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REMARK ON REGULAR MINIMAL SETS

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ABSTRACT. In this paper, we define a subgroup $S(X,\gamma)$ of the group of automorphisms of universal minimal sets and give a necessary and sufficient condition for a minimal transformation group to be regular.

Let (M,T) be a universal minimal transformation group, and let G be the group of automorphisms of (M,T). Given a minimal transformation group (X,T), and a homomorphism $\gamma : M \to X$, J. Auslander [3] defined a subgroup of G as follows.

$$G(X,\gamma) \equiv \{\alpha \in G \mid \gamma\alpha = \gamma\}$$

that is, a homomorphism from (M,T) to a minimal set determines a subgroup $G(X,\gamma)$ of G. He showed that different homomorphisms determines conjugate subgroups and also obtained an information about homomorphisms of distal minimal sets and regular minimal sets.

In this paper, we define a subgroup $S(X,\gamma)$ of G and give a necessary and sufficient condition for a minimal transformation group to be regular minimal.

Throughout this paper, (X,T) will denote a transformation group with compact Hausdorff phase space X. A closed nonempty subset A of X is called a *minimal set* if for every $x \in A$ the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called

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an ***almost periodic point.*** If X is itself minimal, we say that it is a *minimal transformation group* or a *minimal set.*

The compact Hausdorff space X carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X \times X$. The points x and y of X are called *proximal* provided that for each index U of X, there exists a $t \in T$ such that $(xt, yt) \in U$.

Let (X,T) and (Y,T) be transformation groups. If π is a continuous map from X to Y with $\pi(xt) = \pi(x)t \quad (x \in X,\ t \in T)$, then π is called a *homomorphism.* A homomorphism h of X into itself is called an *endomorphism. Automorphism* is defined similarly.

We denote the automorphisms of (X,T) by $A(X)$. If every endomorphism of X is an automorphism, then the transformation group (X,T) is said to be *coalescent.*

Let $\{(X_i,T) \mid i \in I\}$ be a family of transformation groups with the same phase group T. The product transformation group $(\Pi_i X_i, T)$ is defined by the condition that $(x_i \mid i \in I) \in \Pi_i X_i$ and $t \in T$ imply $(x_i \mid i \in I)\, t = (x_i t \mid i \in I)$.

We define E, the *enveloping semigroup* of (X,T), to be the closure of T in X^X, providing X^X with its product topology. The *minimal right ideal* I is the non-empty subset of E with $IE \subset I$, which contains no proper non-empty subset of the same property.

THEOREM 1 ([1], Theorem 3). *Let (X,T) be a minimal set. Then the following are equivalent.*

(1) If I is a minimal right ideal contained in the enveloping semigroup E of (X,T), then the minimal set (X,T) and (I,T) are isomorphic.

(2) (X,T) is isomorphic with (I,T), where I is a minimal right ideal in the enveloping semigroup of some transformation group (Z,T).

(3) If $x, y \in X$, then there is an endomorphism h of (X,T) such that $h(x)$ and y are proximal.

(4) If (x,y) is an almost periodic point of $(X \times X, T)$, then there is an endomorphism h of (X,T) such that $h(x) = y$.

A minimal set which satisfies any one of the properties (1) through (4) will be called *regular minimal.* It is well-known that regular minimal sets are coalescent.

DEFINITION 2 ([5]). Let T be an arbitrary topological group. A minimal transformation group (M,T) is said to be *universal* if every minimal transformation group with acting group T is a homomorphic image of (M,T).

For any group T, a universal minimal set exists and is unique up to isomorphism. ([5] ,[3]).

For a given $H \subset A(X)$, now we define a new subset $S_H(X,\gamma)$ of G, which is motivated by $G(X,\gamma)$.

DEFINITION 3. Let (M,T) be a universal minimal transformation group, which will be fixed from now on. Given a minimal transformation group (X,T), a homomorphism $\gamma : M \longrightarrow X$, and a subset H of $A(X)$, define

$$S_H(X,\gamma) = \{\alpha \in G \mid h\gamma\alpha = \gamma \ for \ some \ h \in H\}$$

If we take $H = \{1_X\}$, the trivial subgroup of $A(X)$, then $S_H(X,\gamma)$ coincides with $G(X,\gamma)$. We denote $S_{A(X)}(X,\gamma)$ by (S,γ), simply.

REMARK. From the definition, the following are verified easily.

1. If H is a subgroup of $A(X)$, then $S_H(X,\gamma)$ is a subgroup of G. In fact, let $\alpha_1, \alpha_2 \in S_H(X,\gamma)$, that is, $h\gamma\alpha_1 = \gamma$ and $g\gamma\alpha_2 = \gamma$ for some h, g in H. Then $(gh\gamma)\alpha_1\alpha_2 = g(h\gamma\alpha_1)\alpha_2 = g\gamma\alpha_2 = \gamma$. This shows that $\alpha_1\alpha_2 \in S_H(X,\gamma)$, because $gh \in H$. Next, let $\alpha \in S_H(X,\gamma)$. Since G is a group, there exists an α^{-1} in G such that $\alpha\alpha^{-1} = 1$. $\alpha \in S_H(X,\gamma)$ implies $h\gamma\alpha = \gamma$ for some $h \in H$, and so, $\gamma = \gamma\alpha\alpha^{-1} = h^{-1}\gamma\alpha^{-1}$. Therefore, $\alpha^{-1} \in S_H(X,\gamma)$.

2. If H and K are subsets of $A(X)$ with $H \subset K$, then $S_H(X,\gamma) \subset S_K(X,\gamma)$, thus we have

$$G(X,\gamma) \subset S_H(X,\gamma) \subset S(X,\gamma) \subset G.$$

THEOREM 4. *Let (M,T) be universal minimal, (X,T) a minimal, and let H be a subgroup of $A(X)$. If $\beta \in G$, then*

$$\beta^{-1}S_H(X,\gamma)\beta = S_H(X,\gamma\beta).$$

PROOF. Let $\alpha \in S_H(X,\gamma)$. It follows that $h\gamma\alpha = \gamma$ for some $h \in H$, and $h\gamma\beta(\beta^{-1}\alpha\beta) = h\gamma\alpha\beta = \gamma\beta$. So, we have $\beta^{-1}\alpha\beta \in S_H(X,\gamma\beta)$. Conversely, let $\alpha \in S_H(X,\gamma\beta)$. Then $h(\gamma\beta)\alpha = \gamma\beta$ for some $h \in H$, and it follows that $h\gamma(\beta\alpha\beta^{-1}) = \gamma$. That is, $\beta\alpha\beta^{-1} \in S_H(X,\gamma)$ and $\alpha \in \beta^{-1}S_H(X,\gamma)\beta$.

If we take $H = \{1_X\}$, then Lemma 2 (ii) ([3]) is a corollary of Theorem 4.

COROLLARY 5. *Let (M,T) be a universal minimal and let (X,T) be minimal. If $\beta \in G$, then $\beta^{-1}G(X,\gamma)\beta = G(X,\gamma\beta)$.*

From the Lemma 1 ([3]), we have the following.

LEMMA 6. *Let (M,T) be a universal minimal transformation group and let $\gamma : M \longrightarrow X$ be a homomorphism. The following hold.*

(i) If X is minimal, then given $h \in A(X)$, there exists an $\alpha \in G$ such that $h\gamma\alpha = \gamma$.

(ii) If X is regular minimal, then given $\alpha \in G$, there exists an $h \in A(X)$ such that $h\gamma\alpha = \gamma$.

PROOF. (i) Let $x \in X$. Since $(h(x),x)$ is an almost periodic point of $(X \times X,T)$, there exists an almost periodic point (m_1,m_2) of $(M \times$

$M, T)$ such that $(\gamma(m_1, \gamma(m_2)) = (h(x), x)$. Let $\alpha \in G$ such that $\alpha(m_1) = m_2$. Then $\gamma\alpha(m_1) = \gamma(m_2) = x$ and hence $h\gamma\alpha(m_1) = h(x) = \gamma(m_1)$. This shows that $h\gamma\alpha = \gamma$.

(ii) is proved similarly as in (i).

LEMMA 7 ([1] Lemma 2). *Let (X, T), (Y, T) be minimal with (Y, T) regular minimal, and let h_1 and h_2 be homomorphisms from (X, T) to (Y, T). Then there is a unique automorphism k of (Y, T) such that $h_2 = kh_1$.*

Let M be universal, X, Y minimal and let $\gamma : M \to X$, $\pi : X \to Y$ be homomorphisms. Then $G(X, \gamma) \subset G(Y, \pi\gamma)$ is always true, but $S(X, \gamma) \subset S(Y, \pi\gamma)$ is not, in general. Regular minimality of Y ensures the following theorem.

THEOREM 8. *Let X be minimal and Y a regular minimal. If $\pi : X \to Y$ is a homomorphism, then $S(X, \gamma) \subset S(Y, \pi\gamma)$.*

PROOF. Let $\alpha \in S(X, \gamma)$. Then $h\gamma\alpha = \gamma$ for some $h \in A(X)$. Given homomorphisms $\pi : X \to Y$, and $\pi h : X \to Y$, there is a unique $k \in A(X)$ such that $\pi h = k\pi$ by Lemma 7, we have $k\pi\gamma\alpha = \pi h\gamma\alpha = \pi\gamma$. Therefore $\alpha \in S(Y, \pi\gamma)$.

Now, we define an equivalent condition for a minimal transformation group to be regular.

THEOREM 9. *Let (X, T) be regular minimal, and let (Y, T) be minimal and let $\pi : X \to Y$ be a homomorphism. Then the following are equivalent ;*

(i) (Y, T) is regular minimal

(ii) $S(X, \gamma) \subset S(Y, \pi\gamma)$

PROOF. (i) implies (ii) follows from Theorem 8. Now, we show that (ii) implies (i). Let $(y_1, y_2) \in (Y \times Y, T)$ be an almost periodic point of $(Y \times Y, T)$. We show that there exists an automorphism k of Y such that $k(y_1) = y_2$. Since (y_1, y_2) is an almost periodic point, there exists an almost periodic point (x_1, x_2) of $(X \times X, T)$ such that

$$\pi^*((x_1, x_2)) = (\pi(x_1), \pi(x_2)) = (y_1, y_2) \tag{1}$$

where $\pi^* : X \times X \longrightarrow Y \times Y$ is the map defined by $\pi^*(x, x') = (\pi(x), \pi(x'))$. There exists also an almost periodic point (m_1, m_2) of $(M \times M, T)$ such that

$$\gamma^*((m_1, m_2)) = (\gamma(m_1), \gamma(m_2)) = (x_1, x_2) \tag{2}$$

where γ^* is defined similarly as π^*. From (1) and (2), we obtain

$$(\pi\gamma(m_1), \pi\gamma(m_2)) = (\pi(x_1), \pi(x_2)) = (y_1, y_2)$$

Since X is regular minimal (and hence X is coalescent), there exists an automorphism h of X such that $h(x_1) = x_2$. Define

$$\alpha(m_2) = m_1$$

Then

$$\gamma\alpha(m_2) = \gamma(m_1) = x_1$$

and

$$h\gamma\alpha(m_2) = h(x_1) = x_2 = \gamma(m_2)$$

which shows that $h\gamma\alpha = \gamma$, and therefore $\alpha \in S(X, \gamma)$. Since $S(X, \gamma) \subset S(Y, \pi\gamma)$, $\alpha \in S(Y, \pi\gamma)$. That is, $k\pi\gamma\alpha = \pi\gamma$ for some automorphism k of Y. It follows that

$$\begin{aligned} ky_1 &= k\pi(x_1) = k\pi\gamma(m_1) = k\pi\gamma\alpha(m_2) \\ &= \pi\gamma(m_2) = \pi(x_2) = y_2 \end{aligned}$$

Therefore, Y is regular minimal.

LEMMA 10 ([3], Lemma 3). *Let (X, T) be minimal, and let $\gamma : M \to X$ be a homomorphism. If (X, T) is regular, and $\sigma \in G$, there is an automorphism h of (X, T) such that $\gamma\sigma = h\gamma$.*

In [3], Auslander showed that if X is regular minimal, then $G(X, \gamma)$ is a normal subgroup of G. Similary, so is $S(X, \gamma)$. In fact, let $\alpha \in S(X, \gamma)$ and let $\sigma \in G$. Then $\sigma \in G$ implies $k\gamma\sigma = \gamma$ for some $k \in A(X)$ by Lemma 10, and since $\alpha \in S(X, \gamma)$, $h\gamma\alpha = \gamma$ for some $h \in A(X)$. Furthermore, $\gamma = k^{-1}\gamma\sigma^{-1}$. Thus,

$$khk^{-1}\gamma(\sigma^{-1}\alpha\sigma) = kh(k^{-1}\gamma\sigma^{-1})\alpha\sigma = k(h\gamma\alpha)\sigma = k\gamma\sigma = \gamma.$$

Since $khk^{-1} \in A(X)$, we have $\sigma^{-1}\alpha\sigma \in S(X, \gamma)$. We conclude that regular minimality of X implies $S(X, \gamma)$ is a normal subgroup of G.

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