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# LEFT DERIVATIONS ON BANACH ALGEBRAS

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ABSTRACT. In this paper we show that every left derivation on a semiprime Banach algebra $A$ is a derivation which maps $A$ into the intersection of the center of $A$ and the Jacobson radical of $A$, and hence every left derivation on a semisimple Banach algebra is zero.

## 1. Introduction

In 1955 Singer and Wermer proved that the range of a continuous derivation on a commutative Banach algebra is contained in the Jacobson radical [6] . In the same paper they conjectured that the assumption of continuity is not necessary. In 1988 Thomas proved the Singer-Wermer conjecture [8]. There are also non-commutative versions of the Singer-Wermer theorem. For example, in [1] Brešar and Vukman proved that every continuous left derivation on a Banach algebra $A$ maps $A$ into its Jacobson radical. Also they proved that every left derivation on a semiprime ring $X$ is a derivation which maps $X$ into its center. The purpose of this paper is to show that every left derivation on a semiprime Banach algebra $A$ is a derivation which maps $A$ into the intersection of the center of $A$ and the Jacobson radical of $A$, and hence every left derivation on a semisimple Banach algebra is zero.

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## 2. Preliminaries

Throughout, $A$ will represent a complex algebra with the center $Z(A)$, $R$ the Jacobson radical of $A$, and $L$ the prime radical of $A$. Recall that $A$ is prime if $xAy = 0$ implies $x = 0$ or $y = 0$, and $A$ is semiprime if $xAx = 0$ implies $x = 0$. A linear mapping $D : A \to A$ is called a derivation if $D(xy) = xD(y) + D(x)y$ $(x, y \in A)$. A linear mapping $D : A \to A$ is called a left derivation if $D(xy) = xD(y) + yD(x)$ $(x, y \in A)$. Let $T$ be a linear mapping from a Banach space $X$ into a Banach space $Y$. Then the separating space of $T$ is defined as

$$S(T) = \{y \in Y : \text{ there exists } x_k \to 0 \text{ in } X \text{ with } T(x_k) \to y\},$$

and $T$ is continuous if and only if $S(T) = \{0\}$ ( see [6]). $N$ will denote the set of all natural numbers.

## 3. The Results

DEFINITION 3.1. Let $A$ be a Banach algebra. A closed 2-sided ideal $J$ of $A$ is a separating ideal if for each sequence $\{a_n\}$ in $A$, there exists $m \in N$ such that $(Ja_n \dots a_1)^- = (Ja_m \dots a_1)^-$ for all $n \geq m$.

By Stability Lemma [3] it is easy to see that every derivation on a Banach algebra has a separating space which is a separating ideal.

The following lemma is due to Cusack [2].

LEMMA 3.2. *Let $A$ be a Banach algebra, and $P$ a minimal prime ideal of $A$ such that $J \not\subset P$, where $J$ is a separating ideal of $A$. Then $P$ is closed.*

The next lemma can be referred to [5].

LEMMA 3.3. *Let $D$ be a left derivation on an algebra $A$. Then*

$$D^n(xy) = \sum_{r=0}^{n-1} \binom{n-1}{r} [D^r(x)D^{n-r}(y) + D^r(y)D^{n-r}(x)] \qquad (n \in N,\ x, y \in A).$$

PROOF. We prove the statement by induction. It is trivial when $n = 1$. We assume that

$$D^n(xy) = \sum_{r=0}^{n-1} \binom{n-1}{r} [D^r(x)D^{n-r}(y) + D^r(y)D^{n-r}(x)].$$

Then

$$\begin{aligned} D^{n+1}(xy) =& D(D^n(xy)) \\ =& \sum_{r=0}^{n-1} \binom{n-1}{r} [D^r(x)D^{n+1-r}(y) + D^{n-r}(y)D^{r+1}(x) \\ &+ D^r(y)D^{n+1-r}(x) + D^{n-r}(x)D^{r+1}(y)] \\ =& xD^{n+1}(y) + \sum_{r=1}^{n-1}[\binom{n-1}{r} + \binom{n-1}{n-r}]D^r(x)D^{n+1-r}(y) \\ &+ D^n(x)D(y) + yD^{n+1}(x) + \sum_{r=1}^{n-1}[\binom{n-1}{r} + \binom{n-1}{n-r}] \\ &D^r(y)D^{n+1-r}(x) + D^n(y)D(x) \\ =& xD^{n+1}(y) + \sum_{r=1}^{n-1} \binom{n}{r} D^r(x)D^{n+1-r}(y) \\ &+ D^n(x)D(y) + yD^{n+1}(x) \\ &+ \sum_{r=1}^{n-1} \binom{n}{r} D^r(y)D^{n+1-r}(x) + D^n(y)D(x) \\ =& \sum_{r=0}^{n} \binom{n}{r} [D^r(x)D^{n+1-r}(y) + D^r(y)D^{n+1-r}(x)]. \end{aligned}$$

The proof of the lemma is complete.

The following lemma is a crucial tool in proving Lemma 3.5.

LEMMA 3.4. *Let $D$ be a left derivation on an algebra $A$. Suppose that $P$ is a minimal prime ideal of $A$ such that $[D^k(x), y] \in P$ for all*

*$x, y \in A$ and all $k \in N$, where $[u, v]$ denotes the commutator $uv - vu$. Then $D$ fixes the minimal prime ideal $P$ of $A$.*

PROOF. We shall prove that the ideal $P' = \{a \in P : D^k(a) \in P \text{ for all } k \in N\}$ is prime again. Since $D(P') \subset P'$, minimality of $P$ therefore yields $D(P) \subset P$. Take $a, b \in A$ such that $a \notin P'$ but $axb \in P'$ for all $x \in A$. Choose $n \in N_0 (= N \cup \{0\})$ with the property $D^n(a) \notin P$ and $D^m(a) \in P$ for all $m \in N_0, m < n$. We now have to prove by induction that $D^k(b) \in P$ for all $k \in N_0$. Using Lemma 3.3, we have

$$\begin{aligned} D^{n+k}(axb) &= \sum_{j=0}^{n+k-1} \binom{n+k-1}{j} [D^j(a)D^{n+k-j}(xb) \\ &\quad + D^j(xb)D^{n+k-j}(a)] \\ &= \sum_{j=0}^{n+k-1} \binom{n+k-1}{j} D^j(a)D^{n+k-j}(xb) \\ &\quad + \sum_{j=0}^{n+k-1} \binom{n+k-1}{j} D^j(xb)D^{n+k-j}(a) \\ &= \sum_{j=0}^{n-1} \binom{n+k-1}{j} D^j(a)D^{n+k-j}(xb) \qquad (1) \\ &\quad + \binom{n+k-1}{n} D^n(a)D^k(xb) \qquad (2) \\ &\quad + \sum_{j=n+1}^{n+k-1} \binom{n+k-1}{j} D^j(a)D^{n+k-j}(xb) \qquad (3) \\ &\quad + \sum_{j=0}^{k-1} \binom{n+k-1}{j} D^j(xb)D^{n+k-j}(a) \qquad (4) \end{aligned}$$

$$(5) \qquad + \binom{n+k-1}{k} D^k(xb)D^n(a)$$

$$(6) \qquad + \sum_{j=k+1}^{n+k-1} \binom{n+k-1}{j} D^j(xb)D^{n+k-j}(a).$$

By assumption, the left-hand side always belongs to $P$. If $k = 0$, then $b$ belongs to $P$ by the choice of n and hypothesis of the lemma. Suppose that $k \geq 1$. Then (1) belongs to $P$ since $D^j(a) \in P$ for all $j \leq n-1$. An application of Lemma 3.3 to (3) yields

$$\sum_{j=n+1}^{n+k-1} \binom{n+k-1}{j} D^j(a)D^{n+k-j}(xb)$$
$$= \sum_{j=n+1}^{n+k-1} \binom{n+k-1}{j} D^j(a)[\sum_{i=0}^{n+k-j-1} \binom{n+k-j-1}{i}$$
$$\cdot (D^i(x)D^{n+k-j-i}(b) + D^i(b)D^{n+k-j-i}(x))],$$

which belongs to $P$ since $D^i(b) \in P$ and $D^{n+k-j-i}(b) \in P$ for $0 \leq i \leq n+k-j \leq k-1$ by the induction hypothesis. Also another application of Lemma 3.3 to (4) yields

$$\sum_{j=0}^{k-1} \binom{n+k-1}{j} D^j(xb)D^{n+k-j}(a)$$
$$= \sum_{j=0}^{k-1} \binom{n+k-1}{j} [\sum_{i=0}^{j-1} \binom{j-1}{i} \quad (D^i(x)D^{j-i}(b)$$
$$+ D^i(b)D^{j-i}(x))]D^{n+k-j}(a),$$

which belongs to $P$ since $D^i(b) \in P$ and $D^{j-i}(b) \in P$ for $0 \leq i \leq j \leq k-1$ by the induction hypothesis. Finally, (6) belongs to $P$ since$D^{n+k-j}(a) \in P$ for $n+k-j \leq n-1$. Then we have

$D^n(a)D^k(xb) \in P$ by hypothesis of the lemma. But

$$D^n(a)D^k(xb) = D^n(a)[xD^k(b) + bD^k(x) + \sum_{i=1}^{k-1} \binom{k-1}{i} (D^i(x)D^{k-i}(b) + D^i(b)D^{k-i}(x))].$$

By the induction hypothesis we have $D^i(b) \in P$ and $D^{k-i}(b) \in P$. Consequently we see that $D^n(a)xD^k(b) \in P$ for all $x \in A$. Since $P$ is a prime ideal, it follows that $D^k(b) \in P$. We complete the proof.

LEMMA 3.5. *Let $D$ be a left derivation on a Banach algebra $A$ with radical $R$. Suppose that the following conditions are satisfied:*

(1) $[D^n(x), y] \in L$ *for all* $x, y \in A$ *and all* $n \in N$;
(2) $S(D) \subset Z(A)$,

*where $S(D)$ is the separating space of the left derivation $D$. Then $D(A) \subset R$.*

PROOF. Let $Q$ be any primitive ideal of $A$. Using Zorn's lemma, we find a minimal prime ideal $P$ contained in $Q$, and hence $D(P) \subset P$ by condition (1) and Lemma 3.4. Suppose first that $P$ is closed. Then we may define a left derivation $\bar{D} : A/P \to A/P$ by $\bar{D}(x + P) = D(x) + P(x \in A)$. Since $A/P$ is prime, Brešar and Vukman's theorem [1] implies that $\bar{D} = 0$ or $A/P$ is commutative. In the second case, $\bar{D}(A/P)$ is contained in the radical of $A/P$ by [8] whence, in both cases, $\bar{D}(A/P) \subset Q/P$. Consequently we see that $D(A) \subset Q$. Observe that $S(D)$ is a separating ideal of $A$ by condition (2). If $P$ is not closed, then we see that $S(D) \subset P$ by Lemma 3.2. Denoting $\pi : A \to A/\bar{P}$ the canonical epimorphism, we have, by [ 6, Chap.1], $S(\pi \circ D) = \overline{\pi(S(D))} = \{0\}$ whence $\pi \circ D$ is continuous.As a result, $(\pi \circ D)(\bar{P}) = \{0\}$, that is, $D(\bar{P}) \subset \bar{P}$. Hence we may also define a continuous left derivation $\widetilde{D} : A/\bar{P} \to A/\bar{P}$ by $\widetilde{D}(x + \bar{P}) = D(x) + \bar{P}$

($x \in A$). Then we see that $\widetilde{D}(A/\bar{P})$ is contained in the radical of $A/\bar{P}$ by [1, Theorem 2.1], and hence $\widetilde{D}(A/\bar{P}) \subset Q/\bar{P}$. So we obtain that $D(A) \subset Q$. It follows that $D(A) \subset Q$ for every primitive ideal Q, that is, $D(A) \subset R$.

Now we prove our main result.

THEOREM 3.6. *Let $D$ be a left derivation on a semiprime Banach algebra $A$ with radical $R$. Then $D$ is a derivation such that $D(A) \subset Z(A) \cap R$.*

PROOF. Note that $D$ is a derivation such that $D(A) \subset Z(A)$ [1, Proposition 1.6], and hence $D^n(A) \subset Z(A)$ for all $n \in N$. Since $Z(A)$ is a closed subalgebra of $A$, we see that $S(D) \subset Z(A)$. Therefore, by Lemma 3.5, we have $D(A) \subset R$. Consequently it follows that $D(A) \subset Z(A) \cap R$.

COROLLARY 3.7. *Let $D$ be a left derivation on a semisimple Banach algebra. Then $D = 0$.*

The following theorem is a generalization of Brešar and Vukman's theorem [1, Theorem 2.1].

THEOREM 3.8. *Let $D$ be a left derivation on a Banach algebra $A$ with radical $R$. If $D^n$ is continuous for some $n \in N$, then $D(A) \subset R$.*

PROOF. Let $P$ be any primitive ideal of $A$. Note that the quotient algebra $A/P$ is semisimple. Let $x \in A$ and $y \in P$ and observe that $xD(y) = D(xy) - yD(x) \in D(P) + P$. This shows that $D(P) + P$ is a left ideal of $A$,hence $\pi(D(P))$ is a left ideal of $A/P$, where $\pi : A \to A/P$ is the canonical epimorphism. A simple modification of the proof of Lemma 2.1 in [4] shows that $\pi(D^m(x^m)) = \pi(m!(D(x))^m)$ holds for all $x \in P$ and $m \in N$. Since $D^n$ is continuous for some

$n \in N$, we have, for each $x \in P$ and $k \in N$,

$$\begin{aligned} ||(\pi(D(x)))^{nk}||^{\frac{1}{nk}} &\le ((nk)!)^{-\frac{1}{nk}}||\pi(D^{nk}(x^{nk}))||^{\frac{1}{nk}} \\ &\le ((nk)!)^{-\frac{1}{nk}}||D^n||^{\frac{1}{n}}||x|| \to 0 \text{ as } k \to \infty. \end{aligned}$$

This shows that $\pi(D(P))$ is a quasinilpotent left ideal of $A/P$, therefore, it is contained in the radical of $A/P$. Semisimplicity forces $D(P) \subset P$. Thus we may define a left derivation $\bar{D} : A/P \to A/P$ by $\bar{D}(x+P) = D(x) + P$ $(x \in A)$. By Corollary 3.7, $\bar{D} = 0$ since $A/P$ is a semisimple Banach algebra. Consequently we see that $D(A) \subset R$ since $P$ was any primitive ideal of $A$.

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