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# A RELATION BETWEEN A GENERAL MAXIMAL OPERATOR AND THE SHARP MAXIMAL OPERATOR

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## 1. Introduction

A function which is integrable over every compact subset of $\mathbf{R}^n$ is called *locally integrable* and the space of such functions is denoted by $L^1_{loc}(\mathbf{R}^n)$. Let $f \in L^1_{loc}(\mathbf{R}^n)$. For $x \in \mathbf{R}^n$, we define

$$\Lambda(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes $Q$ containing $x$, and $|Q|$ denotes the Lebesgue measure of $Q$. Then $\Lambda(f)$ is called the *Hardy-Littlewood maximal function* of $f$, and the operator $\Lambda : f \to \Lambda(f)$, the *Hardy-Littlewood maximal operator.*

Throughout this paper, $Q$ will always mean a compact cube in $\mathbf{R}^n$ with sides parallel to the axes and nonempty interior.

We now define a more general maximal function $\Lambda_\mu(f)$ with respect to $\mu$ on $\mathbf{R}^n$ than the maximal function $\Lambda(f)$ as follows. For $x \in \mathbf{R}^n$ and $1 \leq \mu < \infty$, we define

$$\Lambda_\mu(f)(x) = \sup_{x \in Q} \left( \frac{1}{|Q|} \int_Q |f(y)|^\mu dy \right)^{1/\mu},$$

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where the supremum is taken over all cubes $Q$ containing $x$. Note that $\Lambda_\mu(f)(x)$ coincides with $[\Lambda(|f|^\mu)(x)]^{1/\mu}$, and $\Lambda_1(f)$ is the usual Hardy-Littlewood maximal function $\Lambda(f)$.

In this paper, we study a close relation between a general maximal function $\Lambda_\mu(f)$ and the sharp maximal function $\Lambda^\sharp(f)$. The main tool of the proof is the Calderón-Zygmund decomposition.

## 2. Preliminaries

For $f \in L^1_{loc}(\mathbf{R}^n)$, the *sharp maximal function* $\Lambda^\sharp(f)$ is defined at $x \in \mathbf{R}^n$ by setting

$$\Lambda^\sharp(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the supremum is taken over all cubes $Q$ containing $x$, and $f_Q$ denotes the *average* of $f$ over $Q$, that is,

$$f_Q = \frac{1}{|Q|} \int_Q f(x) dx.$$

The operator $\Lambda^\sharp : f \to \Lambda^\sharp(f)$ will be called the *sharp maximal operator*.

Note that $\Lambda_\mu(f)(x)$ coincides with $[\Lambda(|f|^\mu)(x)]^{1/\mu}$, and $\Lambda_1(f)$ is the usual Hardy-Littlewood maximal function $\Lambda(f)$.

THEOREM 1 ([4]). *If $f \in L^p(\mathbf{R}^n)$ and $1 \le \mu < p \le \infty$, then $\Lambda_\mu(f) \in L^p(\mathbf{R}^n)$ and*

$$(1) \qquad ||\Lambda_\mu(f)||_p \le C_{p,\mu} ||f||_p,$$

*where $C_{p,\mu}$ depends only on $p$, $\mu$ and the dimension $n$.*

REMARK. The inequality (1) of Theorem 1 fails for $p \le \mu$. For details see [4].

## 3. A Relation with $\Lambda_\mu(f)$ and $\Lambda^\sharp(f)$

Let $f \in L^1(\mathbf{R}^n)$ and let $\mathcal{C}_\alpha(f) = \{Q_j\}$ be the collection formed by those maximal diadic cubes over which the average of $|f|$ is greater than $\alpha$. Let $x \notin \bigcup_j Q_j$. Then the average of $|f|$ over any dyadic cube will be not greater than $\alpha$. The following is called the *Calderón-Zygmund decomposition.*

LEMMA 2 ([2]). *Let $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$, then there exists a family of non-overlapping cubes $\mathcal{C}_\alpha(f)$ consisting of those maximal dyadic cubes over which the average of $|f|$ is greater than $\alpha$ such that*

(i) $\alpha < |Q|^{-1} \int_Q |f(x)| dx < 2^n \alpha$ *for every* $Q \in \mathcal{C}_\alpha(f)$,

(ii) $|f(x)| \le \alpha$ *for a.e.* $x \notin \bigcup_{Q \in \mathcal{C}_\alpha(f)} Q$,

*and*

(iii) $\{x \in \mathbf{R}^n : \Lambda(f)(x) > \alpha\} \subset \bigcup_{Q \in \mathcal{C}_{4^{-n}\alpha}(f)} Q^3$ *for every* $\alpha > 0$,

*where $Q^3$ denotes the cube with the same center as $Q$ but with side length three times that of $Q$.*

LEMMA 3 ([2]). *For every $f \in L^1_{loc}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$,*

$$\Lambda^\sharp(|f|)(x) \le 2\Lambda^\sharp(f)(x).$$

There is a close relation between the maximal functions $\Lambda_\mu(f)$ and $\Lambda^\sharp(f)$. It is contained in the following statement:

THEOREM 4. *Let $f$ be such that $\Lambda_\mu(f) \in L^{p_0}(\mathbf{R}^n)$ for some $p_0$, $1 \le \mu < p_0 < \infty$, then there exists a constant $C$ such that*

$$\int_{\mathbf{R}^n} [\Lambda_\mu(f)(x)]^p dx \le C \int_{\mathbf{R}^n} [\Lambda^\sharp(f)(x)]^p dx$$

*for every $p$ with $p_0 \leq p < \infty$.*

PROOF. Without loss of generality, we may assume that $f$ is non-negative by Lemma 3 and $\Lambda(f) = \Lambda(|f|)$. First, the Calderón-Zygmund decomposition in Lemma 2 can be carried out for function $f^\mu$, $1 \leq \mu < \infty$. Let $\alpha^\mu > 0$ and suppose $Q$ is a cube such that $(f^\mu)_Q > \alpha^\mu$. Then, for every $x \in Q$,

$$\alpha^\mu < \frac{1}{|Q|} \int_Q [f(y)]^\mu \, dy \leq [\Lambda_\mu(f)(x)]^\mu$$

and thus

$$\begin{aligned} \alpha^{p_0} &\leq \frac{1}{|Q|} \int_Q [\Lambda_\mu(f)(x)]^{p_0} \, dx \\ &\leq \frac{1}{|Q|} \int_{\mathbf{R}^n} [\Lambda_\mu(f)(x)]^{p_0} \, dx \\ &= \frac{C}{|Q|} \end{aligned}$$

for some $p_0, 1 \leq \mu < p_0 < \infty$. It follows that if $\{Q_k\}$ is an increasing family of dyadic cubes $Q_k$ such that for $k = 1, 2, 3, \ldots$,

$$\frac{1}{|Q_k|} \int_{Q_k} [f(y)]^\mu \, dy > \alpha^\mu,$$

then the family $\{Q_k\}$ is finite since $|Q_k|$ is bounded. Thus, each dyadic cube $Q_k$ satisfying $(f^\mu)_{Q_k} > \alpha^\mu$ is contained in a maximal dyadic cube. Let $\{Q_j\}$ be the family consisting of these maximal dyadic cubes, then for each of them,

$$\alpha^\mu < \frac{1}{|Q_j|} \int_{Q_j} [f(y)]^\mu \, dy \leq 2^n \alpha^\mu. \tag{2}$$

Let $\{Q_{\alpha^\mu,j}\}$ be the family satisfying the condition (2). Then

$$[f(x)]^\mu \leq \alpha^\mu$$

for a.e. $x \notin \bigcup_j Q_{\alpha^\mu,j}$. Observe that if $\alpha^\mu < \beta^\mu$, then $Q_{\beta^\mu,j} \subset Q_{\alpha^\mu,k}$ for some $k$. Given $\alpha^\mu > 0$, we fix $Q_0 = Q_{2^{-n-1}\alpha^\mu,j_0}$ and take $A > 0$. Then there are two possibilities: either

$$Q_0 \subset \{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}$$

or

$$Q_0 \not\subset \{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}.$$

In the first case

$$\sum_{\{j:Q_{\alpha^\mu,j}\subset Q_0\}} |Q_{\alpha^\mu,j}| \leq |\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}|.$$

In the second case

$$\frac{1}{|Q_0|}\int_{Q_0} |f(y) - f_{Q_0}|dy \leq \alpha/A.$$

Note that $f_{Q_0} \leq 2^n 2^{-n-1}\alpha = \alpha/2$. Then we have

$$\begin{aligned}\sum_{\{j:Q_{\alpha^\mu,j}\subset Q_0\}} (\alpha - \alpha^\mu/2)|Q_{\alpha^\mu,j}| &\leq \sum_{\{j:Q_{\alpha^\mu,j}\subset Q_0\}} \int_{Q_{\alpha^\mu,j}} |f(y) - f_{Q_0}|dy \\ &\leq \int_{Q_0} |f(y) - f_{Q_0}|dy \\ &\leq A^{-1}\alpha|Q_0|.\end{aligned}$$

Adding up in all the possible $Q_0$, we get that

$$\sum_j |Q_{\alpha^\mu,j}| \leq |\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}| + 2A^{-1}\sum_k |Q_{2^{-n-1}\alpha^\mu,k}|.$$

Put $\phi(\alpha^\mu) = \sum_j |Q_{\alpha^\mu,j}|$ and $\psi(\alpha^\mu) = |\{x \in \mathbf{R}^n : \Lambda(f^\mu)(x) > \alpha^\mu\}|$. Then we know that

$$\phi(\alpha^\mu) \le \psi(\alpha^\mu)$$

and

$$\psi(\alpha^\mu) \le \sum_j |Q^3_{4^{-n}\alpha^\mu,j}| \le C_1\phi(\alpha^\mu/C_2). \tag{3}$$

In terms of $\phi$, we have

$$\phi(\alpha^\mu) \le |\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}| + 2A^{-1}\phi(2^{-n-1}\alpha^\mu).$$

Now, for $N > 0$,

$$\begin{aligned} I_N &\equiv \int_0^N p\alpha^{p-1}\phi(\alpha^\mu)d\alpha \\ &\le \int_0^N p\alpha^{p-1}\psi(\alpha^\mu)d\alpha \\ &\le pp_0{}^{-1}N^{p-p_0}\int_0^N p_0\alpha^{p_0-1}\psi(\alpha^\mu)d\alpha \\ &\le pp_0{}^{-1}N^{p-p_0}\int_{\mathbf{R}^n}[\Lambda_\mu(f)(x)]^{p_0}dx < \infty \end{aligned}$$

since $\Lambda_\mu(f) \in L^{p_0}(\mathbf{R}^n)$. Also

$$\begin{aligned} I_N \le &\int_0^N p\alpha^{p-1}|\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}|d\alpha \\ &+ 2A^{-1}\int_0^N p\alpha^{p-1}\phi(2^{-n-1}\alpha^\mu)d\alpha \\ = &\int_0^N p\alpha^{p-1}|\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}|d\alpha \\ &+ CA^{-1}\int_0^{2^{-n-1}N} p\alpha^{p-1}\phi(\alpha^\mu)d\alpha. \end{aligned}$$

Thus we have

$$I_N \le \int_0^N p\alpha^{p-1}|\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}|d\alpha + CA^{-1}I_N.$$

Take $A = 2C$, we obtain

$$I_N \le 2\int_0^N p\alpha^{p-1}|\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}|d\alpha.$$

Letting $N \to \infty$, then we arrive at

$$\begin{aligned}(4) \qquad &\int_0^\infty p\alpha^{p-1}\phi(\alpha^\mu)d\alpha \\ &\le 2\int_0^\infty p\alpha^{p-1}|\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha/A\}|d\alpha.\end{aligned}$$

and thus (3) and (4) imply that

$$\begin{aligned}\int_{\mathbf{R}^n} [\Lambda_\mu(f)(x)]^p dx &= \int_0^\infty p\alpha^{p-1}\psi(\alpha^\mu)d\alpha \\ &\le C_1\int_0^\infty p\alpha^{p-1}\phi(\alpha^\mu/C_2)d\alpha \\ &\le C\int_0^\infty p\alpha^{p-1}\phi(\alpha^\mu)d\alpha \\ &\le C\int_0^\infty p\alpha^{p-1}|\{x \in \mathbf{R}^n : \Lambda^\sharp(f)(x) > \alpha\}|d\alpha\ . \\ &= C\int_{\mathbf{R}^n} [\Lambda^\sharp(f)(x)]^p dx.\end{aligned}$$

The proof is therefore complete.

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