International Journal of Management Science
Vol. 1, No. 1, December 1995

An Extended EPQ Model to Relax the Constant Demand Assumption into Periodic Demand

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ABSTACT

This article presents a new model called the periodic square wave(PSW) to describe the material flow of periodic processes involving an intermediate buffer. The material flows into and out of the intermediate buffer are assumed to be periodic square shaped. By using this model, It is proved that the classical economic lot size model with finite supply rate, the so-called EPQ model, can be applicable to the arbitrary periodic demand case. This new model relaxes the original assumption of the constant demand. It is shown, as a unique application example, that the explicit solution for determining both upstream and downstream economic lot size can be obtained with the aid of the PSW model. The PSW model provides more accurate information on analyzing the inventory and production system than the classical approach, without losing simplicity and increasing the computational burden.

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1. INTRODUCTION

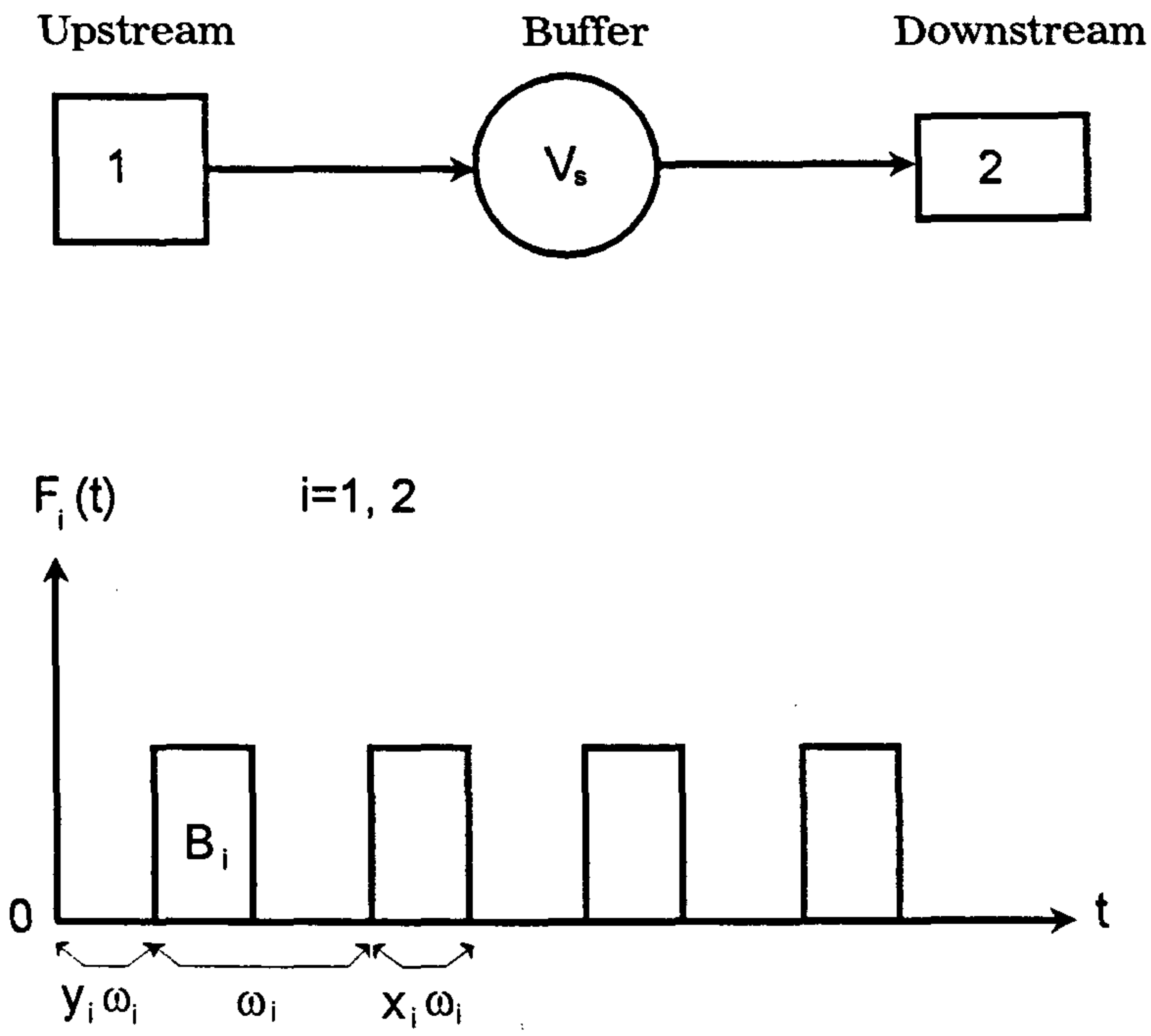
The economic order quantity (EOQ) model has been widely used in practice, in spite of unrealistic assumptions, because it is simple and because total cost in the neighborhood of optimum lot size is relatively insensitive to moderately small variations in input cost data[2]. Many varieties have been developed from the EOQ model to relax some of the assumptions used to derive the simple economic lot size formula and thus enrich the applicability of the model, for example, finite supply rate, backlogging, lost sales, quantity discounts [2].

In this study we will introduce a new model called the periodic square wave (PSW). PSW model resembles the classical economic lot size model with finite supply rate, so called economic production quantity (EPQ) model. Both EPQ and PSW models presume that the ordered up-stream material flow from supplier is periodic square shaped. However, the flow patterns of downstream customer demand of each model is different. EPQ presumes constraint material flows for downsteam customer demand but PSW presumes both material flows into and out of the intermediate buffer are periodic square-wave shaped. We can break the constant demand assumption of the classical economic lot size model by with preserving the a simple analytic solution. With the aid of PSW model, we will prove that the result of the classical economic lot size model with finite supply rate is still optimal under the assumption of arbitrary periodic demand.

2. PERIODIC SQUARE-WAVE MODEL

We will consider a single-product, single input / single output (SISO) intermediate buffer(storage) system. The elementary process components of our study are composed of an upstream unit with periodic square shaped flow pattern intermediate buffer, and a downstream unit also with periodic square shaped flow patterns. A schematic diagram, definition of variables and flow pattern are shown in Fig. 1. The up-stream flow represents the incoming material flow into the intermediate buffer of which order quantity (B_1) and order cycle time (ω_1) should be determined. The downstream flow, of which demand size (B_2), demand cycle time (ω_2) and initial delay time fraction (y_2) are already known, represents the customer demand going out from the intermediate buffer. The system variables are defined as follows;

〈Fig 1〉 Modelling of SISO Buffer with Periodic Square Wave Flow



A : ordering cost, dollars per order

B_1 : upstream batch size, units per lot

B_2 : downstream batch size, units per lot

D : constant demand, units per year

$F_1(t)$: upsteam flow of periodic square form

$F_2(t)$: downsteam flow of periodic square form

H : annual inventory holding cost, dollars per unit of item per year

P : production(supply) rate, units per year

U_1 : upstream flow rate, units per year

U_2 : downstream flow rate, units per year

V_{max} : maximum inventory hold-up, units of item

V_{min} : minimum inventory hold-up, units of item

V_{ub} : upper bound of inventory hold-up, units of item

V_{lb} : lower bound of inventory hold-up, units of item

$V(t)$: inventory hold-up, units of item

$V(0)$: initial inventory hold-up, units of item

V_s : storage size, units of item

$\bar{V}$: time averaged inventory hold-up, units of item

x_1 : transportation time fraction of upstream

x_2 : transportation time fraction of downstream

y_2 : initial delay time fraction of downstream

z : arbitrary real number

Greek Letters

ω_1 : cycle time of upstream, year

ω_2 : cycle time of downstream, year

Subscript

i = 1 for upstream, 2 for downstream

Special Functions

int[.] : truncation function to make integer

res[.] : postitive residual function to be truncated

$u(.)$: unit step function

Special operators are necessary to develop the subsequent results. A real number z can be separated into an integer part and a positive residual part which is less than 1. The integer part will be denoted by int[z] and the residual by res[z].

With the excetption of the constant demand model, the assumptions necessary to develop our results follow the classical economic lot size model with finite supply rate. [2] Customer demand is assumed to be periodic square wave in this section and it will be extended to arbitrary periodic function in next section.

The material balance around the buffer reduces to a simple ordinary differential equation:

$$\frac{dV}{dt}=F_1(t)u(t)-F_2(t-y_2\omega_2)\,u(t-y_2\omega_2) \tag{1}$$

Integration of Eq. (1) gives

$$V(t)=V(0)+\int_0^t F_1(\tau)d\tau-\int_0^{t-y_2\omega_2} F_2(\tau)d\tau \tag{2}$$

Each integral can be calculated by directly taking into account the periodic square form of the funcitons $F_1(t)$ and $F_2(t)$. In order to develop the first integral, as shown in Fig. 2, for the given time t, the number of complete cycles are given by $int\,[\frac{t}{\omega_1}]$ and the mass corresponding to the completed cycles is $B_1\,int\,[\frac{t}{\omega_1}]$. For the remaining time, $res\,[\frac{t}{\omega_1}]$ which is less than one cycle in length, there are two cases to consider. If $res\,[\frac{t}{\omega_1}]$ is larger than x_1, another full cycle has to be added to $B_1\,int\,[\frac{t}{w_1}]$. If it is less than x_1, the hatched area within the cycle, shown in Fig. 2, which is equal to $\frac{B_1}{x_1}\,res\,[\frac{t}{\omega_1}]$, has to be added. These two cases can be combined into one expression using the min{.} function.

$$\int_0^t F_1(\tau)d\tau=B_1(int[\frac{t}{\omega_1}]+min\,\{1,\frac{1}{x_1}\,res[\frac{t}{\omega_1}]\}) \tag{3}$$

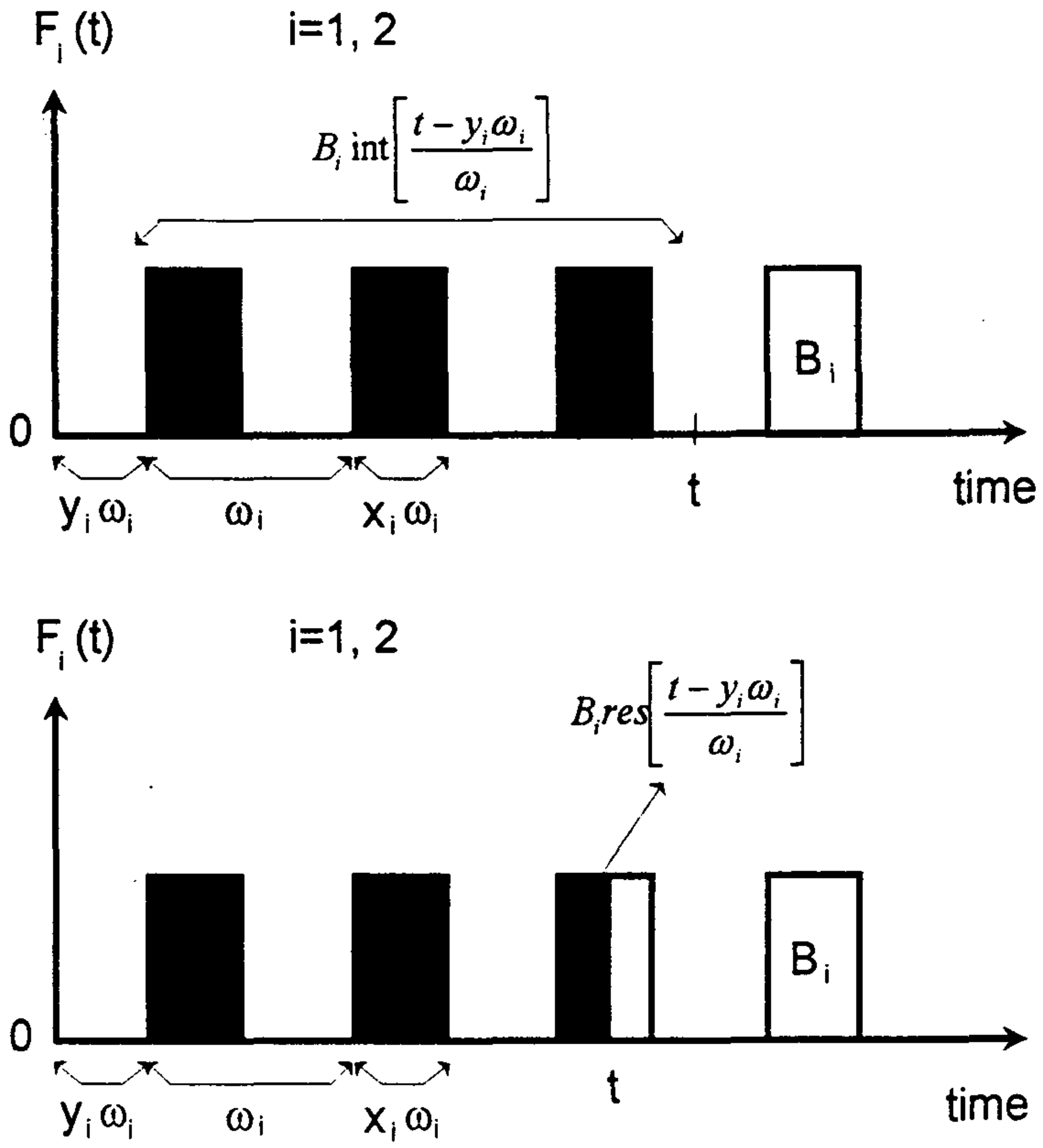
The second integral of Eq. (2) just involves a shift in the time scale. The complete from of the hold-up equation is as follows;

$$\begin{aligned}V(t)=V(0)+B_1\,(int\,[\frac{t}{\omega_1}]+min\{1,\frac{1}{x_1}\,res\,[\frac{t}{\omega_1}]\})\\ -B_2\,(int\,[\frac{t-y_2\omega_2}{\omega_2}]+min\,\{1,\frac{1}{x^2}\,res\,[\frac{t-y_2\omega_2}{\omega_2}]\})\end{aligned} \tag{4}$$

In addition to this equation, there is a relationship which results from a steady state overall material balance, namely:

$$\frac{B_1}{\omega_1}=\frac{B_2}{\omega_2} \tag{5}$$

〈Figure 2〉 Intergration of Periodic Square Wave Flow



Eq. (4) includes a basic functional group, called the Flow Accumulation Function.

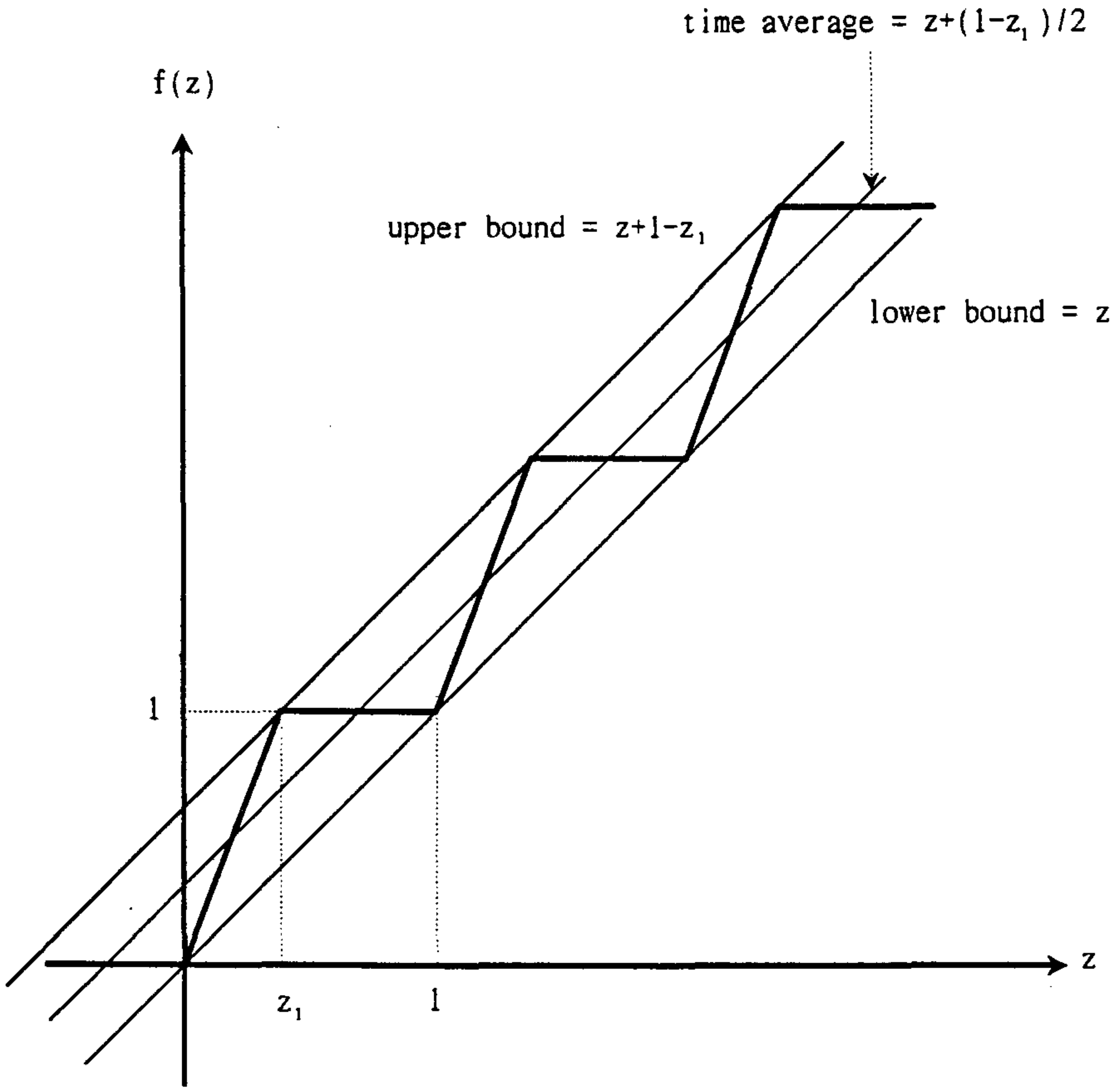
$$f(z)=int[z]+min\{1, \frac{1}{z} res[z]\} \tag{6}$$

The flow accumulation function has the following useful relationships;

$$z\le f(z)\le z+1-z_1 \tag{7}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(z)dz=z+\frac{1-z_1}{2} \tag{8}$$

〈Figure 3〉 Bounds on Flow Accumulation Function



These equations are obvious from Fig. 3.

The lower / upper bound on V(t) can be calculated by using Eq. (5) and (7).

$$V_{ub}=V(0)+B_1(1-x_1)+B_2y_2 \tag{9}$$

$$V_{lb}=V(0)-B_2(1-x_2-y_2) \tag{10}$$

The time averaged hold-up $\overline{V}$ can be calculated by using Eq. (5) and (8).

$$\overline{V}=V(0)+B_1\left(\frac{1-x_1}{2}\right)-B_2\left(\frac{1-x_2}{2}-y_2\right) \tag{11}$$

Now, we are ready to calcuate the total annual cost which is the sum of annual ordering cost and annual inventory holding cost.

$$Total\ Cost = \frac{A}{\omega_1} + H\overline{V} \tag{12}$$

Eq. (12) can be rewritten by using Eq. (5) and (11).

$$Total\ Cost = \frac{AB_2}{\omega_2 B_1} + H(\frac{1-x_1}{2})B_1 + constant \tag{13}$$

The optimum order quantity B^*_1 is obtained by setting the differentiation of total cost with respect to B_1 zero; the solution is

$$B_1^* = \sqrt{\frac{2AB_2}{(1-x_1)\omega_2 H}} \tag{14}$$

Eq. (14) can be compared with the result of the classical economic lot size model with finite supply rate in [2]. The parameters of PSW model have the following relationships with the classical lot size model with finite supply rate.

$$\frac{B_2}{\omega_2} = D,\ x_1 = \frac{D}{P},\ B_1 = Q \tag{15}$$

By applying Eq. (15) into Eq. (14), the same result of the classical economic lot size model with finite supply rate is driven from PSW model.

$$Q^* = \sqrt{\frac{2AD}{H(1-\frac{D}{P})}} \tag{16}$$

This can be interpreted that the classical economic lot size model with finite supply rate can be applicable to not only constant demand but also periodic square wave demand.

The PSW model provides a different result for the buffer capacity limitation. The sufficient condition for buffer size limit, $0 \le V_{lb} \le V_{ub} \le V_s$. with

Eq. (9) and (10) provides the bound for order quantity and initial inventory hold-up.

$$B_1 \leq \frac{V_s - (V(0) + y_2 B_2)}{1 - x_1} \tag{17}$$

$$V(0) \geq B_2(1 - x_2 - y_2) \tag{18}$$

When we presume the intermediate buffer size is limited, the optimal order quantity should be confined by Eq. (17). This can produce a different result from the classical economic lot size model with finite supply rate. The buffer limit of the classical lot size model with finite supply rate is;

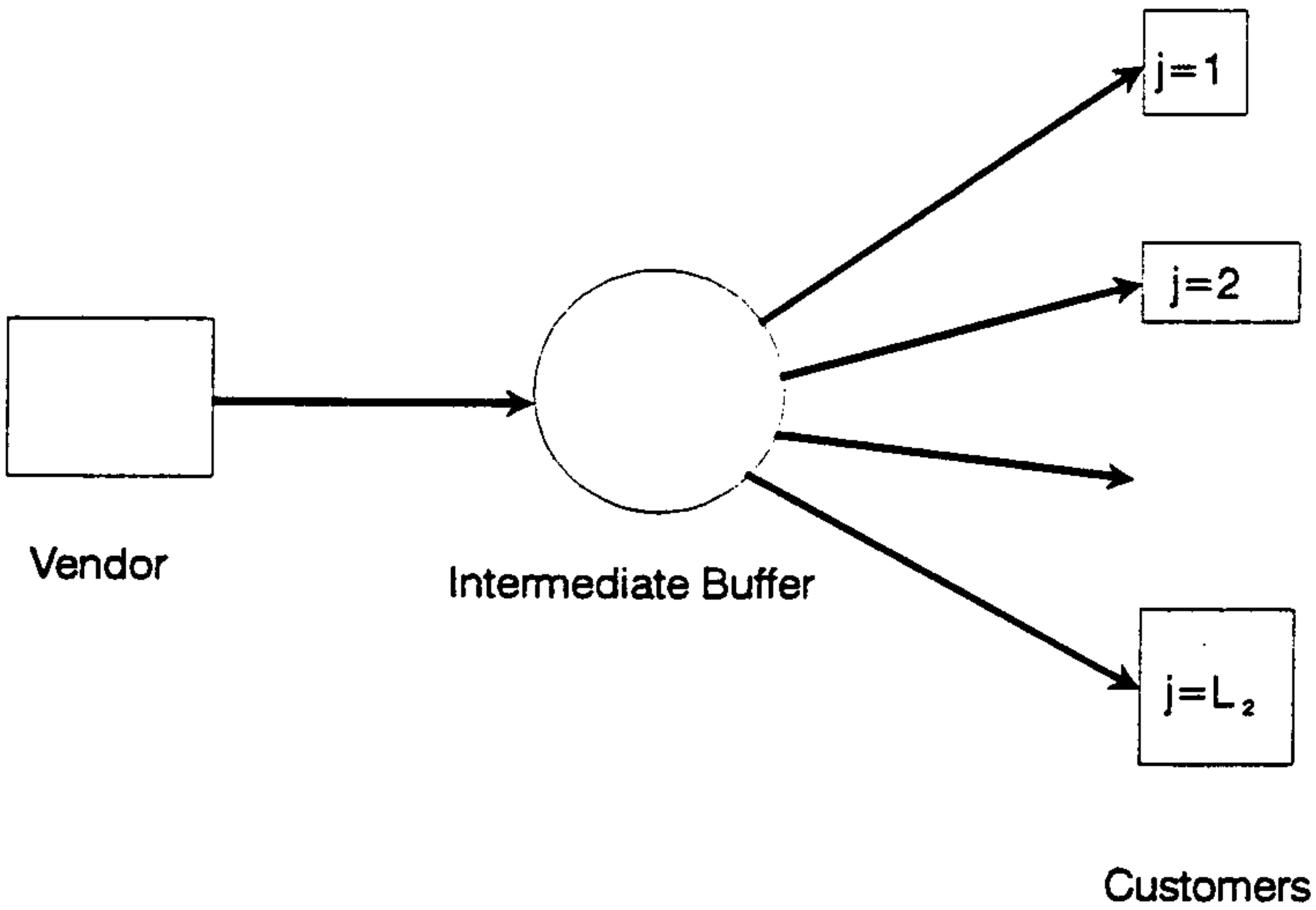
$$\mathrm{B}_1 \leq \frac{V_s}{1 - x_1} \tag{19}$$

3. RELAXING PERIODIC SQUARE WAVE DEMAND INTO ARBITRARY PERIODIC DEMAND

We can apply the PSW model to a single product, the single input/multiple output (SIMO) intermediate buffer system as shown at Fig. 4. Suppose we have L_2 down-streams. Each up or down-stream is presumed to have periodic square wave material flow. Up-stream flow can be represented as the same equation as in Eq. (4) and down-stream flows can be represented the sum of each down-stream which has the same equation in Eq. (4). Therefore, the inventory hold-up is

$$\begin{aligned} V(t) = {} & V(0) + B_1 \left(int\left[\frac{t}{\omega_1}\right] + min\left\{1, \frac{1}{x_1} res\left[\frac{t}{w_1}\right]\right\}\right) \\ & - \sum_{j=1}^{L_2} B_{2j} \left(int\left[\frac{t - y_{2j}\,\omega_{2j}}{\omega_{2j}}\right] + min\left\{1, \frac{1}{x_{2j}} res\left[\frac{t - y_{2j}\,\omega_{2j}}{\omega_{2j}}\right]\right\}\right) \end{aligned} \tag{20}$$

〈Figure 4〉 Schematic Diagram of SIMO Buffer



where subscript *2j* represents the *j-th* stream among down-streams. When we follow the same procedure in the above PSW model, we will have completely the same result of Eq. (14). The differences exist in the time averaged constant demand and the constant part of total cost in Eq. (13) which do not influence the final result.

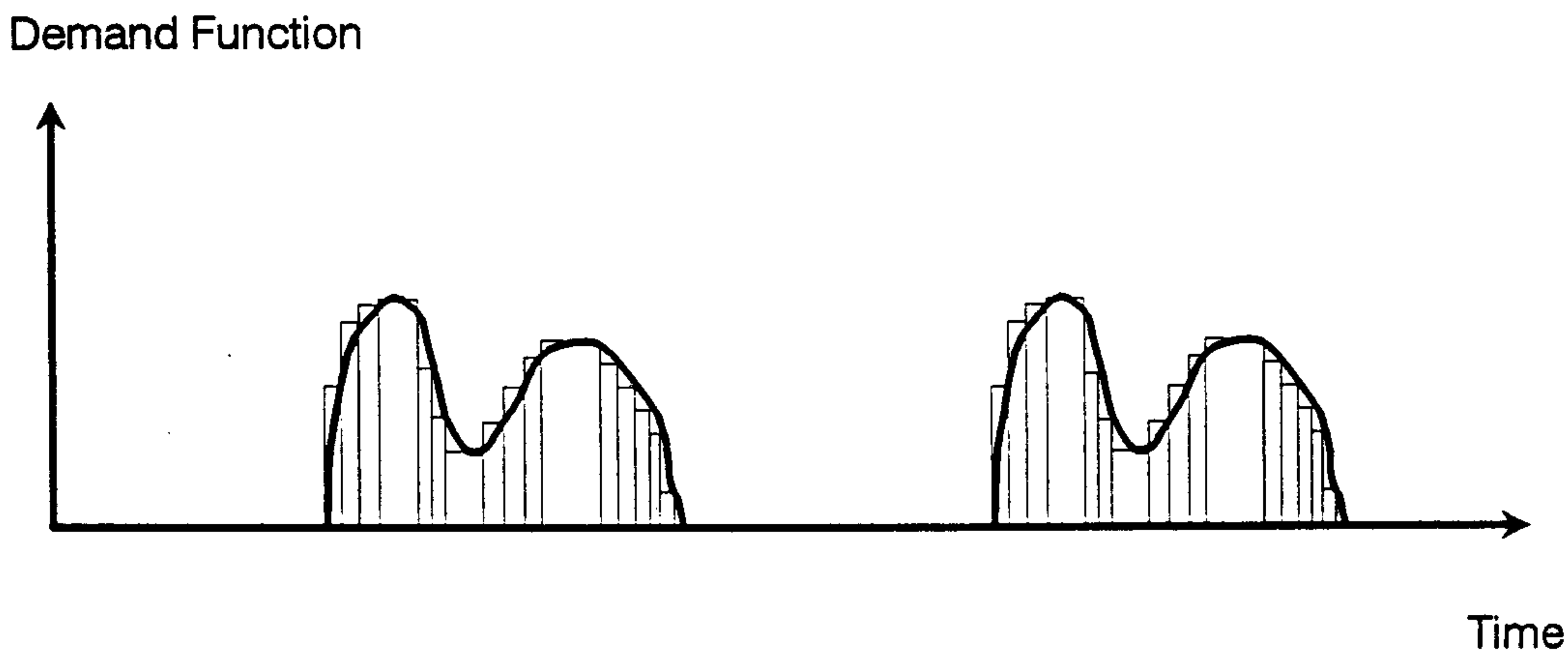
$$D = \frac{B_1}{\omega_1} = \sum_{j=1}^{L_2} \frac{B_{2j}}{\omega_{2j}} \tag{21}$$

The periodic square wave functions of multiple output streams can be utilized to represent an abitrary periodic function. Suppose an arbitrary periodic demand function $\Delta(t)$ where $\Delta(t)=\Delta(t+\omega_2)$. It is obvious from Fig. 5 that $\Delta(t)$ can be approximated to any degree of accuracy by the superposition of sufficient number of proper periodic square waves.

$$\Delta(t) = \lim_{L_2 \to \infty} \sum_{j=1}^{L_2} B_{2j} \left(int \left[\frac{t - y_2\omega_2}{\omega_2}\right] + min\left\{1, \frac{1}{x_{2j}} res\left[\frac{t - y_2\omega_2}{\omega_2}\right]\right\}\right) \tag{22}$$

Theoreticaly, any periodic demand functions can be represented with the superpostion of periodic square waves as the same way as Fourier series development. The procedure and result of finding optimal order quantity is the same as the above single product, SIMO intermediate buffer system except

〈Figure 5〉 Arbitrary Periodic Demand



that the cycle time ω_2 in Eq. (22) is changed into ω_{2j}. We can conclude that the result of the classical economic lot size model, Eq. (16), can be applicable to arbitrary periodic demand.

In the following two sections, we will introduce two simple cases that the PSW model can be solely applicable over the EPQ model.

4. ECONOMIC LOT SIZE MODEL WITH FINITE SUPPLY/SHIPPING RATE

Consider the SISO buffer system in Fig. 1. It was assumed in section 2 that downstream parameters; demand size (B_2), demand cycle time (ω_2) and initial delay time fraction(y_2), were already known and fixed values. In this section it will be assumed that these downstream variables are unknown and should be determined optimally as well as upstream variables; order quantity (B_1) and order cycle time (ω_1). Also, It will be assumed that buffer size V_s is finite constant. This situation exists in the real world when the downstream is within a supplier's system boundary and the supplier can easily manipulate customer delivery patterns with fixed average demand rate. For consistency with the EPQ model, it will be assumed that transportation time fraction of downstream x_2 and demand rate $D=\frac{B_1}{\omega_1}=\frac{B_2}{\omega_2}$ are known constant. A nonnegative variable $t_2 \equiv y_2\omega_2$ is defined for convenience. The ordering cost for upstream is set to A_1 and the ordering cost for downstream is set to A_2. The total cost is composed of the sum of both upstream and downstream ordering costs and inventory holding cost. It can be easily derived as the same way as Eq. (13).

$$\text{Total Cost} = \frac{A_1 D}{B_1} + H\left(\frac{1-x_1}{2}\right) B_1 + \frac{A_2 D}{B_2} + H\left(\frac{1-x_2}{2}\right) B_2 + HDt_2 + HV(0) \tag{23}$$

The inventory hold-up V(t) should be confined within buffer capacity. Sufficient conditions are $0 \le V_{lb} < V_{ub} \le V_s$. Two constraints are obtained from Eqs. (9) and (10).

$$V(0) - B_2(1-x_2) + Dt_2 \ge 0 \tag{24}$$

$$-(1-x_1)B_1 - Dt_2 + V_s - V(0) \ge 0 \tag{25}$$

The problem is defined as minimizing total cost in Eq. (23) subject to the constraints Eqs. (24) and (25) with respect to positive search variables, B_1, B_2 and nonnegative search variable, t_2.

The objective function is convex. This can be shown by examining its Hessian. The Hessian is a diagonal matrix with all nonnegative elements on its main diagonal. Therfore, the Hessian is positive semidefinite and the total cost is convex. The constraints Eq. (24) and (25) are linear with respect to search variables and therefore, are concave. From the Kuhn-Tucker Sufficiency Theorem in [3], if there is a solution that satisfies Kuhn-Tucker conditions, it is optimal and a global minimum in this case.

Let λ_i be the nonnegative Lagrange multiplier for Eqs. (24), (25) and t_2. The Lagrangian is:

$$L = \frac{A_1 D}{B_1} + H\left(\frac{1-x_1}{2}\right) B_1 + \frac{A_2 D}{B_2} + H\left(\frac{1-x_2}{2}\right) B_2 + HDt_2 + HV(0) - \lambda_1 (V(0) - B_2(1-x_2) + Dt_2) - \lambda_2(-(1-x_1)B_1 - Dt_2 - V_s - V(0) - \lambda_3 t_2 \tag{26}$$

The Kuhn-Tucker conditions are Eqs. (24), (25) and the following equations:

$$\frac{\partial L}{\partial B_1}=-\frac{A_1D}{B_1^2}+\frac{H(1-x_1)}{2}+\lambda_2(1-x_1)=0 \tag{27}$$

$$\frac{\partial L}{\partial B_2}=-\frac{A_2D}{B_2^2}-\frac{H(1-x_2)}{2}+\lambda_1(1-x_2)=0 \tag{28}$$

$$\frac{\partial L}{\partial t_2}=HD-\lambda_1D+\lambda_2D-\lambda_3=0 \tag{29}$$

$$\lambda_1(V(0)-B_2(1-x_2)+Dt_2)=0 \tag{30}$$

$$\lambda_2(-(1-x_1)B_1-Dt_2+V_s-V(0))=0 \tag{31}$$

$$\lambda_3t_2=0 \tag{32}$$

The solution can be obtained by trying both $\lambda=0$ and $\lambda>0$. The detail procedure is in APPENDIX B. There are 4 solution sets each of which has different paramertic zone. The final results are summarized in Table 1.

[Table 1] The solution of Kuhn-Tucker conditions for economic lot size with finite supply/ shipping rate.

	Case (1)	Case (2)
Restricition 1	No	No
Restriction 2	$\sqrt{\frac{2A_1D(1-x_1)}{H}}\leq V_s-V(0)$	$\sqrt{\frac{2A_1D(1-x_1)}{H}}>V_s-V(0)$
B_1	$B_1=\sqrt{\frac{2A_1D}{(1-x_1)H}}$	$B_1=\frac{V_s-V(0)}{1-x_1}$
B_2	$B_2=\frac{V(0)}{(1-x_2)}$	$B_2=\frac{V(0)}{1-x_2}$
t_2	$t_2=0$	$t_2=0$

	Case (3)	Case (4)
Restricition 1	$V(0) < \sqrt{\frac{2A_2D(1-x_2)}{H}}$	$V(0) < \sqrt{\frac{2A_2D(1-x_2)}{H}}$
Restriction 2	$V_s \leq \sqrt{\frac{2D}{H}}(\sqrt{A_1(1-x_1)} + \sqrt{A_2(1-x_2)})$	$V_s > \sqrt{\frac{2D}{H}}(\sqrt{A_1(1-x_1)} + (\sqrt{A_2(1-x_2)}))$
B_1	$B_1 = \sqrt{\frac{2A_1D}{(1-x_1)H}}$	$B_1 = \frac{V_s}{\sqrt{A_1(1-x_1)} + \sqrt{A_2(1-x_2)}}\sqrt{\frac{A_1}{1-x_1}}$
B_2	$B_2 = \sqrt{\frac{2A_2D}{(1-x_2)H}}$	$B_2 = \frac{V_s}{\sqrt{A_1(1-x_1)} + \sqrt{A_2(1-x_2)} +}\sqrt{\frac{A_2}{1-x_2}}$
t_2	$t_2 = \sqrt{\frac{2A_2(1-x_2)}{HD}} - \frac{V(0)}{D}$	$t_2 = \frac{V_s\sqrt{A_2(1-x_2)}}{D(\sqrt{A_1(1-x_1)} + \sqrt{A_2(1-x_2)})} - \frac{V(0)}{D}$

The PSW model is flexible enough to represent multiple stages and/or parallel units interconnected with intermediate buffers. It is not difficult to compose an optimization problem for a multiple echelon inventory system with the PSW model. The result of this section says that explicit solutions of Kuhn-Tucker conditions can be obtained for a certain simple structure. It would definitely be worthwhile to explore the whole network structure with the PSW model, which is beyond the scope of this study.

5. LIMITING VOLUME OF INTERMEDIATE BUFFER

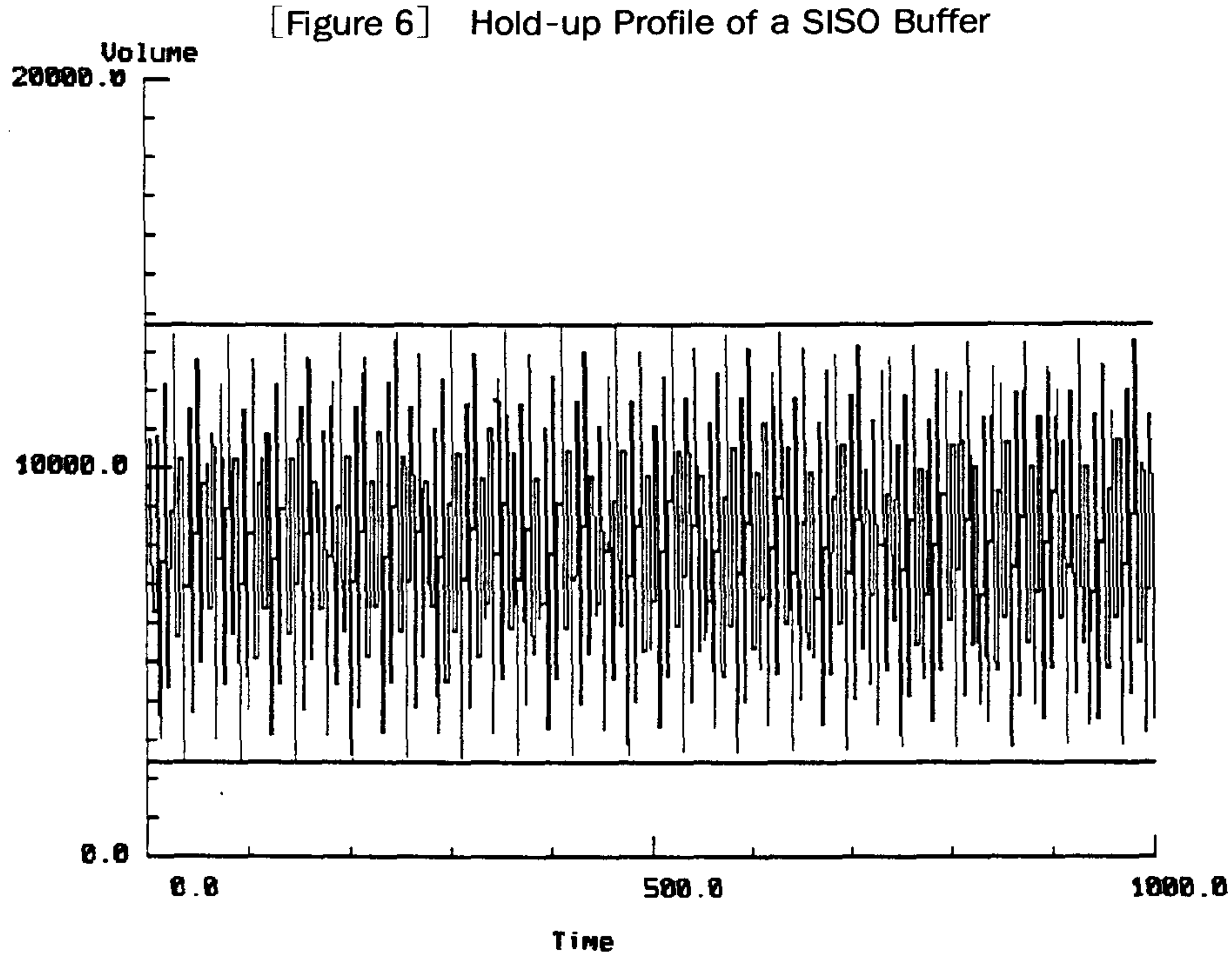
Both PSW and EPQ models result in the same optimal solution for determining order lot size and cycle time but they produce different results for the limiting buffer size. The hold-up profile of the EPQ model looks like a saw shape, and the limiting buffer size with the EPQ model is given in Eq. (19).

However, the hold-up profile of the PSW model looks very complicated as can be seen in Fig 6. The buffer volume V_s and the initial inventory hold-up V(0) can be obtained from the inequalities $0 \le V_{lb} < V_{ub} \le V_s$. The global minimum / maximum of inventory hold-up, V_{max} and V_{min}, for SISO intermediate buffer in Eq. (4) can be obtained by quite involved algebraic manipulation as shown in APPENDIX A.

$$V_{max} = V(0) + B_1 - \frac{B_1}{\beta_2}(\text{int}[x_1\beta_2 - y_2\beta_1] + \min\{1, \frac{1}{x_i}\text{res}[x_1\beta_2 - y_2\beta_1]\}) \quad (33)$$

$$V_{min} = V(0) - B_2 + \frac{B_2}{\beta_1}(\text{int}[(x_2 + y_2)\beta_1] + \min\{1, \frac{1}{x_i}\text{res}[(x_2 + y_2)\beta_1]\}) \quad (34)$$

[Figure 6] Hold-up Profile of a SISO Buffer



where i=1 for $U_1 \le U_2$, i=2 for $U_1 > U_2$. Integer β_1 and β_2 are defined from the following equation:

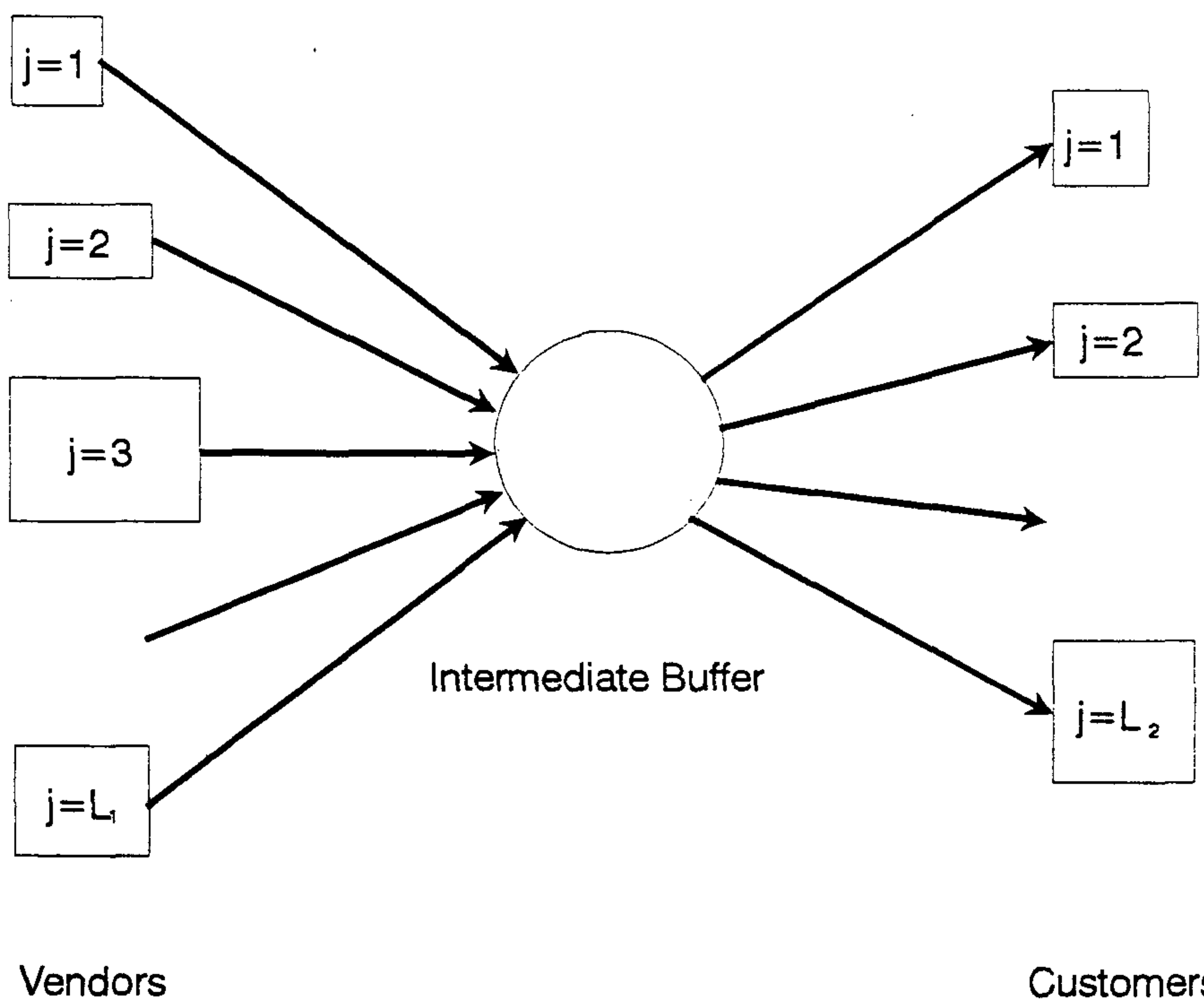
$$LCM(\omega_1\ \omega_2) = \beta_1\ \omega_1 = \beta_2\ \omega_2 \quad (35)$$

where LCM(⋯,⋯) is the least common multiple. The usage of Eq.(33) and (34) is quite limited because of the complexity of calculating the least common multiple of cycle times in Eq. (35).

The structure of the intermediate buffer in Fig. 4 can be generalized into the multiple input / multiple output (MIMO) buffer case. Suppose we have an intermediate buffer interconnected with L_1 upstream and L_2 downstream as is shown at Fig. 7. The PSW model also provides very accurate description for the MIMO intermediate buffer. The hold-up equation for a MIMO buffer can be represented by summing over each stream, which is treated the same way as in the previous section:

$$\frac{dV}{dt} = \sum_{j=1}^{L_1} F_{1j}(t)u(t-y_{1j}\ \omega_{1j}) - \sum_{j=1}^{L_2} F_{2j}(t)u(t-y_{2j}\ \omega_{2j})u(t-y_{2j}\ \omega_{2j}) \tag{36}$$

〈Figure 7〉 Schematic Diagram of MIMO Buffer



where $t \geq max[y_{1j}\ \omega_{1j}]$ and subscript $1j$ represents the *j-th* stream among upstreams. Integration of Eq. (36) yields,

$$V(t)=V(0)+\sum_{j=1}^{L_1} B_{1j}(int[\frac{t-y_{1j}\ \omega_{1j}}{\omega_{1j}}]+min\{1, \frac{1}{x_{1j}} res[\frac{t-y_{1j}\ \omega_{1j}}{\omega_{1j}}]\})$$
$$-\sum_{j=1}^{L_2} B_{2j}(int[\frac{t-y_{2j}\ \omega_{2j}}{\omega_{2j}}]+min\{1, \frac{1}{x_{2j}} res[\frac{t-y_{2j}\ \omega_{2j}}{\omega_{2j}}]\}) \quad (37)$$

The overall material balance corresponding to Eq. (6) in the SISO case is:

$$\sum_{j=1}^{L_1} \frac{B_{1j}}{\omega_{1j}} = \sum_{j=1}^{L_2} \frac{B_{2j}}{\omega_{2j}} \quad (38)$$

The analytical solution of minimum / maximum hold-up is available only for a special structure of the MIMO storage system, named the "symmetrical structure," which has identical batch sizes for all up or downstreams, equally distributed initial delays[4]. The symmetrical structure is an uncommon MIMO buffer system and these maximum / minumum hold-up equations do not seem to be useful in practice. The exact maximum or minimum hold-up equations for the general MIMO buffer system are not explicitly obtained since this would require the systematic manipulation of a number of integer variables that have different scales. Instead of deriving an exact maximum / minmum hold-up equation, we can develop a very tight upper / lower bound on the hold-up equation which is useful for engineering purposes. Eq. (37) contains a basic functional form of the flow accumulation function, Eq. (6). The upper / lower bound of hold-up can be easily obtained by applying Eq. (7) for each stream in Eq. (37).

$$V_{ub}= V(0)-\sum_{j=1}^{L_1} B_{1j}(x_{1j}+y_{1j}-1)+\sum_{j=1}^{L_2} B_{2j}y_{2j} \quad (39)$$

$$V_{lb}= V(0)-\sum_{j=1}^{L_1} B_{1j}y_{1j}+\sum_{j=1}^{L_2} B_{2j}(x_{2j}+y_{2j}-1) \quad (40)$$

The upper bound on the buffer size can be calculated by subtracting Eq. (39) from (40).

$$V_s \leq \sum_{j=1}^{L_1} B_{1j}(1-x_{1j})+\sum_{j=1}^{L_2} B_{2j}(1-y_{2j}) \quad (41)$$

6. DISCUSSIONS

The usefulness of PSW model is not confined to revalidating the classical lot-sizing model. The PSW model has been successfully applied for the design and operation of intermediate storage in noncontinuous chemical processes[4]. A model-based real-time inventory control mechanism has been developed for single-product MIMO intermediate storage. The PSW model has been applied to predict the future behavior of system state variables and control input calculation as well as system parameter identification. This study was extended to a multiple-product MIMO warehouse inventory control system[6].

The advantage of the PSW model comes from the fact that we can easily calculate very close upper / lower bound and time average of inventory hold-up, as in Eq. (9), (10) and (11), as well as that the model can accurately represent the material flow of the inventory system. The PSW model can be applicable to a combined inventory and production system as well as a pure inventory system. We can analyze the inventory and production system in a continuous time domain, not on multiperiod time buckets, without increasing the computational burden. Although the PSW cannot remove the combinatorial nature of the system problem, we can reduce the computational complexity as far as system state variables are periodic.

The PSW model also has several weak points. It seems very difficult or impossible to represent backlogging and lost sales. It is not convenient to model a highly nonperiodic process or a finite horizon. Further investigation is required to reveal fully the characteristics of the PSW model.

7. CONCLUSIONS

The EPQ model presumed constant demand and a periodic square wave material flow for order. Instead, the PSW model presumed that the material flow of both order and demand was periodic-square-wave shaped. The results of optimal order size and frequency in both EPQ and PSW models were the same in spite of different demand pattern assumptions. We could easily prove with the PSW model that the classcial economic-lot-size model gives an optimal solution for arbitrary periodic demand on the contrary to the initial assumption of constant demand. As a unique applicaton example, the PSW model has provided the explicit solution for determining the economic lot size of both upstream and downstream. The limiting buffer size calculated by the PSW model was different from and was obviously more accurate than the limiting buffer size calcuated by the EPQ model. The potential of the PSW model was discussed. Some outstanding application examples were referred. The modeling accuracy, easy generalization and analytic simplicity of the PSW model provide a powerful theoretical tool for production and inventory system analysis.

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APPENDIX A : The Solution Procedure for Eqs. (33) and (34).

First of all, the division theorem for integers should be introduced before we start the procedure.

Integer Division Theorem

For the given real z and integers β_1, β_2, the integr variable α_1 maps one to one and onto the integer variables (q_1, a_1) such that;

$$\alpha_1 \beta_1 + int[z] = q_1\beta_2 + a_1 \quad \text{(A1)}$$

where

$$\begin{aligned} \alpha_1 &= \{0, 1, 2, \cdots, \beta_2 - 1\} \\ a_1 &= \{0, 1, 2, \cdots, \beta_2 - 1\} \end{aligned} \quad \text{(A2)}$$

if and only if

$$GCD(\beta_1, \beta_2) = 1 \quad \text{(A3)}$$

For proof, see Burton [1].

The following working expression can be developed using Eq. (A1).

$$\begin{aligned} int\left[\frac{\alpha_1\beta_1 + z}{\beta_2}\right] &= q_1 = \frac{\alpha_1\beta_1 + int[z] - a_1}{\beta_2} \\ res\left[\frac{\alpha_1\beta_1 + z}{\beta_2}\right] &= \frac{a_1 + res[z]}{\beta_2} \end{aligned} \quad \text{(A4)}$$

The hold-up function V(t) in Eq. (4) in the main text is periodical with a period of $\omega_i\beta_i$, that is, $V(t) = V(t + \omega_i\beta_i)$. Therefore, it is necessary to search extreme points within a period. The local minima of the hold-up function in Eq. (4) in the main text occur at the starting moments of the feed flow if the upstream flow rate is greater than if the downstream flow rate, or at the ending moments of the discharging flow if the downstream flow rate is greater than the upstream flow rate. In an analogous fashion, the local maxima of the hold-up function occur at the ending moments of the feed flow or at the starting moments of the discharging flow. The case in which the upstream flow rate is greater than the downstream flow rate will be called the upstream dominant case and the other the downstream dominant case. In each case, the local minimum / maximum values of the hold-up can be calculated by Eq. (4) in the main text using the following time values within a period of $\omega_i \beta_i$.

Up-stream Dominant Case($U_1 \geq U_2$)

$$t_{min} = \alpha_1 \omega_1 \tag{A5}$$

$$t_{max} = (\alpha_1 + x_1)\omega_1 \tag{A6}$$

where $\alpha_1 = \{0, 1, 2, \cdots, \beta_1 - 1\}$

Down-stream Dominant Case(($U_1 < U_2$)

$$t_{min} = (\alpha_2 + x_2 + y_2)\omega_2 \tag{A7}$$

$$t_{max} = (\alpha_2 + y_2)\omega_2 \tag{A8}$$

where $\alpha_2 = \{0, 1, 2, \cdots, \beta_2 - 1\}$

The continuous search variable t in terms of which the hold-up is expressed can be transformed into finite integer variables α_1 or α_2 by inserting Eq. (A5)~(A8) into Eq. (4) in the main text. The resulting equations for the minimum / maximum can be segregated into piecewise linear forms with respect to independent variables with the aid of the integer division theorem in (A1). Then, the global minimum / maximum can be readily obtained by selecting integer variable values that yield extreme points of each piecewise linear interval. The procedure to get the minimum or maximum hold-up is quite the same and thus only the minimum hold-up case will be discussed here.

Upstream Dominant Case ($U_1 \geq U_2$)

$$\frac{t_{min}}{\omega_1} = \alpha_1 \tag{A9}$$

$$\frac{t_{min} - y_2 \omega_2}{\omega_2} = \frac{\alpha_1 \beta_2 - int[y_2 \beta_1] - 1 + 1 - res[y_2 \beta_1]}{\beta_1} \tag{A10}$$

The integer part of Eq. (A10) can be transformed by using Eq. (A4) into;

$$\alpha_1 \beta_2 - \mathrm{int}[y_2\beta_1] - 1 = q_1\beta_1 + a_1 \tag{A11}$$

where $a_1 = \{0, 1, 2, \cdots, \beta_1 - 1\}$.

Based on the integer division theorem, there exists a one-to-one correspondence between, α_1 and a_1 because β_1 and β_2 are relatively prime.

$$int\left[\frac{t_{min} - y_2\omega_2}{\omega_2}\right] = q_1 = \frac{\alpha_1\beta_2 - int[y_2\beta_1] - 1 - a_1}{\beta_1} \tag{A12}$$

$$res\left[\frac{t_{min} - y_2\omega_2}{\omega_2}\right] = \frac{1 - res[y_2\beta_1] + a_1}{\beta_1} \tag{A13}$$

Inserting Eq. (A9), (A12) and (A13) into Eq. (4) in the main text gives;

$$\begin{aligned} V(t_{min}) &= V(0) + \beta_1\alpha_1 \\ &- B_2\left[\frac{\alpha_1\beta_2 - int[y_2\beta_1] - 1 - \alpha_1}{\beta_1} + min\{1, \frac{a_1 + 1 - res[y_2\beta_1]}{x_2\beta_1}\}\right] \end{aligned} \tag{A14}$$

The second and third terms cancel each other using the relation of $B_1\beta_1 = B_2\beta_2$.

There are two cases for min{⋯}.

$$\text{i)}\ \frac{a_1 + 1 - res[y_2\beta_1]}{x_2\beta_1} \geq 1 \tag{A15}$$

Then, the min{⋯} of Eq. (A14) is 1 and Eq. (A14) changes to;

$$V(t_{min}) = V(0) + B_2\left[\frac{int[y_2\beta_1] + 1 + a_1}{\beta_1}\right] - B_2 \tag{A16}$$

The integer search variable α_1 is transformed into a_1. Minimization $V(t_{min})$ with respect to a_1 can be easily found by finding the minimum of a_1 since $V(t_{min})$ is a linearly increasing function with respect to a_1 in Eq. (A16). Integer a_1 can be minimized from inequality Eq. (A15).

$$\min a_1 = int[(x_2 + y_2)\beta_1] - int[y_2\beta_1] \tag{A17}$$

Inserting Eq. (A17) into Eq. (A16) gives a minmum hold-up for the domain of a_1 satisfying Eq. (A15).

$$V^1_{min} = V(0) + B_2\left[\frac{int[x_2 + y_2)\beta_1] + 1}{\beta_1}\right] - B_2 \tag{A18}$$

$$\text{ii)}\ \frac{a_1 - 1 + res[y_2\beta_1]}{x_2\beta_1} < 1 \tag{A19}$$

Then, the min{…} of Eq. (A14) is $(a_1 + 1 - res[y_2\beta_1])/x_2\beta_1$ and Eq. (A14) changes to;

$$V(t_{min}) = V(0) + \frac{B_2}{\beta_1}\left[int[y_2\beta_1] + 1 + (1 - \frac{1}{x_2})a_1 - \frac{1 - res[y_2\beta_1]}{x_2}\right] \tag{A20}$$

Minimizing $V(t_{min})$ with repect to a_1 can be easily done by finding maximum of a_1 since $V(t_{min})$ is linearly decreasing function with respect to a_1 in Eq. (A20). Integer a_1 can be minimized from inequality Eq. (A19).

$$\max a_1 = int[(x_2 + y_2)\beta_1] - int[y_2\beta_1] - 1 \tag{A21}$$

Inserting Eq. (A21) into Eq. (A20) gives a minmum hold-up for the domain of a_1 satisfying Eq. (A19)

$$V^2_{min} = V(0) + \frac{B_2}{\beta_1}\left[int[(x_2 + y_2)\beta_1] + \frac{res[(x_2 + y_2)\beta_1]}{x_2}\right] - B_2 \tag{A22}$$

Eq. (A18) and Eq. (A22) both serve to determine the minimum hold-up. Combining two equations with the min{…} function results in the following equation;

$$V_{min} = V(0) + \frac{B_2}{\beta_1}\left[int[(x_2 + y_2)\beta_1] + min\{1, \frac{res[(x_2 + y_2)\beta_1]}{x_2}\}\right] - B_2 \tag{A23}$$

B. Downstream Dominant Case ($U_1 < U_2$)

The procedure is similar to the upstream dominant case and the result is;

$$V_{min} = V(0) + \frac{B_2}{\beta_1}\left[\text{int}[(x_2+y_2)\beta_1] + \min\{1, \frac{res[(x_2+y_2)\beta_1]}{x_1}\}\right] - B_2 \quad (A24)$$

Two minmum hold-up equations, Eq. (A23) and Eq. (A24) are almost the same except for the variables x_1 and x_2 for each case.

APPENDIX B: The Solution of Kuhn-Tucker Conditions

Eqs. (27) and (28) can be rewritten to:

$$B_1 = \sqrt{\frac{2A_1D}{(1-x_1)(H+2\lambda_2)}} \qquad \text{(B1)}$$

$$B_2 = \sqrt{\frac{2A_2D}{(1-x_2)(-H+2\lambda_1)}} \qquad \text{(B2)}$$

From Eq. (B2), $\lambda_1 > H/2$. Since either the inequality constraint or its Lagrange multiplier should be zero, Eq.(24) should be active.

$$V(0) - B_2(1-x_2) + Dt_2 = 0 \qquad \text{(B3)}$$

We should consider 4 cases; (1)$\lambda_3 > 0$ and $\lambda_2=0$, (2) $\lambda_3>0$ and $\lambda_2>0$, (3) $\lambda_3=0$ and $\lambda_2=0$, (4) $\lambda_3=0$ and $\lambda_2>0$.

Case (1) $\lambda_3 > 0$ and $\lambda_2=0$

The case that λ_3 is not zero means $t_2=0$. Eq. (B3) can be solved with respect to B_2:

$$B_2 = \frac{V(0)}{1-x_2} \qquad \text{(B5)}$$

B_1 can be obtained by substituting 0 for λ_2 in Eq. (B1)

$$B_1 = \sqrt{\frac{2A_1D}{(1-x_1)H}} \qquad \text{(B5)}$$

By inserting Eqs. (B4) and (B5) into Eq. (25) in the main text a parametric restriction can be obtained for this case.

$$\sqrt{\frac{2A_1D(1-x_1)}{H}} \leq V_s - V(0) \qquad \text{(B6)}$$

Case (2) $\lambda_3 > 0$ and $\lambda_2 > 0$

The solution $t_2=0$ and Eq. (B4) holds true in this case. Since λ_2 is not zero, the constraint Eq. (25) in the main text is binding and can be solved with respect to B_1.

$$B_1 = \frac{V_s - V(0)}{1-x_1} \tag{B7}$$

λ_2 can be obtained from Eqs. (B1) and (B7). The condition that λ_2 is positive gives the opposite inequality of Eq. (B7).

$$\sqrt{\frac{2A_1D(1-x_1)}{H}} > V_s - V(0) \tag{B8}$$

Case (3) $\lambda_3=0$ and $\lambda_2=0$

From Eq. (29), $\lambda_2=H$. By plugging $\lambda_2=H$ into Eqs. (B1) and (B2), the standard EPQ solution can be obtained for both B_1 and B_2.

$$B_2 = \sqrt{\frac{2A_2D}{(1-x_2)H}} \tag{B9}$$

The solution for B_1 is the same as Eq. (B5). t_2 can be obtained by inserting Eq. (B9) into Eq. (B3).

$$t_2 = \sqrt{\frac{2A_2D(1-x_2)}{HD}} - \frac{\sqrt{V(0)}}{D} \tag{B10}$$

The condition that t_2 should be positive in this case gives a parametric restriction.

$$V(0) < \sqrt{\frac{2A_2D(1-x_2)}{H}} \tag{B11}$$

The condition $\lambda_2=0$ means that the inequality constraint Eq. (25) is not binding. Another parametric restriction can be obtained by inserting Eqs. (B5) and (B10) into Eq. (25).

$$V_s \le \sqrt{\frac{2D}{H}}\left(\sqrt{A_1(1-x_1)}+\sqrt{A_2(1-x_2)}\right) \qquad \text{(B12)}$$

Case (4) $\lambda_3=0$ and $\lambda_2>0$

Since λ_1, λ_2 are positive in this case, both Eqs. (24) and (25) are binding. The following equation is obtained by removing Dt_2 and $V(0)$ terms from both Eqs. (24) and (25).

$$V_s = (1-x_1)B_1+(1-x_2)B_2 \qquad \text{(B13)}$$

Eq. (B13) can be solved with respect to λ_2 by substituting B_1 and B_2 for Eqs. (B1) and (B2) with the condition $\lambda_1 = H+\lambda_2$.

$$\lambda_2=\frac{1}{2}\left(\frac{\sqrt{2A_1D(1-x_1)}+\sqrt{2A_2D(1-x_2)}}{V_s}\right)^2-\frac{H}{2} \qquad \text{(B14)}$$

The condition that λ_2 should be positive in this case gives a parametric restriction which is opposite to Eq. (B12).

$$V_s > \sqrt{\frac{2D}{H}}\left(\sqrt{A_1(1-x_1)}+\sqrt{A_2(1-x_2)}\right) \qquad \text{(B15)}$$

The solution for B_1 and B_2 can be obtained by plugging Eq. (14) into Eqs. B(1) and (B2) with the condition $\lambda_1=H+\lambda_2$.

$$B_1 = \frac{V_s}{\sqrt{A_1(1-x_1)} + \sqrt{A_2(1-x_2)}}\sqrt{\frac{A_1}{1-x_1}} \tag{B16}$$

$$B_2 = \frac{V_s}{\sqrt{A_1(1-x_1)} + \sqrt{A_2(1-x_2)}}\sqrt{\frac{A_2}{1-x_2}} \tag{B17}$$

Finally, t_2 can be obtained from Eq. (B3).

$$t_2 = \frac{V_s\sqrt{A_2(1-x_2)}}{D(\sqrt{A_1(1-x_1)} + \sqrt{A_2(1-x_2)})} - \frac{V(0)}{D} \tag{B18}$$

The condition that t_2 should be positive in this case gives another parametric restriction.

$$\frac{V_s}{V(0)} > \frac{\sqrt{A_1(1-x_1)} + \sqrt{A_2(1-x_2)}}{\sqrt{A_2(1-x_2)}} \tag{B19}$$

Combining Eq. (B19) and (B15) gives the same condition as Eq. (B11).