)u(t_0), t \ge t_0$. Therefore $u(t) = R(t,t_0)u(t_0)$.

We consider the integral equation of Volterra type given by

$$u(t) = h(t) + \int_{t_0}^{t} K(t,s,u(s))ds. \tag{10}$$

where $h \in C(\mathbb{R}^+,\mathbb{R})$ and $K \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\mathbb{R})$.

The following is the comparison result relative to the equation (10).

THEOREM 7 [2, Theorem 1.6.3]. *Suppose that*

$$m(t) \le h(t) + \int_{t_0}^{t} K(t,s,m(s))ds, \quad t \ge t_0,$$

where $m \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $K(t,s,u)$ is nondecreasing in u for each (t,s). Assume that $r(t)$ is the maximal solution of (10) existing on $[t_0,\infty)$. Then $m(t) \le r(t), t \ge t_0$.

COROLLARY 8. *If, in Theorem 7, $K(t,s,u) = K(t,s)u$, then $m(t) \leq r(t), t \geq t_0$, where $r(t)$ is the solution of (10) satisfying*

$$r(t) = h(t) - \int_{t_0}^{t} R(t,s)h(s)ds,$$

$R(t,s)$ being the resolvent kernel given by

$$R(t,s) = -k(t,s) + \int_{s}^{t} R(t,\sigma)K(\sigma,s)d\sigma.$$

PROOF. We prove that for any solution $r(t)$ of (10).

$$r(t) = h(t) - \int_{t_0}^{t} R(t,s)h(s)ds, \quad t \geq t_0.$$

Note that

$$R(t,s)r(s) = \left[h(s) + \int_{t_0}^{s} K(s,\tau)r(\tau)d\tau\right] R(t,s), \quad t_0 \leq s \leq t.$$

Integrating from t_0 to t, we obtain

$$\begin{aligned}\int_{t_0}^{t} R(t,s)r(s)ds - \int_{t_0}^{t} R(t,s)h(s)ds &= \int_{t_0}^{t} R(t,s)\left[\int_{t_0}^{s} K(s,\tau)r(\tau)d\tau\right] ds \\ &= \int_{t_0}^{t}\left[\int_{\tau}^{t} R(t,\sigma)K(\sigma,\tau)d\sigma\right] r(\tau)d\tau \\ &= \int_{t_0}^{t} \left[R(t,\tau) + K(t,\tau)\right] r(\tau)d\tau.\end{aligned}$$

Thus we have

$$-\int_{t_0}^{t} R(t,s)h(s)ds = \int_{t_0}^{t} K(t,\tau)r(\tau)d\tau.$$

This implies that

$$r(t) = h(t) - \int_{t_0}^{t} R(t,s)h(s)ds, \quad t \geq t_0.$$

Finally, we can prove the following [2, Theorem 1.3.3] as a corollary of Theorem 7.

COROLLARY 9. *Suppose that*

$$m(t) \leq h(t) + \int_{t_0}^{t} K(t,s)g(m(s))ds, \quad t \geq t_0.$$

where $m, h \in C(\mathbb{R}^+, \mathbb{R}^+), g \in C((0,\infty),(0,\infty)), g(u)$ *is nondecreasing in* $u, K \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+), K_t(t,s)$ *exists and it continuous nonnegative. Assume that* g *is subadditive. Then*

$$m(t) \leq h(t) + G^{-1}\left[G(c) + \int_{t_0}^{t} v_1(s)ds\right], \quad t_0 \leq t \leq T_0 < T,$$

where $G(u) - G(u_0) = \int_{u_0}^{u} \frac{1}{g(s)}ds, G^{-1}(u)$ *is the inverse of* $G(u)$,

$$T = \sup\left\{t \geq t_0 : G(u) + \int_{t_0}^{t} v(s)ds \in dom\ G^{-1}\right\}, \quad c = \int_{t_0}^{T_0} v_2(s)ds,$$

$$v_1(t) = K(t,t) + \int_{t_0}^{t} K_t(t,s)ds, v_2(t) = K(t,t)g(h(t)) + \int_{t_0}^{t} K_t(t,s)g(h(s))ds.$$

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