JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 8, June 1995

ON NEARNESS SPACE

SEUNG ON LEE AND EUN AI CHOI

ABSTRACT. In 1974 H.Herrlich invented nearness spaces, a very fruitful concept which enables one to unify topological aspects. In this paper, we introduce the Lindelöf nearness structure, countably bounded nearness structure and countably totally bounded nearness structure. And we show that $(X,\ \xi_L)$ is concrete and complete if and only if $\xi_L = \xi_t$ in a symmetric topological space $(X,\ t)$. Also we show that the following are equivalent in a symmetric topological space $(X,\ t)$:

(1) $(X,\ \xi_L)$ is countably totally bounded.
(2) $(X,\ \xi_t)$ is countably totally bounded.
(3) $(X,\ t)$ is countably compact.

1. Introduction

NOTATION 1.1. Let X be a set. For $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ and $A, B \subset X$ the following notation is used:

(1) $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$.
(2) $\mathcal{A}$ *corefines* $\mathcal{B}$ means that for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $B \subset A$, and denoted by $\mathcal{A} < \mathcal{B}$.
(3) $\mathcal{A}$ *refines* $\mathcal{B}$ means that for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$, and denoted by $\mathcal{A} \prec \mathcal{B}$.

DEFINITION 1.2. Let X be a set and $\xi \subset \mathcal{P}^2(X)$ where $\mathcal{P}^2(X)$ is the power set of the power set of X. Then ξ is said to be a nearness structure on X if it satisfies the following :

Received by the editors on June 30, 1995.
1991 *Mathematics subject classifications*: Primary 54E15, 54E17.

(N_1) $\mathcal{A} < \mathcal{B} \in \xi$ implies $\mathcal{A} \in \xi$.

(N_2) $\cap\mathcal{A} \neq \emptyset$ implies $\mathcal{A} \in \xi$.

(N_3) $\emptyset \neq \xi \neq \mathcal{P}^2(X)$.

(N_4) If $\mathcal{A} \vee \mathcal{B} \in \xi$, then $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$.

(N_5) $Cl_\xi\mathcal{A} = \{Cl_\xi A : A \in \mathcal{A}\} \in \xi$ implies $\mathcal{A} \in \xi$, where $Cl_\xi A = \{x \in X : \{\{x\}, A\} \in \xi\}$.

In this case, the pair $(X,\ \xi)$ is called a *nearness space* or shortly an *N-space*, and $\mathcal{A}$ is said to be *near* if $\mathcal{A} \in \xi$.

ξ is called a *quasi-nearness structure* or shortly a *Q-nearness structure* on X if ξ satisfies (N_1), (N_2), (N_3) and (N_4).

Given a nearness space $(X,\ \xi)$, the operator Cl_ξ is a closure operator on X. Hence there exists a topology associated with each nearness space in a natural way. This topology is denoted by $t(\xi)$ or t_ξ. This topology is symmetric, i.e., if $x \in \overline{\{y\}}$ then $y \in \overline{\{x\}}$.

DEFINITION 1.3. A nearness structure ξ is *compatible* with a topology t on a set X if $t = t_\xi$, where t_ξ is a topology generated by ξ.

Conversely, given any symmetric topological space $(X,\ t)$ there exists a compatible nearness structure ξ_t given by

$$\xi_t = \{\mathcal{A} \subset \mathcal{P}(X) : \cap\overline{\mathcal{A}} \neq \emptyset\},$$

where $\overline{\mathcal{A}} = \{\overline{A} : A \in \mathcal{A}\}$.

DEFINITION 1.4. Let $(X,\ \xi)$ be a nearness space.

(1) $(X,\ \xi)$ is *topological* if $\mathcal{A} \in \xi$ implies $\cap\overline{\mathcal{A}} \neq \emptyset$.

(2) A non-empty subset $\mathcal{A}$ of $\mathcal{P}(X)$ is a *ξ-cluster* if $\mathcal{A}$ is a maximal element of the set ξ, ordered by inclusion.

(3) $(X,\ \xi)$ is *concrete* if each near collection is contained in some ξ-cluster.

(4) $(X,\ \xi)$ is *complete* if $\cap\overline{\mathcal{A}} \neq \emptyset$ for each maximal element $\mathcal{A}$ in ξ.

(5) $(X,\ \xi)$ is *contigual* if $\mathcal{A} \notin \xi$ implies that there exists finite $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B} \notin \xi$.
(6) $(X,\ \xi)$ is *totally bounded* if $\mathcal{A} \notin \xi$ implies that there exists finite $\mathcal{B} \subset \mathcal{A}$ such that $\cap\mathcal{B} = \emptyset$.
(7) For $\mathcal{A} \subset \mathcal{P}(X)$, $\overline{\mathcal{A}}$ has the *f.i.p.* if for any finite subfamily $\mathcal{B}$ of $\mathcal{A}$, $\cap\overline{\mathcal{B}} \neq \emptyset$.

DEFINITION 1.5. Let $(X,\ t)$ be a symmetric topological space and

$$\xi_p = \{\mathcal{A} \subset \mathcal{P}(X) : \overline{\mathcal{A}} \text{ has the f.i.p.}\}.$$

Then $(X,\ \xi_p)$ is called the *Pervin nearness space* on $(X,\ t)$.

PROPOSITION 1.6. *Every contigual nearness space is concrete.*

PROOF. See reference [6].

PROPOSITION 1.7. *Let $(X,\ t)$ be a T_1 topological space. Then ξ_p is a compatible contigual nearness structure on X.*

PROOF. See reference [3].

2. The Lindelöf Nearness Space

For $\mathcal{A} \subset \mathcal{P}(X)$, $\overline{\mathcal{A}}$ has the *c.i.p.* if for any countable subfamily $\mathcal{B}$ of $\mathcal{A}$, $\cap\overline{\mathcal{B}} \neq \emptyset$.

DEFINITION 2.1. Let $(X,\ t)$ be a symmetric topological space and

$$\xi_L = \{\mathcal{A} \subset \mathcal{P}(X) : \overline{\mathcal{A}} \text{ has the c.i.p.}\}.$$

Then ξ_L is called the *Lindelöf nearness structure* on $(X,\ t)$, and $(X,\ \xi_L)$ the *Lindelöf nearness space* on $(X,\ t)$.

THEOREM 2.2. *Let $(X,\ t)$ be a symmetric topological space. Then $(X,\ \xi_p)$ is concrete and complete if and only if $\xi_p = \xi_t$.*

PROOF. Suppose that $(X,\ \xi_p)$ is concrete and complete. It is obvious that $\xi_t \subset \xi_p$. To show $\xi_p \subset \xi_t$, take any $\mathcal{A} \in \xi_p$. Then $\mathcal{A}$ is contained in some ξ_p-cluster $\mathcal{B}$ and $\cap\overline{\mathcal{B}} \neq \emptyset$; and hence $\cap\overline{\mathcal{A}} \neq \emptyset$. Thus $\xi_p \subset \xi_t$ implies $\xi_p = \xi_t$. Conversely, suppose $\xi_p = \xi_t$ then $(X,\ \xi_p)$ is contigual. Hence $(X,\ \xi_p)$ is concrete by Proposition 1.6.. And for any $\mathcal{A} \in \xi_p$-cluster, $\mathcal{A} \in \xi_t$, and hence $\cap\overline{\mathcal{A}} \neq \emptyset$. Hence $(X,\ \xi_p)$ is complete.

PROPOSITION 2.3. *Let $(X,\ t)$ be a symmetric topological space. Then ξ_L is a compatible nearness structure on $(X,\ t)$.*

PROOF. See reference [3].

THEOREM 2.4. *Let $(X,\ t)$ be a symmetric topological space. Then $(X,\ \xi_L)$ is concrete and complete if and only if $\xi_L = \xi_t$.*

PROOF. Suppose $\xi_L = \xi_t$. To show $(X,\ \xi_L)$ is concrete, take any $\mathcal{A} \in \xi_L$, then $\cap\overline{\mathcal{A}} \neq \emptyset$. Pick $x \in \cap\overline{\mathcal{A}}$. Let $\xi_L(x) = \{B \subset X : x \in Cl_{\xi_L}B\}$, then$\cap\overline{\xi_L(x)} \neq \emptyset$ implies $\xi_L(x) \in \xi_L$. To show $\xi_L(x)$ is maximal, assume that $\xi_L(x) \subset \mathcal{D} \in \xi_L$ and take any $D \in \mathcal{D}$. Since $x \in \overline{\{x\}} = Cl_{\xi_L}\{x\}$, $\{x\} \in \xi_L(x) \subset \mathcal{D} \in \xi_L$. Then $\{\{x\}, D\} \in \xi_L$ implies $x \in Cl_{\xi_L}D$. Thus $D \in \xi_L(x)$ and hence $\mathcal{D} \subset \xi_L(x)$. Hence $\xi_L(x)$ is ξ_L-cluster. Assume that $A \in \mathcal{A}$ but $A \notin \xi_L(x)$, then $x \notin Cl_{\xi_L}A$. But for each $A \in \mathcal{A}$, $x \in \overline{A} = Cl_{\xi_L}A$. This is a contradiction. Hence $\mathcal{A} \subset \xi_L(x)$. Thus $(X,\ \xi_L)$ is concrete. Next, we will show that $(X,\ \xi_L)$ is complete. Let $\mathcal{A} \in \xi_L$-cluster, then $\mathcal{A} \in \xi_L = \xi_t$, and hence $\cap\overline{\mathcal{A}} \neq \emptyset$. Thus $(X,\ \xi_L)$ is complete. Conversely, if $(X,\ \xi_L)$ is concrete and complete, then $\xi_t \subset \xi_L$. To show $\xi_L \subset \xi_t$, let $\mathcal{A} \in \xi_L$. Then there is a ξ_L-cluster $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{B}$ since $(X,\ \xi_L)$ is concrete. Because $(X,\ \xi_L)$ is complete, $\cap\overline{\mathcal{B}} \neq \emptyset$; hence $\cap\overline{\mathcal{A}} \neq \emptyset$. Thus $\mathcal{A} \in \xi_t$.

COROLLARY 2.5. *Let $(X,\ t)$ be a symmetric topological space. If $(X,\ \xi_p)$ is concrete and complete, then $(X,\ \xi_L)$ is concrete and complete.*

NOTATION 2.6. Let $(X,\ \xi)$ be a nearness space.

(1) $\mu_p = \{\mathcal{A} \subset \mathcal{P}(X) : \{X - A : A \in \mathcal{A}\} \notin \xi_p\}$.
(2) $\mu_L = \{\mathcal{B} \subset \mathcal{P}(X) : \{X - B : B \in \mathcal{B}\} \notin \xi_L\}$.
(3) $\mu_t = \{\mathcal{C} \subset \mathcal{P}(X) : \{X - C : C \in \mathcal{C}\} \notin \xi_t\}$.

In this paper, a compact space need not be Hausdorff.

COROLLARY 2.7. *Let $(X,\ t)$ be a symmetric topological space. Then :*

(1) $\xi_t \subset \xi_L \subset \xi_p$ and $\mu_p \subset \mu_L \subset \mu_t$.
(2) $\mu_p = \mu_L$ *if and only if* $(X,\ t)$ *is countably compact.*
(3) $\mu_L = \mu_t$ *if and only if* $(X,\ t)$ *is Lindelöf.*
(4) $\mu_p = \mu_L = \mu_t$ *if and only if* $(X,\ t)$ *is compact.*

PROOF. See reference [3].

COROLLARY 2.8. *Let $(X,\ t)$ be a symmetric topological space. Then :*

(1) $(X,\ \xi_L)$ *is concrete and complete if and only if* $(X,\ t)$ *is Lindelöf.*
(2) $(X,\ \xi_p)$ *is concrete and complete if and only if* $(X,\ t)$ *is compact.*

Definition 2.9. Let $(X,\ \xi)$ be a Q-nearness space. Then:

(1) $(X,\ \xi)$ is *countably contigual* if $\mathcal{A} \notin \xi$ implies that there exists a countable $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B} \notin \xi$.
(2) $(X,\ \xi)$ is *countably bounded* if $\mathcal{A} \notin \xi$ implies that there exists a countable $\mathcal{B} \subset \mathcal{A}$ such that $\cap\mathcal{B} = \emptyset$.

(3) $(X,\ \xi)$ is ***countably totally bounded*** if every countable $\mathcal{A} \subset \mathcal{P}(X)$ with the finite intersection property is near.

PROPOSITION 2.10. *Let $(X,\ t)$ be a symmetric topological space. Then :*

(1) *$(X,\ \xi_L)$ is countably contigual.*
(2) *$(X,\ \xi_L)$ is countably bounded.*

PROOF. See reference [3].

THEOREM 2.11. *Let $(X,\ t)$ be a symmetric topological space. Then:*

(1) *If $(X,\ \xi_L)$ is contigual then $(X,\ t)$ is countably compact.*
(2) *If $(X,\ \xi_t)$ is countably bounded then $(X,\ t)$ is Lindelöf.*

PROOF. (1) Suppose $(X,\ \xi_L)$ is contigual and take any countable open cover $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ of X. Then $\{X - G_\alpha : \alpha \in \Lambda\} \notin \xi_L$, and since $(X,\ \xi_L)$ is contigual there exists a finite $\mathcal{B} = \{X - G_{\alpha_i} : G_{\alpha_i} \in \mathcal{G},\ \ i = 1, 2, .., n\}$ such that $\mathcal{B} \notin \xi_L$. Hence $\mathcal{G}$ has a finite subcover $\{G_{\alpha_i} : i = 1, 2, .., n\}$ for X.

(2) Take any open cover $\mathcal{G} = \{\ G_\alpha : \alpha \in \Lambda\}$ of X. Then $\{X - G_\alpha : \alpha \in \Lambda\} \notin \xi_t$, and since $(X,\ \xi_t)$ is countably bounded there exists a countable $\mathcal{D} = \{X - G_{\alpha_i}\ :\ i \in$ I, I is a countable set$\} \subset \{X - G_\alpha : \alpha \in \Lambda\}$ such that $\cap\mathcal{D} = \emptyset$. Hence $\mathcal{G}$ has a countable subcover $\mathcal{D}^* = \{G_{\alpha_i} : i \in$ I, I is a countable set$\}$ for X.

THEOREM 2.12. *Let $(X,\ t)$ be a symmetric topological space. Then the following are equivalent :*

(1) *$(X,\ \xi_L)$ is countably totally bounded.*
(2) *$(X,\ \xi_t)$ is countably totally bounded.*
(3) *$(X,\ t)$ is countably compact.*

PROOF. (2)$\Longrightarrow$(3). Suppose $(X,\ \xi_t)$ is countably totally bounded. Take any countable open cover $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ of X. Since $\xi_t = \{\mathcal{A} \subset \mathcal{P}(X) : \cap\overline{\mathcal{A}} \neq \emptyset\}$, $\{X - G_\alpha : \alpha \in \Lambda\} \notin \xi_t$ and hence there exists a finite $\mathcal{B} \subset \{X - G_\alpha : \alpha \in \Lambda\}$ such that $\cap\mathcal{B} = \emptyset$. Hence $(X,\ t)$ is countably compact.

(3)$\Longrightarrow$(2). Suppose $(X,\ t)$ is countably compact. Let $\mathcal{A} \notin \xi_t$ and $\mathcal{A}$ a countable subfamily of $\mathcal{P}(X)$. Then $\cap\overline{\mathcal{A}} = \emptyset$; and hence $\cup\{X - \overline{A} : A \in \mathcal{A}\} = X$. Thus there exists a finite $\mathcal{B} \subset \{X - \overline{A} : A \in \mathcal{A}\}$ with $\cup\mathcal{B} = X$. Hence $(X,\ \xi_t)$ is countably totally bounded.

(1)$\Longrightarrow$(3). Suppose $(X,\ \xi_L)$ is countably totally bounded. Take any countable open cover $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ of X. Since $\xi_L = \{\mathcal{A} \subset \mathcal{P}(X) : \overline{\mathcal{A}}$ has the c.i.p.$\}$, $\{X - G_\alpha : \alpha \in \Lambda\} \notin \xi_L$ and hence there exists a finite $\mathcal{B} \subset \{X - G_\alpha : \alpha \in \Lambda\}$ such that $\cap\mathcal{B} = \emptyset$. Hence $(X,\ t)$ is countably compact.

(3)$\Longrightarrow$(1). Suppose $(X,\ t)$ is countably compact. Let $\mathcal{A} \notin \xi_L$ and $\mathcal{A}$ a countable subfamily of $\mathcal{P}(X)$. Then $\cap\overline{\mathcal{A}} = \emptyset$; and $\cup\{X - \overline{A} : A \in \mathcal{A}\} = X$. Thus there exists a finite $\mathcal{B} \subset \{X - \overline{A} : A \in \mathcal{A}\}$ with $\cup\mathcal{B} = X$. Hence $(X,\ \xi_L)$ is countably totally bounded.

DEFINITION 2.13. Let $(X,\ \xi)$ be a nearness space and k a regular infinite cardinal.

Then :

(1) $(X,\ \xi)$ is *k-contigual* if $\mathcal{A} \notin \xi$ implies that there exists $\mathcal{B} \subset \mathcal{A}$ with $|\ \mathcal{B}\ | \leq k$ such that $\mathcal{B} \notin \xi$.

(2) $(X,\ \xi)$ is *k-bounded* if $\mathcal{A} \notin \xi$ implies that there exists $\mathcal{B} \subset \mathcal{A}$ with $|\ \mathcal{B}\ | \leq k$ such that $\cap\mathcal{B} = \emptyset$.

(3) For $\mathcal{A} \subset \mathcal{P}(X)$, $\overline{\mathcal{A}}$ has the *k.i.p.* if for any $\mathcal{B} \subset \mathcal{A}$ with $|\ \mathcal{B}\ | \leq k$, $\cap\overline{\mathcal{B}} \neq \emptyset$.

For a symmetric topological space $(X,\ t)$, let

$$\xi_k = \{\mathcal{A} \subset \mathcal{P}(X) : \overline{\mathcal{A}} \text{ has the k.i.p.}\},$$

where k is a regular infinite cardinal.

PROPOSITION 2.14. *Let* $(X,\ t)$ *be a symmetric topological space and* k *a regular infinite cardinal. Then* ξ_k *is a compatible* k*-contigual nearness structure on* X.

PROOF. First, we will show that ξ_k is a compatible nearness structure on X. For each $A \subset X$, $x \in Cl_{\xi_k} A$ if and only if $\{\{x\}, A\} \in \xi_k$. Thus $\overline{\{x\}} \cap \overline{A} \neq \emptyset$. Let $y \in \overline{\{x\}} \cap \overline{A}$, then $x \in \overline{\{y\}} \subset \overline{A}$; and hence $Cl_{\xi_k} A \subset \overline{A}$. Conversely, let $x \in \overline{A}$. Then $\overline{\{x\}} \cap \overline{A} \neq \emptyset$ implies $\{\{x\}, A\} \in \xi_k$;and hence $x \in Cl_{\xi_k} A$. Thus $\overline{A} \subset Cl_{\xi_k} A$. Next, to show that $(X,\ \xi_k)$ is k-contigual, let $\mathcal{A} \notin \xi_k$. Then there exists $\mathcal{B} \subset \mathcal{A}$ such that $|\ \mathcal{B}\ | \leq k$ and $\cap \overline{\mathcal{B}} = \emptyset$, and then $\mathcal{B} \notin \xi_k$; and hence $(X,\ \xi_k)$ is k-contigual. Lastly, it is obvious that $(X,\ \xi_k)$ is a nearness space.

THEOREM 2.15. *Let* $(X,\ t)$ *be a symmetric topological space. Then* $(X,\ \xi_k)$ *is concrete and complete if and only if* $\xi_k = \xi_t$.

PROOF. Suppose $\xi_k = \xi_t$. To show $(X,\ \xi_k)$ is concrete, take any $\mathcal{A} \in \xi_k$, then $\cap \overline{\mathcal{A}} \neq \emptyset$. Pick $x \in \cap \overline{\mathcal{A}}$. Let $\xi_k(x) = \{B \subset X : x \in Cl_{\xi_k} B\}$. Assume that $\xi_k(x) \subset \mathcal{D} \in \xi_k$ and take any $D \in \mathcal{D}$. Since $x \in \overline{\{x\}} = Cl_{\xi_k}\{x\}$, $\{x\} \in \xi_k(x) \subset \mathcal{D} \in \xi_k$. Then $\{\{x\}, D\} \in \xi_k$ and hence $x \in Cl_{\xi_k} D$. Thus $D \in \xi_k(x)$ implies $\mathcal{D} \subset \xi_k(x)$. Hence $\xi_k(x)$ is ξ_k-cluster. Assume that $A \in \mathcal{A}$ but $A \notin \xi_k(x)$, then $x \notin Cl_{\xi_k} A$. But for each $A \in \mathcal{A}, x \in \overline{A} = Cl_{\xi_k} A$. This is a contradiction. Hence $\mathcal{A} \subset \xi_k(x)$. Thus $(X,\ \xi_k)$ is concrete. Next, we will show that $(X,\ \xi_k)$ is complete. Let $\mathcal{A} \in \xi_k$-cluster, then $\mathcal{A} \in \xi_k = \xi_t$, and then $\cap \overline{\mathcal{A}} \neq \emptyset$. Thus $(X,\ \xi_k)$ is complete. Conversely, it is obvious that $\xi_t = \xi_k$.

REMARK. In a Q-nearness space, every countably contigual nearness space must be countably bounded. But every countably bounded nearness space need not be countably contigual.

EXAMPLE 2.16. Let $X = R \times \{0,1\}$ and let

$$\mathcal{D} = \{R \times \{0\}\} \cup \{R \times \{1\}\} \cup \{\{r\} \times \{0,1\} : r \in R\}.$$

Define

$$\mu = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{D} \prec \mathcal{A}\}.$$

Then $(X,\ \mu)$ is a Q-nearness space and is countably bounded, but not countably contigual. For if $\mathcal{A} = \mathcal{D}$ then there exist no countable subset $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B} \in \mu$; and hence $(X,\ \mu)$ is not countably contigual.

REFERENCES

1. B.Banaschewski, *Extensions of topological spaces*, Can. Math. Bull. **7** (1964), 1-22.
2. H. L. Bently and H. Herrlich, *Extensions of topological spaces,* Topology, Proc. Memphis State Univ. Conference, Marcel-Dekker, New York, 1976, pp. 129-184.
3. J. W. Carlson, *Lindelöf nearness structures*, Proc. Conference Toledo, Ohio, 1983, pp. 185-196.
4. J. W. Carlson, *Topological properties in nearness spaces*, Gen. Top. Appl. **8** (1978), 111-118.
5. J. W. Carlson, preprint.
6. H. Herrlich, *Topological structures*, Mathematical Centre Tracts 52, Amsterdam, 1974.
7. H. Herrlich, *A Concept of Nearness*, Gen. Top. Appl. **5** (1974).
8. Bourbaki, *General Topology*, Hermann, 1966.
9. Pinter, *Set Theory*, Addison-Wesley Publishing Company, 1971.

DEPARTMENT OF MATHEMATICS
CHUNGBUK NATIONAL UNIVERSITY
CHEONGJU 360-763, KOREA