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SOME PROPERTIES OF ONE–SIDED STOPPING TIMES

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ABSTRACT. Let τ_a be the first time that a perturbed random walk with extended real-valued independent and identically distributed (i.i.d.) random variables crosses a constant boundary $a \geq 0$. For the stopping times τ_a we investigate some basic properties and obtain its limiting distribution as $a \to \infty$ and an upper bound of the expected stopping times $E(\tau_a)$.

1. Introduction

Let $Z_1, Z_2, \cdots$ be i.i.d. random variables with a distribution function F and a positive finite mean μ and let $\xi_1, \xi_2, \cdots$ random variables for which $(Z_1, \xi_1), (Z_2, \xi_2), \cdots, (Z_n, \xi_n)$ are independent of $Z_k, k > n$, for every $n \geq 1$. In addition, $\epsilon_1, \epsilon_2, \cdots$ are i.i.d. and independent of $(Z_1, \xi_1), (Z_2, \xi_2), \cdots$ such that

$$P\{\epsilon_1 = 0\} = p_0, \qquad P\{\epsilon_1 = +\infty\} = 1 - p_0.$$

Define $X_i = Z_i + \epsilon_i (i = 1, 2, \cdots)$, and put $S_n = \sum_{i=1}^n Z_i$, $\hat{S}_n = S_n + \xi_n$, and $\tilde{S}_n = \sum_{i=1}^n X_i + \xi_n$. It is convenient to let $S_0 = \hat{S}_0 = \tilde{S}_0 = 0$. For given $a \geq 0$ define

$$\zeta_a = \inf\{n \geq 1 \mid S_n > a\},$$

$$t_a = \inf\{n \geq 1 \mid \hat{S}_n > a\},$$

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and

$$\tau_a = \inf\{n \geq 1 \mid \tilde{S}_n > a\}.$$

For the stopping times of ζ_a and t_a, several studies such as asymptotic normality of the stopping times (Sigmund 1968, Feller 1971, Section 11.5, Woodroofe 1982, Lemma 4.2, Gut 1988, Theorem III 5.1), asymptotic expansion for the expected stopping times (Hagwood and Woodroofe 1982, Lai and Siegmund 1979), and the limiting distribution of the residual waiting times (Woodroofe 1982, Section 2.2, Lai and Siegmund 1977) were done.

In this paper, we investigate some properties of the stopping times τ_a. First we examine the relations between t_a and τ_a and use them to find the limiting distribution of τ_a and an upper bound of $E(\tau_a)$ in terms of $E(t_a)$. In remarks, we refer to the limiting distribution of the residual waiting times $\tilde{R}_a$, where $\tilde{R}_a = \tilde{S}_{\tau_a} - a$.

Throughout this paper, $\xrightarrow{p}$ and $\xrightarrow{d}$ denote convergence in probability and convergence in distribution, respectively.

2. Some Properties of One-Sided Stopping Times

LEMMA 1. *(1) If $\{Y_n\}, n \geq 1$, is uniformly integrable (u.i.) and $Y_n \xrightarrow{d} Y$, then $E|Y| < +\infty$ and $E(Y_n) \to E(Y)$.*

(2) If $Y_n \geq 0$ and $Y_n \xrightarrow{d} Y$, then $E(Y_n) \to E(Y) < +\infty$ if and only if $\{Y_n\}$ is u.i. .

PROOF. See [5], p. 183.

LEMMA 2. *Suppose that $\sup_n E|Y_n|^{1+\epsilon} < +\infty$ for some $\epsilon > 0$. Then $\{Y_n\}$ is u.i. .*

PROOF. Observe that

$$c^\epsilon \int_{||Y_n|>c} |Y_n| dP \leq \int_{|Y_n|>c} |Y_n|^{1+\epsilon} \leq E|Y_n|^{1+\epsilon}.$$

Divide by c^ϵ and take the supremum over the set of positive integers n. Then by assumtion

$$\sup_n \int_{|Y_n|>c} |Y_n| dP \le c^{-\epsilon} \sup_n E|Y_n|^{1+\epsilon} \to 0 \text{ as } c \to \infty.$$

Thus $\{Y_n\}$ is u.i. .

LEMMA 3. *Suppose that* $\xi_n/n \xrightarrow{p} 0$ *as* $n \to \infty$. *Then*
(1) $t_a < +\infty$ *with probability 1 (w.p. 1), and* $\tau_a \le t_a$ *w.p. 1.*
(2) $E(t_a^\alpha p_0^{t_a}) \to 0$ *as* $a \to \infty$ *for any fixed* $\alpha \ge 0$ *and* $p_0 \in [0,1)$.

PROOF. (1) By the law of large numbers and the definition of $\hat{S}_n$, $\hat{S}_n/n = S_n/n + \xi_n/n \xrightarrow{p} \mu$ as $n \to \infty$, so there is a subsequence (n_k) for which $\hat{S}_{n_k}/n_k \to \mu$ w.p. 1 as $k \to \infty$. It follows that $\sup_n \hat{S}_n = +\infty$ w.p. 1 and this implies $t_a < +\infty$ w.p. 1 for all $a \ge 0$. On the other hand, it holds that for each $n,\ k \in N$

$$\begin{aligned} P\{t_a = k,\ \tau_a = n+k\} &= P\{\hat{S}_i \le a \text{ for } 0 \le i \le k-1, \hat{S}_k > a, \tilde{S}_j \le a \\ &\qquad \text{for } 1 \le j \le n+k-1,\ \tilde{S}_{n+k} > a\} \\ &\le P\{\hat{S}_k > a,\ \tilde{S}_k \le a\} \\ &\le P\{\tilde{S}_k > a,\ \hat{S}_k \le a\} = 0, \end{aligned}$$

so that

$$P\{\tau_a - t_a > 0\} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} P\{t_a = k,\ \tau_a = n+k\} = 0.$$

(2) t_a increases strictly to a limit $t_\infty \le +\infty$ w.p. 1 as $a \to \infty$, and $P\{t_\infty \le n\} = \lim_{a\to\infty} P\{\max_{1\le k\le n} \hat{S}_k > a\} = 0$ for all $n \ge 1$. Therefore $t_a \to +\infty$ w.p.1 as $a \to \infty$. Now define a function $u(x)$ by $x^\alpha p_0^x$. Since it is continuous and $u(x) \to 0$ as $x \to \infty$, we have

$u(t_a) \to 0$ w.p. 1 as $a \to \infty$. Next, we shall show that $\{u(t_a) : a \geq 0\}$ is u.i. . Note that $u(x)$ is bounded on $[0, \infty)$, that is, $\sup_{x \geq 0} u(x) < B$ for some $B > 0$. Since $1 \leq t_a < +\infty$ w.p.1 for all $a \geq 0$, we have $\sup_{a \geq 0} u(t_a) \leq B$ w.p. 1. This implies that $\sup_n E[u(t_a)^2] \leq B^2 < +\infty$. By Lemma 2 with $\epsilon = 1$, $\{u(t_a) : a \geq 0\}$ is u.i. and hence, by Lemma 1 (2), $E[u(t_a)] \to 0$ as $a \to \infty$.

The following theorem gives the relations between t_a and τ_a.

THEOREM 1. *(1)* $(1-p_0 e^{it})E(e^{it\tau_a}) = (1-p_0)e^{it} + (1-e^{it})E(p_0^{t_a} e^{itt_a})$. *(2)* $E(\tau_a) = \frac{1-E(p_0^{t_a})}{1-p_0}$ *and* $Var(\tau_a) = \frac{p_0-[E(p_0^{t_a})]^2}{(1-p_0)^2} + \frac{E(p_0^{t_a})-2E(t_a p_0^{t_a})}{1-p_0}$, *where* $p_0 \neq 1$.

PROOF. (1) If $p_0 = 0$, then $\tau_a = 1$ w.p. 1 for all $a \geq 0$. If $p_0 = 1$, then $\tau_a = t_a$ w.p. 1. Hence the equality holds for $p_0 = 0$ or 1. Observe that for each $n \geq 1$

$$P\{\tau_a > n\} = p_0^n P\{t_a > n\},$$

so that

$$P\{\tau_a = n\} = p_0^{n-1} P\{t_a \geq n\} - p_0^n P\{t_a \geq n+1\}.$$

It follows that for $0 < p_0 < 1$

$$\begin{aligned}
E(e^{it\tau_a}) &= \sum_{n=1}^{\infty} e^{itn}(p_0^{n-1} P\{t_a \geq n\} - p_0^n P\{t_a \geq n+1\}) \\
&= q/p_0 \sum_{n=1}^{\infty} q^{n-1} P\{t_a \geq n\} - \sum_{n=1}^{\infty} q^n P\{t_a \geq n+1\} \quad (q = p_0 e^{it}) \\
&= (q/p_0 - 1) \sum_{n=1}^{\infty} q^{n-1} P\{t_a \geq n\} + P\{t_a \geq 1\} \\
&= \frac{(q-p_0)[1-E(q^{t_a})]}{p_0(1-q)} + 1 \qquad (|q| < 1) \\
&= \frac{(1-p_0)}{(1-p_0 e^{it})} e^{it} + \frac{(1-e^{it})}{(1-p_0 e^{it})} E(p_0^{t_a} e^{itt_a}).
\end{aligned}$$

Thus the equality holds for all $p_0 \in [0,1]$.

(2) Differentiate both sides of (1) with respect to t twice. Then we have

(2.1)
$$\begin{aligned}&(1-p_0e^{it})E(i\tau_a e^{it\tau_a}) - ip_0e^{it}E(e^{it\tau_a})\\ &= i(1-p_0)e^{it} - ie^{it}E(p_0^{t_a}e^{itt_a}) + (1-e^{it})E(it_a p_0^{t_a}e^{itt_a}),\end{aligned}$$

and

(2.2)
$$\begin{aligned}&(1-p_0e^{it})E(\tau^2 e^{it\tau_a}) - 2p_0e^{it}E(\tau_a e^{it\tau_a}) - p_0e^{it}E(e^{it\tau_a})\\ &= (1-p_0)e^{it} - e^{it}E(p_0^{t_a}e^{itt_a}) - 2e^{it}E(t_a p_0^{t_a}e^{itt_a})\\ &\quad + (1-e^{it})E(t_a^2 p_0^{t_a}e^{itt_a}).\end{aligned}$$

Putting $t=0$ in (2.1) and (2.2), we obtain that for $p_0 \neq 1$

$$\begin{aligned}E(\tau_a) &= \frac{1-E(p_0^{t_a})}{1-p_0},\\ E(\tau_a^2) &= \frac{1}{(1-p_0)^2}[1+p_0-(1+p_0)E(p_0^{t_a}) - 2(1-p_0)E(t_a p_0^{t_a})],\end{aligned}$$

and these moments yield the variance of τ_a.

REMARKS. (1) τ_a has a finite limit w.p. 1 as $a \to \infty$. Let $\tau_\infty = \lim_{a\to\infty}\tau_a$. Then $P\{\tau_\infty > n\} = \lim_{a\to\infty} P\{\max_{1\le k\le n}\tilde{S}_k < a\} = \lim_{a\to\infty} p_0^n P\{\max_{1\le k\le n}\hat{S}_k < a\} \le p_0^n \to 0$ as $n \to \infty$ if $p_0 \in [0,1)$. Therefore $P\{\tau_\infty = +\infty\} = 0$.

(2) Let $\tilde{R}_a$ be the residual waiting times $\tilde{S_{\tau_a}} - a$. Under the assumption of Lemma 3, $\tilde{R}_a \xrightarrow{d} \tilde{R}$ as $a \to \infty$, where $P\{\tilde{R} = +\infty\} = 1$. For $P\{\tilde{R}_a = +\infty\} = 1 - E(p_0^{t_a})$ and by Lemma 3 (2), $E(p_0^{t_a}) \to 0$ as $a \to \infty$ if $p_0 \in [0,1)$.

THEOREM 2. *Under the assumption of Lemma 3, if $c(a) \to c, d(a) \to d$ as $a \to \infty$ and $p_0 \in [0,1)$, then $\frac{\tau_a - c(a)}{d(a)}$ has the limiting distribution F_∞, where F_∞ is the distribution corresponding to the characteristic function $\phi_\infty(t) = \frac{1-p_0}{1-p_0 e^{it/d}} e^{i(1-c)t/d}$. In particular, if $c = d = 1$, then F_∞ is the geometric distribution, $Geo(1-p_0)$.*

PROOF. By Theorem 1, we have

(2.3)

$$E[e^{i(\tau_a - c(a))t/d(a)}]$$
$$= \frac{1-p_0}{1-p_0 e^{it/d(a)}} e^{i(1-c(a))t/d(a)} + \frac{1-e^{it/d(a)}}{1-p_0 e^{it/d(a)}} E[p_0^{t_a} e^{i(t_a - c(a))t/d(a)}].$$

The second term of the right-hand side of (2.3) goes to zero as $a \to \infty$, because $|\frac{1-e^{it/d(a)}}{1-p_0 e^{it/d(a)}}| \le \frac{2}{1-p_0}$ and by Lemma 3 (2)

$$|E[p_0^{t_a} e^{i(t_a - c(a))t/d(a)}]| \le E(p_0^{t_a}) \to 0 \text{ as } a \to \infty.$$

Hence the left-hand side and the first term of the right-hand side of (2.3) have the same limiting function corresponding to the characteristic function $\phi_\infty(t) = \frac{1-p_0}{1-p_0 e^{it/d}} e^{i(1-c)t/d}$. Since $\phi_\infty(t)$ is continuous and $\phi_\infty(0) = 1$, it is a characteristic function. Furthermore, if $c = d = 1$, then $\phi_\infty(t) = \frac{1-p_0}{1-p_0 e^{it}}$, which is the characteristic function of $Geo(1-p_0)$.

THEOREM 3. *If $0 < p_0 < 1$, then*

$$\log_{p_0}[1-(1-p_0)E(\tau_a)] \le E(t_a).$$

PROOF. Define $u(x) = p_0^x$. Clearly it is convex. Hence by Jensen's inequality $p_0^{E(t_a)} \le E(p_0^{t_a})$. By Theorem 1 (2), we have $E(\tau_a) \le$

$\frac{1-p_0^{E(t_a)}}{1-p_0}$. Taking the logarithm with the base p_0, we obtain the above inequality.

REMARK. For simplicity, we may regard the upper bound of $E(\tau_a)$ as $\frac{1}{1-p_0}$ instead of $\frac{1-p_0^{E(t_a)}}{1-p_0}$ for small p_0, since the difference between both values is small for small p_0. For example, if $0 < p_0 < 2^{-1}$, then the difference is less than 1. On the other hand, if p_0 is, relatively to any fixed a, very close to 1, then the difference is large. So in this case $\frac{1-p_0^{E(t_a)}}{1-p_0}$ is better than $\frac{1}{1-p_0}$. Also, for sufficiently large a, we can use the approximation of $E(t_a)$ under some additional conditions (see [7], p. 48) in computing the upper bound.

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