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A NOTE ON THE LIPSCHITZ CLASSES OF PERIODIC STOCHASTIC PROCESSES

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ABSTRACT. In this paper, in order to generalize the Theorem[1], it was attempted to generalize the method to prove the Theorem.

1. Introduction

Throughout this paper, $(\Omega, \mathcal{F}, P)$ is the underlying probability space and $X(t,\omega)$, $t \in \mathbf{R}$, is a stochastic process of the r-th order, $r \in [1, \infty)$, that is,

$$||X(t,\omega)||_r = (E|X(t,\omega)|^r)^{\frac{1}{r}} < \infty.$$

We say $X(t,\omega)$ is periodic with period 2π, if

$$||X(t+2\pi,\omega) - X(t,\omega)||_r = 0 \qquad \text{for every } t$$

and

$$\int_{-\pi}^{\pi} ||X(t,\omega)||_r^r \, dt < \infty.$$

The class of 2π-periodic processes of the r-th order will be denoted by L_p^r.

Let $\phi(h)$ be a positive nondecreasing function of h∈(0,1]. Write

$$\Delta_h^j X(t,\omega) = \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{j}{\nu} X(t+\nu h, \omega),$$

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where j is a positive integer. The class of $X(t,\omega)$ with the property

$$\sup_{h>0}\int_{-\pi}^{\pi}\left(\frac{\|\Delta_h^j X(t,\omega)\|_r}{\phi(h)}\right)^r dt < \infty$$

is denoted by $\Delta_{p,r}^*(\phi)$ and is called the Lipschitz class $\Delta_{p,r}^*(\phi)$.

For a stochastic process $X(t,\omega)$ of L_p^r which belong to $\Delta_{p,r}^*(\phi)$, we consider the Fourier series

$$X(t,\omega) \sim \sum_{n=-\infty}^{\infty} C_n(\omega)e^{int},$$

where

$$C_n(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(t,\omega)e^{-int}\,dt.$$

In this study, in order to generalize the Theorem[1], it was attempted to generalize the method to prove the Theorem.

Let $I(h)$ be a positive non decreasing function on $(0,1]$ such that

$$I(2h) \le K_1 I(h) \tag{1}$$

and

$$\int_0^h \frac{u^{jp}}{I(u)}\,du \le K_2\frac{h^{jp+1}}{I(h)} \tag{2}$$

$$\int_h^1 \frac{du}{I(u)} \le K_3\frac{h}{I(h)} \tag{3}$$

where K_1, K_2, K_3 are constants. We put

$$A_p(X,I,a) = \int_0^1 \frac{1}{I(h)}\left(\int_{-\pi}^{\pi}\|\Delta_h^j X(t,\omega)\|_a^a\,dt\right)^{\frac{p}{a}} dh$$

$$B_p(X,I,a') = \sum_{n=1}^{\infty} \frac{n^{-2}}{I(\frac{1}{n})} \left(\sum_{|k|>n} ||C_k(\omega)||_a^{a'} \right)^{\frac{p}{a'}}$$

$$C_p(X,I,a') = \sum_{n=1}^{\infty} \frac{n^{-(2+jp)}}{I(\frac{1}{n})} \left(\sum_{|k|=1}^{n} k^{ja'} ||C_k(\omega)||_a^{a'} \right)^{\frac{p}{a'}}$$

where $a \geq 1, \frac{1}{a} + \frac{1}{a'} = 1$, and $0 < p \leq a'$.

We give the following Lemma.

LEMMA. *The conditions $B_p(X,I,a') < \infty$ and $C_p(X,I,a') < \infty$ are, respectively, equivalent to the relations*

$$B_p^*(X,I,a') = \sum_{k=0}^{\infty} \frac{2^{-k}}{I(\frac{1}{2^k})} \left(\sum_{\nu=2^k}^{\infty} ||C_\nu(\omega)||_a^{a'} \right)^{\frac{p}{a'}} < \infty$$

and

$$C_p^*(X,I,a') = \sum_{k=0}^{\infty} \frac{2^{-(jp+1)k}}{I(\frac{1}{2^k})} \left(\sum_{\nu=1}^{2^k} \nu^{ja'} ||C_\nu(\omega)||_a^{a'} \right)^{\frac{p}{a'}} < \infty.$$

From now on K is a constant which may be different in each occurrence.

THEOREM 1. *(I) The relations $B_p(X,I,a') < \infty$ and $C_p(X,I,a') < \infty$ are equivalent. (II) If $a \geq 2, \frac{1}{a} + \frac{1}{a'} = 1$ and $0 < p \leq a'$, then the fact that one of the expressions $B_p(X,I,a')$ and $C_p(X,I,a')$ is finite implies that $A_p(X,I,a)$ is finite. (III) If $1 < a \leq 2, \frac{1}{a} + \frac{1}{a'} = 1$ and $0 < p \leq a'$, then the fact that $A_p(X,I,a)$ is finite implies that $B_p(X,I,a')$ and $C_p(X,I,a')$ are finite.*

PROOF. (I) It is sufficient to show that $B_p^*(X,I,a') < \infty$ is equivalent to $C_p^*(X,I,a') < \infty$. We know that

$$C_p^*(X,I,a') \le \sum_{k=0}^{\infty} \frac{2^{-(jp+1)k}}{I(\frac{1}{2^k})} \left[\sum_{m=0}^{k} \left(\sum_{\nu=2^m}^{2^{m+1}} \nu^{ja'} ||C_\nu(\omega)||_a^{a'} \right) \right]^{\frac{p}{a'}}$$

$$\le K \sum_{k=0}^{\infty} \frac{2^{-(jp+1)k}}{I(\frac{1}{2^k})} \left[\sum_{m=0}^{k} 2^{mja'} \sum_{\nu=2^m}^{2^{m+1}} ||C_\nu(\omega)||_a^{a'} \right]^{\frac{p}{a'}}.$$

Since $0 < p \le a'$, we find from the last inequality that

$$C_p^*(X,I,a') \le K \sum_{k=0}^{\infty} \frac{2^{-(jp+1)k}}{I(\frac{1}{2^k})} \left[\sum_{m=0}^{k} 2^{mjp} \left(\sum_{\nu=2^m}^{2^{m+1}} ||C_\nu(\omega)||_a^{a'} \right)^{\frac{p}{a'}} \right]$$

$$= K \sum_{m=0}^{\infty} 2^{mjp} \left(\sum_{\nu=2^m}^{2^{m+1}} ||C_\nu(\omega)||_a^{a'} \right)^{\frac{p}{a'}} \sum_{k=m}^{\infty} \frac{2^{-(jp+1)k}}{I(\frac{1}{2^k})}.$$

In addition,

$$\sum_{k=m}^{\infty} \frac{2^{-(jp+1)k}}{I(\frac{1}{2^k})} \le K \sum_{k=m}^{\infty} \int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} \frac{u^{jp}}{I(u)}\,du \le K \int_0^{2^{-m}} \frac{u^{jp}}{I(u)}\,du.$$

From this last and (2) we have

$$\sum_{k=m}^{\infty} \frac{2^{-(jp+1)k}}{I(\frac{1}{2^k})} \le K \frac{2^{-(jp+1)m}}{I(\frac{1}{2^m})}.$$

Hence,

$$C_p^*(X,I,a') \le K \sum_{m=0}^{\infty} \frac{2^{-m}}{I(\frac{1}{2^m})} \left(\sum_{\nu=2^m}^{2^{m+1}} ||C_\nu(\omega)||_a^{a'} \right)^{\frac{p}{a'}}$$

$$\le K \sum_{m=0}^{\infty} \frac{2^{-m}}{I(\frac{1}{2^m})} \left(\sum_{\nu=2^m}^{\infty} ||C_\nu(\omega)||_a^{a'} \right)^{\frac{p}{a'}}$$

$$= K B_p^*(X,I,a') < \infty.$$

Let us prove the converse;

$$\begin{aligned} B_p^*(X,I,a') &= \sum_{m=0}^{\infty} \frac{2^{-m}}{I(\frac{1}{2^m})} \left(\sum_{\nu=2^m}^{\infty} \|C_\nu(\omega)\|_a^{a'} \right)^{\frac{p}{a'}} \\ &= \sum_{m=0}^{\infty} \frac{2^{-m}}{I(\frac{1}{2^m})} \left(\sum_{k=m}^{\infty} \sum_{\nu=2^k}^{2^{k+1}} \|C_\nu(\omega)\|_a^{a'} \right)^{\frac{p}{a'}} \\ &\le \sum_{m=0}^{\infty} \frac{2^{-m}}{I(\frac{1}{2^m})} \left(\sum_{k=m}^{\infty} 2^{-ja'k} \sum_{\nu=2^k}^{2^{k+1}} \nu^{ja'} \|C_\nu(\omega)\|_a^{a'} \right)^{\frac{p}{a'}}. \end{aligned}$$

Hence, we find that, for $0 < p \le a'$,

$$\begin{aligned} B_p^*(X,I,a') &\le \sum_{m=0}^{\infty} \frac{2^{-m}}{I(\frac{1}{2^m})} \sum_{k=m}^{\infty} 2^{-jpk} \left(\sum_{\nu=2^k}^{2^{k+1}} \nu^{ja'} \|C_\nu(\omega)\|_a^{a'} \right)^{\frac{p}{a'}} \\ &= \sum_{k=0}^{\infty} 2^{-jpk} \left(\sum_{\nu=2^k}^{2^{k+1}} \nu^{ja'} \|C_\nu(\omega)\|_a^{a'} \right)^{\frac{p}{a'}} \sum_{m=0}^{k} \frac{2^{-m}}{I(\frac{1}{2^m})}. \end{aligned}$$

Now from the relations (1) and (3), we can show that

$$\sum_{m=0}^{k} \frac{2^{-m}}{I(\frac{1}{2^m})} \le K \sum_{m=0}^{k} \int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}} \frac{du}{I(u)} = K \int_{\frac{1}{2^{k+1}}}^{1} \frac{du}{I(u)} \le K \frac{2^{-k}}{I(\frac{1}{2^k})}.$$

Consequently,

$$B_p^*(X,I,a') \le K \sum_{k=0}^{\infty} \frac{2^{-(jp+1)k}}{I(\frac{1}{2^k})} \left(\sum_{\nu=1}^{2^{k+1}} \nu^{ja'} \|C_\nu(\omega)\|_a^{a'} \right)^{\frac{p}{a'}} = K C_p^*(X,I,a').$$

(II) The Fourier coefficient of $\Delta_h^j X(t,\omega)$ is $C_n(\omega)(1-e^{inh})^j$. For $a \geq 2, \dfrac{1}{a}+\dfrac{1}{a'}=1, 0<p\leq a'$, by the Hausdorff-Young inequality, we have

$$\left(\int_{-\pi}^{\pi} |\Delta_h^j X(t,\omega)|^a \, dt\right)^{\frac{1}{a}} \leq \left(\sum_{k=-\infty}^{\infty} |C_k(\omega)|^{a'} |e^{ikh}-1|^{ja'}\right)^{\frac{1}{a'}}.$$

Thus

$$\left(\int_{-\pi}^{\pi} |\Delta_h^j X(t,\omega)|^a \, dt\right)^{\frac{a'}{a}} \leq \sum_{k=-\infty}^{\infty} |C_k(\omega)|^{a'} |e^{ikh}-1|^{ja'}.$$

Taking expectations of both sides, we have

$$E\left[\int_{-\pi}^{\pi} |\Delta_h^j X(t,\omega)|^a \, dt\right]^{\frac{a'}{a}} \leq \sum_{k=-\infty}^{\infty} ||C_k(\omega)||_a^{a'} |e^{ikh}-1|^{ja'}.$$

But

$$\begin{aligned} E\left[\int_{-\pi}^{\pi} |\Delta_h^j X(t,\omega)|^a \, dt\right]^{\frac{a'}{a}} &= E\left(\int_{-\pi}^{\pi} (|\Delta_h^j X(t,\omega)|^{a'})^{\frac{a}{a'}} \, dt\right)^{\frac{a'}{a}} \\ &\geq \left(\int_{-\pi}^{\pi} (E|\Delta_h^j X(t,\omega)|^{a'})^{\frac{a}{a'}} \, dt\right)^{\frac{a'}{a}} \\ &= \left(\int_{-\pi}^{\pi} (||\Delta_h^j X(t,\omega)||_a^{a'})^{\frac{a}{a'}} \, dt\right)^{\frac{a'}{a}} \\ &= \left(\int_{-\pi}^{\pi} ||\Delta_h^j X(t,\omega)||_a^{a} \, dt\right)^{\frac{a'}{a}}. \end{aligned}$$

Consequently,

$$\begin{aligned} A_p(X,I,a) &= \int_0^1 \frac{1}{I(h)} \left(\int_{-\pi}^{\pi} ||\Delta_h^j X(t,\omega)||_a^a \, dt\right)^{\frac{p}{a}} dh \\ &\leq K \int_0^1 \frac{1}{I(h)} \left(\sum_{k=-\infty}^{\infty} ||C_k(\omega)||_a^{a'} |e^{ihk}-1|^{ja'}\right)^{\frac{p}{a'}} dh \\ &= K \int_0^1 \frac{1}{I(h)} \left(\sum_{k=-\infty}^{\infty} ||C_k(\omega)||_a^{a'} |sin\frac{kh}{2}|^{ja'}\right)^{\frac{p}{a'}} dh. \end{aligned}$$

Since $0 < p \leq a'$, we find from the last inequality that

$$
\begin{aligned}
A_p(X,I,a) &\leq K\int_0^1 \frac{1}{I(h)}\left(\sum_{|k|=1}^{n} ||C_k(\omega)||_a^{a'} |sin\frac{kh}{2}|^{ja'}\right)^{\frac{p}{a'}} dh \\
&+ K\int_0^1 \frac{1}{I(h)}\left(\sum_{|k|>n} ||C_k(\omega)||_a^{a'} |sin\frac{kh}{2}|^{ja'}\right)^{\frac{p}{a'}} dh \\
&\leq K\sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{h^{jp}}{I(h)}\,dh \left(\sum_{|k|=1}^{n} k^{ja'}||C_k(\omega)||_a^{a'}\right)^{\frac{p}{a'}} \\
&+ K\sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{dh}{I(h)} \left(\sum_{|k|>n} ||C_k(\omega)||_a^{a'}\right)^{\frac{p}{a'}} \\
&\leq K\sum_{n=1}^{\infty} \frac{n^{-(jp+2)}}{I(\frac{1}{n})} \left(\sum_{|k|=1}^{n} k^{ja'}||C_k(\omega)||_a^{a'}\right)^{\frac{p}{a'}} \\
&+ K\sum_{n=1}^{\infty} \frac{n^{-2}}{I(\frac{1}{n})} \left(\sum_{|k|>n} ||C_k(\omega)||_a^{a'}\right)^{\frac{p}{a'}} \\
&= KC_p(X,I,a') + KB_p(X,I,a').
\end{aligned}
$$

(III) In case $1 < a \leq 2, \dfrac{1}{a} + \dfrac{1}{a'} = 1$, the Hausdorff-Young inequality gives us

$$\left(\sum_{k=-\infty}^{\infty} |C_k(\omega)|^{a'} |sin\frac{kh}{2}|^{ja'}\right)^{\frac{1}{a'}} \leq \left(\int_{-\pi}^{\pi} |\Delta_h^j X(t,\omega)|^a\,dt\right)^{\frac{1}{a}}.$$

Thus we have

$$\left(\sum_{k=-\infty}^{\infty} |C_k(\omega)|^{a'} |sin\frac{kh}{2}|^{ja'}\right)^{\frac{a}{a'}} \leq \int_{-\pi}^{\pi} |\Delta_h^j X(t,\omega)|^a\,dt.$$

Taking expectations of both sides, we have

$$E\left(\sum_{k=-\infty}^{\infty} |C_k(\omega)|^{a'} |sin\frac{kh}{2}|^{ja'}\right)^{\frac{a}{a'}} \le \int_{-\pi}^{\pi} ||\Delta_h^j X(t,\omega)||_a^a \, dt.$$

By the Minkowski inequality, the left hand side of this inequality is not less than

$$\left\{\sum_{k=-\infty}^{\infty} \left[E|C_k(\omega)|^a |sin\frac{kh}{2}|^{ja}\right]^{\frac{a'}{a}}\right\}^{\frac{a}{a'}} = \left\{\sum_{k=-\infty}^{\infty} ||C_k(\omega)||_a^{a'} |sin\frac{kh}{2}|^{ja'}\right\}^{\frac{a}{a'}}.$$

So,

$$\left\{\sum_{k=-\infty}^{\infty} ||C_k(\omega)||_a^{a'} |sin\frac{kh}{2}|^{ja'}\right\}^{\frac{1}{a'}} \le \left(\int_{-\pi}^{\pi} ||\Delta_h^j X(t,\omega)||_a^a \, dt\right)^{\frac{1}{a}}.$$

Consequently,

$$\begin{aligned} A_p(X,I,a) &= \int_0^1 \frac{1}{I(h)} \left(\int_{-\pi}^{\pi} ||\Delta_h^j X(t,\omega)||_a^a \, dt\right)^{\frac{p}{a}} dh \\ &\ge \int_0^1 \frac{1}{I(h)} \left(\sum_{k=-\infty}^{\infty} ||C_k(\omega)||_a^{a'} |sin\frac{kh}{2}|^{ja'}\right)^{\frac{p}{a'}} dh \\ &\ge K \int_0^1 \frac{1}{I(h)} \left(\sum_{|k|=1}^{n} ||C_k(\omega)||_a^{a'} |sin\frac{kh}{2}|^{ja'}\right)^{\frac{p}{a'}} dh. \end{aligned}$$

Since $\sin kh \ge Kkh$ for $kh \le 1$, we find from the last inequality that

$$\begin{aligned} A_p(X,I,a) &\ge K \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{h^{jp}}{I(h)} dh \left(\sum_{|k|=1}^{n} k^{ja'} ||C_k(\omega)||_a^{a'}\right)^{\frac{p}{a'}} \\ &\ge K \sum_{n=1}^{\infty} \frac{n^{-(jp+2)}}{I(\frac{1}{n})} \left(\sum_{|k|=1}^{n} k^{ja'} ||C_k(\omega)||_a^{a'}\right)^{\frac{p}{a'}} \\ &= KC_p(X,I,a'). \end{aligned}$$

We write the best approximation by trigonometric polynomials in L_p^2 by $E_N^{(2)}$, that is,

$$E_N^{(2)} = \left[\sum_{|n| \geq N} ||C_n(\omega)||_2^2 \right]^{\frac{1}{2}}.$$

Then we have

THEOREM 2. *The following two relations are equivalent:*

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}} E_n^{(2)} < \infty$$

and

$$_2||\,||C_n(\omega)||_2\,||_1 < \infty.$$

PROOF. Take $a = 2, p = 1$ and $I(h) = h^{\frac{3}{2}}$. Then $B_1(X, I, 2) = \sum_{n=1}^{\infty} n^{-\frac{1}{2}} E_n^{(2)}$ and we have the conclusion by the above Theorem 1 and Theorem 3[3].

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