International Journal of Management Science
Vol. 1, No. 1, December 1995

Benefits of Using Imperfect Information in Controlling an M/M/1 Queueing System

Ick-Hyun Nam [1)]

Abstract

In this paper, we analyze an M / M / 1 queueing system where there are incentive conflicts among customers. Self-interested customers' decisions whether to join the system or not may not necessarily induce a socially optimal congestion level. As a way to alleviate the over-congestion, toll imposition was used in Naor's paper [3]. Instead of using a toll mechanism, we study the usefulness of imperfect information on system state (queue size, for example) as a way to reduce the over-congestion by self-interested customers.

The main conclusion of this paper is that by purposefully giving fuzzy or imperfect information on the current queue size we can improve the congestion in the system. This result might look contradictory to rough intuition since perfect information should give better performance than imperfect information. We show how this idea is verified. In deriving this result, we use the concept of Nash equilibrium (pure and mixed strategy) as introduced in game theory. In some real situations, using imperfect information is easier to apply than imposing a toll, and thus the result of this paper has practical implications.

---

1) Mailing address: Room 501, College of Business Administration, Seoul National University, Seoul, Korea 151-742

1. Introduction

In this paper we analyze a queueing system where a socially optimal congestion level cannot be achieved. An arriving customer cannot observe the number of customers in the service system before he joins the system, but he may obtain (imperfect) information on such number through an information system. It is assumed that the customer is not allowed to renege after joining the system.

We seek to study the implications of implementing such a system and to choose the system to optimize the overall net benefit when the service to the customers is viewed as a public good. Ultimately, we are interested in how far an information system can alleviate the congestion externality under two scenarios. The first scenario is on the benefit of providing information to the customers when they cannot observe such information themselves (the base case is no information). The second scenario is on the benefit of controlling information when they can observe perfect information without such control (the base case is perfect information).

In the next section we review basic results for an M / M / 1 system, Naor's paper [3], and definition of Nash equilibrium. And then we derive our main conclusion.

2. Review

2.1 Some Results for an M / M / 1 System

Let λ=arrival rate, μ=service rate, $\rho=\lambda/\mu$=traffic intensity, L=mean number of customers in the system, W=mean throughput time, and p_k=steady state probability of finding k customers in the system. Then for an M / M / 1 queueing system, we get the following results:

$$p_k = (1-\rho)\rho^k$$

$$L = \frac{\rho}{1-\rho}$$

$$W = \frac{1}{\mu-\lambda}$$

For an M / M / 1 / K with $\rho<1$ queueing system, the corresponding results are:

$$P_k = \begin{cases} \dfrac{(1-\rho)\rho^k}{1-\rho^{K+1}} & \text{for } k \le K \\ 0 & \text{otherwise} \end{cases}$$

$$L = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}$$

$$W = \frac{L}{\lambda(1-PK)}$$

2.2 Naor's Model

We use the following notations as defined in [3].

- R = Reward for receiving the service,
- C = waiting cost per unit time,
- $v_s = \dfrac{R\mu}{C}$,
- $n_s = [v_s]$ = the critical number of customers in the system below which a customer will choose to join the system if he has perfect information,
- $E[i]$ = expected number of customers in the system,
- P = expected total net gain,
- n_o = control limit which achieves overall optimality of the service system.

In Naor's paper and this one, the selected strategy of n means the following: if the observed value of customers in the system falls short of n, the newly arrived customer will join the system; if the observed value is equal to n, the new customer is diverted and does not join the system. Therefore under this strategy the observed number of customers cannot exceed n.

2.2.1 Self-Optimization

The newly arrived customer weighs the two alternatives—to join or not to join the queueing system—by the net gains associated with them. The expected net gain (given there are i customers in front of the new arrival), in the first case, is equal to

$$G_i = R-(i+1)C\frac{1}{\mu}$$

In the alternative case, the net gain is zero. Hence self-interest is served if a strategy is established in the following fashion. An integer, n_s, is found which satisfies simultaneously two inequalities

$$R-n_sC\frac{1}{\mu} \geq 0$$

and

$$R-(n_s+1)C\frac{1}{\mu} < 0.$$

These two inequalities can be represented by

$$n_s \leq \frac{R\mu}{C} = v_s < n_s + 1,$$

thus giving $n_s=[v_s]$, *where* $[x]$ is the largest integer not exceeding x.

2. 2. 2 Social Optimization

If the viewpoint is taken that the expected sum of the net gains accruing to customers in unit time is the public good which should be optimized, we must proceed in a different mode from that in the self-optimization case. The expected total net gain per unit time, P, under strategy n is given by

$$P = \lambda(1-p_n)R-CE[i] \tag{1}$$

$$= \lambda R\frac{1-\rho^n}{1-\rho^{n+1}} - C[\frac{\rho}{1-\rho} - \frac{(n+1)\rho^{n+1}}{1-\rho^{n+1}}] \tag{2}$$

Using the fact that P is unimodal in n, the socially optimal n_o should satisfy

$$\frac{n_o(1-\rho)-\rho(1-\rho^{n_o})}{(1-\rho)^2} \leq \frac{R\mu}{C} < \frac{(n_o+1)(1-\rho)-\rho(1-\rho^{n_o+1})}{(1-\rho)^2}.$$

We define v_o such that

$$\frac{v_o(1-\rho)-\rho(1-\rho^{v_o})}{(1-\rho)^2}=v_s.$$

Letting $n_o=[v_o]$, we have

$$n_o \leq n_s.$$

And this result gives us that the socially optimal cut off level n_o is smaller than n_s.

2. 2. 3 Socially Optimal Toll

Suppose we impose a toll of θ on customers joining the queue. Then their expected net gain is reduced such that n_o is the current criterion of newly arrived customers based on their present comparison of alternatives. The optimality condition now becomes

$$n \leq \frac{(R-\theta)\mu}{C} < n+1.$$

We need to find the optimal toll θ^* which induces n_o. Thus, the optimal relationship for an optimal toll should be

$$R-\frac{C(n_o+1)}{\mu} < \theta^* \leq R-\frac{Cn_o}{\mu}.$$

2. 3 Nash Equilibrium

We consider a game where n players participate. Player i chooses action a_i in A_i(action set of i), and his payoff is denoted by Π^i. A set of strategies $\{a_i^*\}_{i=1}^n$ is a pure strategy Nash equilibrium if and only if, for all a_i in A_i,
$\Pi^i(a_i^*, a_{-i}^*) \geq \Pi^i(a_i^*, a_{-i}^*)$,
for all i, where $a_{-i}^*=(a_1^*,\cdots,a_{i-1}^*, a_{i+1}^*,\cdots,a_n^*)$. In other words, a Nash equilibrium is a set of actions such that no player taking his opponents' actions as given, wishes to change his own action. This definition is, of course, straightforwardly extended to allow mixed strategies by letting $\tilde{A}_{\bar{a}}$(the set of probability distribution over A_i) be player i's strategy set and letting Π^i denote the expectation over the mixed strategies.

3. Base Cases

We consider first two basic cases. The first case is where we have no information on the system state, the other where we have perfect information on the queue size.

3. 1 No Information

When there is no information, an arriving customer will choose to join the system if his net expected reward is nonnegative. There are two cases. Consider first the case of

$$v_s \geq \frac{1}{1-\rho} \tag{3}$$

In this case, if every customer chooses to join the service system, the expected net reward for an arriving customer to join the system is $R-\frac{C}{\mu}(\frac{\rho}{1-\rho}+1)=(v_s-\frac{1}{1-\rho})\frac{C}{\mu}$ which is nonnegative. Therefore, the pure strategy of joining the system constitutes a Nash equilibrium. The expected number of customers in the system and the expected total net gain are given by

$$E[i] = \frac{\rho}{1-\rho}$$

$$P = \lambda R - C\frac{\rho}{1-\rho} = C\rho(v_s - \frac{1}{1-\rho}).$$

Secondly, suppose

$$v_s < \frac{1}{1-\rho} \tag{4}$$

In this case, if every customer chooses to join the service system, the expected net reward for an arriving customer to join the system is $R-\frac{C}{\mu}(\frac{\rho}{1-\rho}+1)=(v_s-\frac{1}{1-\rho})\frac{C}{\mu}$ which is negative and so it is not optimal to join. On the other hand, if every customer chooses not to join the service system, then it is optimal for an arriving customer to join the system. Therefore, there is no pure strategy Nash equilibrium. A mixed strategy for each customer joining the system with probability p will constitute a Nash equilibrium where $0<p<1$ satisfies the following equation:

$$R-\frac{C}{\mu}\frac{1}{1-p\rho}=0.$$

That is,

$$p=\frac{1}{\rho}\frac{v_s-1}{v_s}.$$

The expected number of customers in this system and the expected total net gain is given by

$$E[i]=\frac{p\rho}{1-p\rho}=v_s-1$$

$$P=p\lambda R-C(v_s-1)=0.$$

3. 2 Perfect Information

In this case, the sysem behaves as an $M/M/1/n_s$ queueing system which has been analyzed in [3]. Let $E[i]_s$ and P_s denote the expected number of customers in the system and the expected total net gain under this scenario. And $E[i]_s$ and P_s are given by

$$E[i]_s=\frac{\rho}{1-\rho}-\frac{(n_s+1)\rho^{n_s+1}}{1-\rho^{n_s+1}}$$

$$P_s=\lambda R\frac{1-\rho^{n_s}}{1-\rho^{n_s+1}}-C\left[\frac{1}{1-\rho}-\frac{(n_s+1)\rho^{n_s+1}}{1-\rho^{n_s+1}}\right].$$

Proposition 1 $P_s\geq 0$.

Proof: For notational simplicity, use n for n_s and v for v_s.

$$P_s=\lambda R\frac{1-\rho^{n}}{1-\rho^{n+1}}C\left[\frac{\rho}{1-\rho}-\frac{(n+1)\rho^{n+1}}{1-\rho^{n+1}}\right] \tag{5}$$

$$=\frac{1}{1-\rho^{n+1}}\left[Cn\rho^{n+1}-(\lambda R+C)\rho^{n}-C(\rho^{n-1}+\cdots+\rho)+\lambda R\right] \tag{6}$$

$$\geq\frac{1}{1-\rho^{v+1}}\left[R\mu\rho^{v+1}-(\lambda R+C)\rho^{v}-C(\rho^{n-1}+\cdots+\rho)+\lambda R\right] \tag{7}$$

since P_s is decreasing for $n>n_o$ and $n_s\leq v$. Denoting $R\mu\rho^{v+1}-(\lambda R+C)\rho^{v}-C(\rho^{n-1}+\cdots+\rho)+\lambda R$ as A for now,

$$A=\lambda R\rho^{v}-(\lambda R+C)\rho^{v}-C(\rho^{n-1}+\cdots+\rho)+\lambda R \tag{8}$$

$$=\lambda R-C(\rho^{v}+\rho^{n-1}+\cdots+\rho) \tag{9}$$

$$\geq\lambda R-C(\rho^{n}+\rho^{n-1}+\cdots+\rho) \tag{10}$$

where the last inequality comes from $n \leq v$. Thus, if we show $\lambda R - C\frac{\rho(1-\rho^n)}{1-\rho} \geq 0$, then $P_s \geq 0$. Let

$$g(n) = \frac{1}{1-\rho}\left(\frac{1-\rho}{\rho}\lambda R - C(1-\rho^n)\right) \tag{11}$$

$$\geq \frac{C\rho}{1-\rho}((1-\rho)v - 1 + \rho^v) \tag{12}$$

$$\equiv \frac{C\rho}{1-\rho}h(v). \tag{13}$$

Using the fact that $\ln\rho \leq \rho - 1$, $h(v)$ can be shown to be increasing in v for $v \geq 0$. This implies $A \geq 0$, and thus $P_s \geq 0$.

4. Imperfect Information

We consider an information system represented by a deterministic information η with N distinct signals. Let the state of the system, s, be the number of customers in the service system. Then the state space, S, is given by

$$S = \{0, 1, 2, \cdots\}$$

and η partitions S into subsets S_i, where

$$\eta(s) = y_i$$

for $s \in S_i$, $i = 1, 2, \cdots, N$, and y_i's are distinct signals from the information system. This is called *imperfect* information system, since only y_i's which are coarser than s, are observable.

4. 1 Pure Strategy Equilibrium

Let β denote a customer's strategy.

$$\beta : Y \rightarrow A = \{0, 1\}.$$

For pure strategy, β is given by $\beta(\eta(s)) = 0$ if the customer does not join the service system and $\beta(\eta(s)) = 1$ if the customer joins the service system. And s_0 is defined as follows.

$$s_0 = \min\{s : \beta(\eta(s)) = 0\}.$$

For example, consider the case where $S_1=\{0, 1, 3, 8\}$, $S_2=\{2, 5\}$, $S_3=\{4, 6, 9\}$, $S_4=\{7, 10\}$. And suppose that $\beta(y_1)=1$, $\beta(y_2)=1$, $\beta(y_3)=0$, $\beta(y_4)=0$. Then we can easily verify that $s_0=4$.

A pure strategy is completely characterized by a critical number s_0 because given s_0, the state of the system can never exceed s_0. Define

$$\alpha_i = \min\{s : \eta(s)=y_i\} \quad i=1, 2,\cdots,N$$

$$H = \{\alpha_1, \alpha_2,\cdots,\alpha_N\}.$$

A pure strategy is equivalent to choosing a critical number s_0 from the set H, where s_0 is the number of customers in the service system below which an arriving customer will choose to join the system. We order the α_i's so that they are in ascending order. Let α_{s-1} and α_s be such that $\alpha_{s-1}<n_s$, $\alpha_s\geq n_s$, i. e.

$$\alpha_1 < \cdots < \alpha_{s-1} < \mathrm{n}_s \leq \alpha_s < \alpha_{s+1} < \cdots < \alpha_N.$$

Note that $\alpha_1=0$ and n_s is the critical number for perfect information case. If every customer chooses $s_0\leq\alpha_{s-1}$, an arriving customer can do better by always joining the sytem and in this case there is no pure-strategy Nash equilibrium. If every customer chooses $s_0>\alpha_s$, an arriving customer can do better by choosing his own s_0 to be α_s. So the only possible pure-strategy Nash equilibrium, if it exists, is to set $s_0=\alpha_s$.

The case for $\alpha_s=\mathrm{n}_s$ is trivial because under this condition, $s_0=\alpha_s$ constitutes a Nash equilibrium under condition (14). That is, if every customer chooses $s_0=\alpha_s$, it is also optimal for an arriving customer to choose α_s. This can be verified as follows. It is always optimal to have $\beta(\eta(s))=1$ for $s<\alpha_{s-1}$ and $\beta(\eta(s))=0$ for $s\geq\alpha_s$. A pure-strategy Nash equilibrium exists if it is optimal for the arriving customer to choose $\beta(\eta(s))=1$ given $s\in\{\alpha_{s-1},\cdots, n_s,\cdots, \alpha_s-1\}$.

Since every customer chooses α_s, the system becomes an $M/M/1/\alpha_s$ queueing system. Let the steady-state distribution of the number of customers in the system be π_k. Then it is given by

$$\pi_k = \frac{\rho^k(1-\rho)}{1-\rho^{\alpha_s+1}},\ 0 \leq k \leq \alpha_s$$

$$\sum_{k=\alpha_{s-1}}^{\alpha_s-1}\pi_k = \frac{1-\rho}{1-\rho^{\alpha_s+1}}\sum_{k=\alpha_{s-1}}^{\alpha_s-1}\rho^k.$$

The expected number of customers in the system given the state $s \in \{\alpha_{s-1}, \cdots, n_s, \cdots, \alpha_s - 1\}$, denoted by $E[i|S_{s-1}]$, is given by

$$E[i|S_{s-1}] = \frac{1}{\sum_{k=\alpha_{s-1}}^{\alpha_s-1} \rho^k} \sum_{k=\alpha_{s-1}}^{\alpha_s-1} k\rho^k.$$

So $s_0 = \alpha_s$ constitutes a Nash equilibrium if the expected reward given the state $s \in \{\alpha_{s-1}, \cdots, n_s, \cdots, \alpha_s - 1\}$ is nonnegative, i. e.

$$R - \frac{C}{\mu}(E[i \mid S_{s-1}] + 1) \geq 0,$$

$$E[i|S_{s-1}] \leq v_s - 1,$$

which after simplification reduces to

$$\alpha_{s-1} + \frac{\rho}{1-\rho} - \frac{(\alpha_s - \alpha_{s-1})\rho^{\alpha_s - \alpha_{s-1}}}{1-\rho^{\alpha_s - \alpha_{s-1}}} \leq v_s - 1. \tag{14}$$

However, for a pure-strategy equilibrium, the critical number s_0 cannot be smaller than n_s even under a general deterministic information system. From Naor's paper, the expected net gain is discretely bunimodal and hence decreasing in n for $n \geq n_o$ where $n_o \leq n_s$. Therefore, it is not possible to alleviate congestion externality if the information system induces a pure-strategy equilibrium.

4. 2 Mixed Strategy Equilibrium

We look at the more special information function η which partitions the state space into N subsets with consecutive elements, i. e.

$$S_i = \{\alpha_i, \cdots, \alpha_{i+1} - 1\}, \; i = 1, 2, \cdots, N-1,$$

$$S_N = \{\alpha_N, \cdots\}.$$

Given such an information function, the optimal strategy for a customer must be such that $\beta(\eta(s)) = 1$ for $s < \alpha_{s-1}$ and $\beta(\eta(s)) = 0$ for $s \geq \alpha_s$. Suppose $\beta(\eta(s)) = p$(meaning to join with probability p) for $s \in \{\alpha_{s-1}, \cdots, n_s, \cdots, \alpha_s - 1\}$ where $0 < p < 1$. The system becomes an $M/M/1/\alpha_s$ queueing system with state-dependent arrival rate:

$$\lambda_s = \begin{cases} \lambda & s < \alpha_{s-1} \\ p\lambda & \alpha_{s-1} \leq s < \alpha_s - 1 \\ 0 & s \geq \alpha_s \end{cases}$$

The condition for a Nash equilibrium under such a mixture of pure and mixed strategies is that if there exists p such that

$$R - \frac{C}{\mu}(E[i \mid S_{s-1}, p] + 1) = 0.$$

That is,

$$E[i | S_{s-1}, p] = v_s - 1, \tag{15}$$

where $E[i|S_{s-1}, p]$ denotes the expected number of customers in the system given the state is in S_{s-1} and every customer adopts a mixed strategy of $\beta(\eta(s)) = p$ for $s \in S_{s-1}$.

Obviously any information system of this form is completely characterized by α_{s-1} and α_s. For notational convenience, let $a = \alpha_{s-1}$ and $b = \alpha_s$. The steady state distribution is given by

$$\pi_k = \begin{cases} \rho^k p_0 & 0 \leq k < a \\ \rho^{k-a} \rho^k p_0 & a \leq k \leq b \end{cases}$$

where $p_0 = \{\sum_{k=1}^{a-1} \rho^k + \sum_{k=a}^{b} p^{k-a} p^k\}^{-1}$, and

$$E[i|S_{s-1}, p] = \frac{1}{\sum_{i=a}^{b-1} \pi_i} \sum_{k=a}^{b-1} k\pi_k \tag{16}$$

$$= \frac{1}{\sum_{i=a}^{b-1} \pi_i (p\rho)^i} \sum_{k=a}^{b-1} k(p\rho)^k \tag{17}$$

$$= a + \frac{p\rho}{1-p\rho} - \frac{(b-a)(p\rho)^{b-a}}{1-(p\rho)^{b-a}} \tag{18}$$

Clearly $E[i|S_{s-1}, p]$ is strictly increasing in p. Therefore condition (15) will be satisfied if there is $0<p<1$ and $E[i|S_{s-1}, p] = v_s - 1$, or equivalently,

$$E[i|S_{s-1}, 0] < v_s - 1 < E[i|S_{s-1}, 1]. \tag{19}$$

The constraint above is

$$a < v_s - 1 < a + \frac{\rho}{1-\rho} - \frac{(b-a)p\rho^{b-a}}{1-\rho^{b-a}}. \tag{20}$$

Under the above conditions, the solution is unique since $E[i|S_{s-1}, p]$ is strictly increasing in p. Given a and b which satisfy condition (20), let $\hat{p}$ be the corresponding solution to (15). Then we can solve for π_k. The expected reward rate of the system is given by

$$R(a, b) = (\lambda\sum_{0}^{a-1}\pi_k + \hat{p}\lambda\sum_{a}^{b-1}\pi_k)R \tag{21}$$

$$= \lambda R p_0 (\frac{1-\rho^a}{1-\rho} + \hat{p}\rho^a \frac{1-(\hat{p}\rho)^{b-a}}{1-\hat{p}\rho} \tag{22}$$

and the expected delay cost rate is given by

$$C(a, b) = (\sum_{0}^{a-1} k\pi_k + \sum_{0}^{b-1} k\pi_k)C \tag{23}$$

$$= Cp_0\{\frac{\rho(1-\rho^a)}{(1-\rho)^2} - \frac{a\rho^a}{1-\rho} + \frac{1}{\hat{p}^a}[\frac{a\rho^a(1-(\hat{p}\rho)^{b-a+1})}{1-\hat{p}\rho} \tag{24}$$

$$+ \frac{(\hat{p}\rho)^{a+1}(1-(\hat{p}\rho)^{b-a+1}}{(1-\hat{p}\rho)^2} - \frac{(b-a+1)(\hat{p}\rho)^{b-a+1}}{1-\hat{p}\rho}]\} \tag{25}$$

where $p_0=[\frac{1-\rho^a}{1-\rho} + \rho^a\frac{1-(\hat{p}\rho)^{b-a}}{1-\hat{p}\rho}]^{-1}$.

To optimize expected total net gain of the system, we seek to solve the following problem:

$$\max_{a,\, b \in I+} \{R(a, b)-C(a, b)\}$$

subject to

$$a < v_s - 1 \tag{26}$$

$$a + \frac{\rho}{1-\rho} - \frac{(b-a)\rho^{b-a}}{1-\rho^{b-a}} > v_s - 1 \tag{27}$$

$$b \geq n_s \tag{28}$$

where I_+ is the set of non-negative integers.

In the next section, we consider the simple case where we have binary system. The first case is where a is set to zero. The other case is where $b=\infty$.

5. Binary System

5. 1 Case One

We first consider the case where $a=0$. We set $b \geq n_s$ to control the system. It is always optimal for a customer to have $\beta(\eta(s))=0$ for $s \geq b$. From condition (14), b will induce a pure strategy of $\beta(\eta(s))=1$ for $s<b$ as an equilibrium if b satisfies the following condition:

$$\frac{\rho}{1-\rho} - \frac{b\rho^b}{1-\rho^b} \leq v_s - 1.$$

In this case, we cannot do better than n_s since $b \geq n_s$ and the expected net gain is decreasing in n for $n>n_0$.

On the other hand, if b is such that

$$\frac{\rho}{1-\rho} - \frac{b\rho^b}{1-\rho^b} > v_s - 1,$$

a mixed strategy of $\beta(\eta(s))=p$ for $s<b$ as an equilibrium is induced. Under the mixed strategy, the expected benefit should be zero. But under perfect information, n_s induces non-negative-expected benefit by Proposition 1.

5. 2 Case Two

Here we deal with the case where $b=\infty$. We set $a<n_s$ to control the system. It is always optimal for a customer to have $\beta(\eta(s))=1$ for $s<a$. For $s \geq a$, if the expected net reward conditional on $s \geq a$ is non-negative, then a pure strategy of $\beta(\eta(s))=1$ for $s \geq a$ constitutes a Nash equilibrium. This requires a to satisfy the following condition:

$$a + \frac{\rho}{1-\rho} \leq v_s - 1.$$

In this case, the system becomes an $M/M/1/\infty$ queueing system which must be worse than the case of perfect information(i.e. n_s) since the expected total net gain is decreasing in n for $n>n_0$. However, if a is such that $a+\frac{\rho}{1-\rho}>v_s-1$, a mixed strategy of $\beta(\eta(s))=p$ for $s>a$ will constitute a Nash equilibrium. The equilibrium condition requires p to satisfy the following:

$$a + \frac{p\rho}{1-p\rho} = v_s - 1.$$

That is,

$$p = \frac{1}{\rho}(1 - \frac{1}{v_s - a}).$$

Note that p is strictly decreasing in a. Since p must be such that $0<p<1$, the condition for a mixed strategy equilibrium reduces to

$$a<v_s-1$$

$$a>v_s-1-\frac{\rho}{1-\rho}=v_s-\frac{1}{1-\rho}.$$

Note that when $v_s-\frac{1}{1-\rho}<0$, there is no lower bound for a.

Under such a mixed strategy equilibrium, the expected reward rate $R(a)$ is given by

$$R(a) = R(\lambda\sum_0^{a-1}\pi_k + p\lambda\sum_a^{\infty}\pi_k) \tag{29}$$

$$= p_0\lambda R(\frac{1-\rho^a}{1-\rho} + \frac{p\rho^a}{1-p\rho}) \tag{30}$$

$$= p_0\lambda R(\frac{1-\rho^a}{1-\rho} + \rho^{a-1}(v_s-\mathrm{a}-1)) \tag{31}$$

$$= p_0\lambda R(\frac{1}{1-\rho} - \rho^{a-1}(\mathrm{a} + \frac{1}{1-\rho} - v_s)) \tag{32}$$

where $p_0=\frac{1}{1-\rho}-\rho^a(a+\frac{1}{1-\rho}-v_s)$. And the expected delay cost rate $C(a)$ is given by

$$C(a) = C(\sum_0^{a-1}k\pi_k + \sum_0^{\infty}k\pi_k) \tag{33}$$

$$= Cp_0(\frac{\rho(1-\rho^a)}{(1-\rho)^2} - \frac{a\rho^a}{1-\rho} + \frac{1}{p^a}[\frac{a(p\rho)^a}{1-p\rho} + \frac{(p\rho)^{a+1}}{(1-p\rho)^2}]) \tag{34}$$

$$= Cp_0(\frac{\rho(1-\rho^a)}{(1-\rho)^2} - \frac{a\rho^a}{1-\rho} + a\rho^a(v_s-a)+\rho^a(v_s-a-1)(v_s-a)) \tag{35}$$

$$= Cp_0(\frac{\rho}{(1-\rho)^2} + \rho^a[v_s(v_s-1) - \frac{\rho}{(1-\rho)^2} - \mathrm{a}(v_s + \frac{\rho}{(1-\rho)})]. \tag{36}$$

Therefore the expected total net gain $f(a)$ is given by

$$f(a) = R(a) - C(a) \tag{37}$$

$$= C\frac{\rho}{1-\rho}\frac{B+(a-B)\rho^a}{A-(a-B)\rho^a} \tag{38}$$

where $A=\frac{1}{1-\rho}$ and $B=v_s-\frac{1}{1-\rho}$.

To optimize the expected total net gain, we seek to solve

$$\max_a f(a)$$

subject to

$$v_s - \frac{1}{1-\rho} < a < v_s - 1$$

$$a \in I_+.$$

Proposition 2 If the set $\{n \in I_+ : v_s - \frac{1}{1-\rho} < n < v_s - 1\}$ is not empty, the optimal binary information system under mixed strategy equlilibrium is given by

$$a^* = \begin{cases} n_s - 1 \\ or \\ n_s - 2 \end{cases}$$

Proof: It suffices to show that $f(a)$ is increasing for the feasible region.

$$f'(a) = C\frac{\rho}{1-\rho}\frac{1}{[A-(a-B)\rho^a]^2}\{\rho^a + (a-B)\rho^a \ln\rho)(A-(a-B)\rho^a) \quad (39)$$

$$-(B+(a-B)\rho^a)(-\rho^a - (a-B)\rho^a \ln\rho)\}. \quad (40)$$

$$f'(a) \geq 0 \longleftrightarrow 1 + (a-B)\ln\rho \geq 0 \quad (41)$$

$$\longleftrightarrow a \leq \frac{B\ln\rho - 1}{\ln\rho} \quad (42)$$

$$= \frac{1}{\ln\rho}(v_s \ln\rho - \frac{\ln\rho}{1-\rho} - 1) \quad (43)$$

$$= v_s - \frac{1}{1-\rho} - \frac{1}{\ln\rho} \quad (44)$$

Now, we need to show

$$v_s - 1 \leq v_s - \frac{1}{1-\rho} - \frac{1}{\ln\rho},$$

which is equivalent to

$$\ln\rho \geq 1 - \frac{1}{\rho}.$$

And this is true since $\ln\frac{1}{\rho} \leq \frac{1}{\rho} - 1$ from Taylor expansion of $\ln\frac{1}{\rho}$. Therefore, we know $f'(a) > 0$ for $v_s - \frac{1}{1-\rho} - 1 < a < v_s - 1$, This implies that f(a) is increasing in a for the feasible region and the result follows.

The expected total net gain using such a binary information system is given by

$$P_{a}^{*} = C(\frac{\rho}{1-\rho})\,[\frac{(v_s - \frac{1}{1-\rho})(1-\rho^{a^*}) + a^*\rho^{a^*}}{v_s\rho^{a^*} + \frac{1}{1-\rho}(1-\rho^{a^*}) + a^*\rho^{a^*}}]$$

One question is under what conditions will

$$P_{a^*} \geq P_s$$

be satisfied. We consider the case where $v_s > [v_s]$, that is, v_s is not integers. In this case, $a^* = n_s - 1$. And $P_{a^*} \geq P_s$ is(using v for v_s and n for n_s) equal to

$$\frac{v(1-\rho^{n-1})(1-\rho) - (1-\rho^{n-1}) + (n-1)\rho^{n-1}(1-\rho)}{v\rho^{n-1}(1-\rho) + 1 - \rho^{n-1} + (n-1)\rho^{n-1}(1-\rho)} \geq \frac{v(1-\rho^n) - \frac{1-\rho^n}{1-\rho} + n\rho^n}{\frac{1-\rho^{n+1}}{1-\rho}}.$$

After some simplification, the condition above becomes

$$A(\rho, n)v^2 + B(\rho, n)v + C(\rho, n) \leq 0,$$

where

$$A(\rho, n) = \rho^{n-1}(1-\rho)(1-\rho)^n,$$
$$B(\rho, n) = \rho^{n-1}(-2\rho^{n+1} + 3\rho^n + \rho^2 - n\rho + n - 2),$$
$$C(\rho, n) = \frac{1}{1-\rho}[-2(n-1)\rho^{n-1} + (3n-1)\rho^n - (n+1)\rho^{n+1} + (n^2-n-2)\rho^{2n-1} + (-2n^2+3n+1)\rho^{2n} + (n-1)^2\rho^{2n+1}].$$

(45)

for ρ near 1, we can characterize more specifically the region where $P_{a^*} \geq P_s$ is satisfied.

Proposition 3 $A_s\ \rho \longrightarrow 1$, $P_{a^*} \geq P_s$ *for* $n > 4$ *and* $v_s > [v_s]$.

Proof: It is easy to see that as $\rho \longrightarrow 1$

$$A(\rho, n) \longrightarrow 0,$$
$$B(\rho, n) \longrightarrow 0.$$

Now consider $C(\rho, n)$. Define

$$D(\rho, n) = \frac{(1-\rho)C(\rho, n)}{\rho^{n-1}}.$$

Then we can show that $D(1, n)=0$, $D'(1, n)=0$, and $D''(1, n)=0$. The third derivative at $\rho=1$ is

$$D'''(1, n) = (n+1)(n-1)n(-n^2+5n-3)$$

and it is negative for $n>4$. Therefore, $C(\rho, n)$ goes to $-\infty$ as $\rho \longrightarrow 1$ for $n>4$.

This conclusion is shown in the numerical example in the next section. And this proposition tells us that under heavy traffic conditions we can construct a binary information system that is better than a perfect information sytem.

6. Numerical Example

We use a computer program to calculate the expected total net gain and the expected number of customers in the system under three scenarios:

1. System under perfect information specified by n_s;
2. System under imperfect binary information with mixed strategy specified by a^*; and
3. Socially optimal system specified by n_o as in [3].

The results of calculation are in the following table. It can be seen that when v_s and ρ are large, the binary information system which induces mixed strategy equilibrium achieves a higher expected total net gain than that corresponding to perfect information. This outcome can be predicted from Proposition 3. For example, for $v_s \geq 6$ and $\rho \geq 0.8$, P_{a^*} is consistently larger than P_s. In the table, the first two numbers in each cell are P_s and P_{a^*} respectively and the third one is the socially optimal expected total net gain. And *nf* means *not feasible*. In the example, C (waiting cost per unit time) was taken to be 100.

ρ / v_S	0.5	0.6	0.7	0.8	0.9
7	252,nf,255	279,277,289	293,292,317	293,297,340	282,294,360
7.5	277,276,279	309,309,317	327,328,347	331,335,373	323,331,395
8	301,nf,303	336,335,345	354,354,379	355,360,407	339,353,431
8.5	326,326,328	366,366,373	389,390,411	393,397,441	380,389,466
9	351,nf,352	394,393,402	418,418,443	419,424,476	397,412,501
9.5	376,375,377	424,424,431	452,453,475	457,461,511	439,448,539
10	400,nf,402	453,452,460	483,483,507	485,490,546	457,473,577
10.5	425,425,426	482,483,489	518,518,540	524,527,581	500,509,615

7. Conclusion and Further Studies

We can summarize our results as follows. Under a deterministic information function, pure strategy cannot alleviate the congestion externality. A binary information function sometimes alleviates the congestion if a mixed strategy is feasible, and if we supply partial characterization on the region where we achieve such an alleviation. We also offer an example which supports this. When we can control the information, we can be better off in social benefit criterion by sending imperfect information (e.g. binary) rather than complete information. This has practical implications since there are many cases where we can only offer rough information on a system state. For example, when we consider a road traffic system where the system state denotes the number of cars on the road, it is impossible to give the exact number of cars. Instead, we can use a few signals such as congested, moderate, or alright according to the rough numbers of cars on the road. We can also control the information by using radio broadcasting for traffic information.

For further research, we need to get more explicit conditions in terms of v_s and ρ under which $P_{a^*} > P_{ns^*}$. The more general tertiary information function will also be analyzed. Clearly it cannot be worse than the binary one but we did not show whether the fertiary system would be strictly better. We should also note that the tertiary information function is the most general form in terms of expected benefit.

References

[1] Gross, D. and Harris, C.M. *Fundamentals of Queueing Theory*, John Wiley and Sons, 1985

[2] Kleinrock, L. *Queueing Systems*, Vol. I, John Wiley and Sons, 1975

[3] Naor, P. The Regulation of Queue Size By Levying Tolls, *Econometrica*, Vol. 37, No. 1(1969)

[4] Tirole, J. *The Theory of Industrial Organization*, The MIT Press, 1988