The Pure and Applied Mathematics 2 (1995), No 2, pp. 103–110
J. Korea Soc. of Math. Edu. (Series B)

FEYNMAN–KAC FUNCTIONALS ASSOCIATED WITH REGULAR DIRICHLET FORM

KI SEONG CHOI

1. Introduction

In their recent paper[2], they show that the existence theory for the analytic operator-valued Feynman path integral can be extended by making use of recent developments in the theory of Dirichlet forms and Markov process. In this field, there is the necessity of studying certain generalized functionals of the process (of Feynman-Kac type). Their study have been concerned with Feynman-Kac type functionals related with smooth measures associated with the classical Dirichlet form (associated with the Laplacian).

In this paper, I will initiate the study of properties of Feynman-Kac type functional associated with general (regular) Dirichlet form.

We consider a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ where X is a locally compact separable metric space and m is a positive Radon measure on X with $\text{supp}[m] = X$. Let $M = (\Omega, X_t, \zeta, P_x)$ be a Hunt process on X which is m-symmetric and associated with $(\mathcal{E}, \mathcal{F})$.

A function $A : [0, \infty) \times \Omega \to [-\infty, \infty]$ is said to be an AF (*additive functional*) if

(1) $A_t(\cdot)$is F_t-measurable, where F_t is the smallest completed σ-algebra which contains $\sigma\{X_s : s \leq t\}$;

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(2) there exist a defining set $\Lambda \in F_\infty$ and an exceptional set $N \subset X$ with $\mathrm{Cap}(N) = 0$ such that $P_x(\Lambda) = 1$ for all $x \in X - N$, $\theta_t\Lambda \subset \Lambda$ for all $t > 0$ (θ_t denotes the shift operator on Ω) and for each $\omega \in \Lambda$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_\cdot(\omega)$ is right contiuous and has left limit, $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

An additive functional A is called PCAF (*positive continuous* AF) if A is an additive functional and $A_\cdot(\omega)$ is non-negative and continuous function for each ω in its defining set Λ. Given a PCAF A, there exists a unique Borel measure μ on X, which is called the *Revuz measure of* A such that

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}[\int_0^t (f(X_s)\,dA_s] = \langle f\cdot\mu, h\rangle := \int_X h(x)(f\cdot\mu)\,(dx)$$

for all γ-excessive functions h and $f \in \mathcal{B}^+$ ($\mathcal{B}^+$ denotes all non-negative Borel functions on $X, \gamma \geq 0$ is a constant).

Denote by S the totality of the associated Revuz measures of PCAF's. The elements in S are called smooth measures. A simple analytical description of S has been given in [7]. For a given smooth measure μ, we denote by A^μ the unique (up to equivalence) positive continuous additive functional such that μ is the Revuz measure of A^μ. For a signed Borel measure $\mu = \mu^+ - \mu^-$, we write $\mu \in S - S$ if $\mu^+ \in S$ and $\mu^- \in S$. Also we denote $A^{\mu^+} - A^{\mu^-}$ by A^μ.

Let us introduce the notation

$$P_t^\mu f(x) = E_x[e^{-A_t^\mu} f(X_t)]$$

provided the right hand side makes sense. Notice that $(P_t^\mu)_{t>0}$ is the so called Feynman-Kac functional. If μ is the smooth measure in Kato class for the Dirichlet form associated with the Laplacian, it is realized in [6] that $(P_t^\mu)_{t>0}$ is a strongly continuous semigroup on $L^2(\mathbb{R}^d)$. In [4], they show that if $\mu \in S - S_{k_0}$, then $(P_t^\mu)_{t>0}$ is a strongly continuous symmetric semigroup on $L^2(m)$.

In this paper, I will show that $(P_t^\mu)_{t>0}$ is a strongly continuous symmetric semigroup on $L^2(m)$ for μ which belongs to the larger class (where $\mu \in C(\mu^+)$) than $S - S_{k_0}$.

2. Symmetric strongly continuous semigroup $(P_t^\mu)_{t>0}$ on $L^2(X,m)$

Throughout this paper, $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(X,m)$, where X is a locally compact separable metric space. Let us denote by $\mathcal{B}(X)$ the family of Borel functions on X. For $f \in \mathcal{B}(X)$, set

$$\|f\|_q = \inf_{Cap(N)=0} \sup_{x\in X-N} |f(x)| \ .$$

Definition 1. *A smooth measure μ is said to be in Kato class ($\mu \in S_k$ in notation) , if*

$$\lim_{t\downarrow 0} \|E.A_t^\mu\|_q = 0$$

Note 1. *Let $(\mathcal{E},\mathcal{F})$ be the classical Dirichlet form generated by Brownian motion (X_t) on $\mathbb{R}^d$. In this case, S_k coincides with the generalized Kato class introduced in [6].*

Let S_0 be the totality of the (positive Radon) measures of finite energy integral. Let us introduce the family S_{k_0} as follows.

$$S_{k_0} = \{\mu \in S_k \cap S_0 \ : \ \mu(X) < \infty\}$$

Theorem 1. *A positive Borel measure μ on X is smooth if and only if there exists an increasing seguence $\{F_n\}_{n\geqslant 1}$ of compact sets satisfying the following properties.*

(1) $I_{F_n} \cdot \mu \in S_{k_0} \quad \forall n \geqslant 1$
(2) $\mu(X - \cup F_n) = 0$
(3) $\lim_{n\to\infty} Cap(K - F_n) = 0$ *for all compact set* K.

Proof. See [3]Theorem 2.4.

For $\alpha \geq 0$, μ and v in $S-S$, $f \in \mathcal{B}(X)$, we set

$$U_v^{\alpha+\mu} f(x) = E_x[\int_0^\infty e^{-\alpha t - A_t^\mu} f(X_t)\, dA_t^v]$$

provided the right-hand side makes sense. When $v = m$, we simply write $U^{\alpha+\mu} f$ for $U_v^{\alpha+\mu} f$.

Theorem 2. *Let $\mu, v \in S$, Then the following two statements are equivalent*

(1) $inf_{\alpha>0} \|U_v^{\mu+\alpha} 1\|_q < \infty$ *and* $inf_{\alpha>0,\ n\geq 1} \|(U_v^{\mu+\alpha})^n 1\|_q < 1$
(1 stands for the constant function with value 1)

(2) *There exists a set N with $Cap(N) = 0$ such that for all $0 < T < \infty$,*

$$\sup_{t\leq T} \sup_{x\in X-N} E_x[e^{-A_t^\mu + A_t^v} + \int_0^t e^{-A_s^\mu + A_s^v} d(A_s^\mu + A_s^v)] < \infty$$

Proof. see [1].

Definition2. *For a given $\mu \in S$, smooth measure v is compatible with μ if v satisfies one of conditions in Theorem 2. The set of all smooth measures which are compatible with μ will be denoted by $C(\mu)$*

Theorem 3. *For any $\mu \in S$, $S_k \subset C(\mu)$.*

Proof. Let O be the vanishing measure on X. If $v \in S_k$, then

$$\inf_{\alpha>0} \|U_v^{o+\alpha} 1\|_q = 0$$

Thus we have $v \in C(O) \subset C(\mu)$

Note 2. *An example which does not belong to Kato class but belong to $C(\mu)$ was given in [5] appendix 1.*

Lemma 1. *Let $\mu \in S - S$ be such that $\mu^- \in C(\mu^+)$. Then there exists constant c and β such that*

$$\|E.e^{-A_t^\mu}\|_q \leq ce^{\beta t}.$$

Proof. Since $\mu^- \in C(\mu^+)$, by Theorem 2 (2), there exists a constant $\lambda \geq 1$ such that

$$\sup_{0\leq t\leq 1} \|E.e^{-A_t^\mu}\|_q \leq \lambda < \infty$$

For $t = n + s$, n being a natural number and $0 \leq s \leq 1$,

$$E_x e^{-A_t^\mu} = E_x[e^{-A_n^\mu} E_{X_n} e^{-A_s^\mu}] \leq \lambda^{n+1}, q.e. \;\; x \in X$$

puting $c = \lambda$ and $\beta = log\lambda$,

$$\|E.e^{-A_t^\mu}\|_q \leq ce^{\beta t}$$

From now on, let us use the short notation $L^2(\mu)$ for $L^2(X, \mu)$. For $\mu \in S$, we set

$$Q_\mu(f,g) = \int_X f(x) \cdot g(x)\mu(dx), \quad \forall f,g \in L^2(|\mu| + m)$$

one can show that $L^2(|\mu| + m)$ is dense in $L^2(m)$. Hence Q_μ is a quadratic form on $L^2(m)$. We put

$$\mathcal{E}^\mu(f,g) = \mathcal{E}(f,g) + Q_\mu(f,g), \quad \forall f,g \in \mathcal{F}^\mu$$

where

$$\mathcal{F}^\mu = \mathcal{F} \cap L^2(|\mu| + m)$$

Theorem 4. *Let $\mu \in S - S$. Then the following assertions are equivalent to each other.*

(1) $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ *is lower semibounded*

(2) $(P_t^\mu)_{t>0}$ *is a strongly continuous semigroup on* $L^2(X, m)$

(3) *There exists constants c and β such that*

$$\|P_t^\mu f\|_{L^2(m)} \leq ce^{\beta t}\|f\|_{L^2(m)}, \ \forall f \in L^2(m)$$

(4) *There exists $\alpha > 0$ such that*

$$U^{\alpha+\mu}(L^2(m)) \subset L^2(m)$$

(5) Q_{μ^-} *is relatively form bounded with respect to* $(\mathcal{E}^{\mu^+}, \mathcal{F}^{\mu^+})$ *with bound* ≤ 1

If any one of the above conditions holds, then P_t^μ is a bounded symmetric operator on $L^2(m)$ for each $t > 0$.

Proof. See [4]. Theorem 4.1

Theorem 5. *Let $\mu \in S - S$ be such that $\mu^- \in C(\mu^+)$. Then P_t^μ is a symmetric bounded operator on $L^2(m)$ for each $t > 0$, i.e. there exist constants c and β such that*

$$\|P_t^\mu f\|_{L^2(m)} \leq ce^{\beta t}\|f\|_{L^2(m)}$$

Proof. By Theorem 1 , there exists an increasing sequence $\{F_n\}_{n\geq 1}$ of compact sets such that $I_{F_n} \cdot \mu^- \in S_{k_0}, \mu(X - \cup F_n) = 0$ and $\lim_{n\to\infty} Cap(K - F_n) = 0$ for any compact set K . Let us put $\mu_n = \mu^+ - I_{F_n} \cdot \mu^-$ and $f_n = (|f| \wedge n) I_{F_n}$ for an arbitrary function f on X. $I_{F_n} \cdot \mu^- \in C(2\mu^+)$. By Lemma 1 , there exists constants c and β such that $\|E.e^{-2A_t^\mu}\|_q \leq c^2 e^{2\beta t}$. We have

$$|P_t^\mu f_n|^2 \leq (E.e^{-2A_t^\mu})(E.|f_n|^2) \leq c^2 e^{2\beta t} E.|f_n|^2 \quad q.e. \ \ x \in X$$

$$\|P_t^{\mu} f_n\|_{L^2(m)} \leq ce^{\beta t}\|f_n\|_{L^2(m)} \leq ce^{\beta t}\|f\|_{L^2(m)}$$

By the dominated convergence theorem,

$$\|P_t^{\mu} f\|_{L^2(m)} \leq ce^{\beta t}\|f\|_{L^2(m)}.$$

Theorem 6. *Let $\mu \in S - S$ be such that $\mu^- \in C(\mu^+)$. Then $(P_t^{\mu})_{t>0}$ is a strongly continuous symmetric semigroup on $L^2(m)$*

Proof. $\mu^- \in C(\mu^+)$ implies $\lambda\mu^- \in C(\mu^+)$ for some $\lambda > 1$ (Theorem 2 (1)).

Since $\lambda\mu^- \in C(\mu^+)$. By Theorem 5, there exist constants c_1 and β_1 such that $\|P_t^{(\mu^+-\lambda\mu^-)} f\|_{L^2(m)} \leq c_1 e^{\beta_1 t}\|f\|_{L^2(m)}$. By Theorem 4, Q_{μ^-} is relatively form bounded with respect to $(\mathcal{E}^{\mu^+}, \mathcal{F}^{\mu^+})$ with bound $\leq \lambda^{-1} < 1$. $(\mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ is bounded from below and closed by the KLMN Theorem [10]. By Theorem 4, $(P_t^{\mu})_{t>0}$ is a strongly continuous symmetric semigroup on $L^2(m)$.

REFERENCES

1. Albeverio, S. Blanbchard, Ph. and Ma, Zhi-Ming, *Singular Schrödinger operators associated with Feynman-Kac semigroups*, Proc. "Random Partial Differential Equations", International Series of Numerical Math. **102, Birkhäuser** (1991).
2. Albeverio, S. Johnson, G.W., Ma, Zhi-Ming, *The analytic operator-valued Feynman intrgral via additive functionals of Brownian motion*, SFB-237 preprint (1993).
3. Albeverio, S. and Ma, Zhi-Ming, *Additive functional, nowhere Radon and Kato class smooth measures associated with Dirichlet forms*, Osaka J. Math. **29** (1991), 247-265.
4. Albeverio, S. and Ma, Zhi-Ming, *Perturbation of Dirichlet forms*, J. Funct. Anal. **99** (1991), 332-356.
5. Aizenman, M and Simon, B., *Brownian motion and Harnack's inequality for Schrödinger operators,*, Commum. Pure Appl. Math. **35** (1982), 209-273.
6. Blanchard, Ph. and Ma, Zhi-Ming, *Semigroup of Schrödinger operators with potentials given by Radon measures, in "Stochastic Processes-Physics and Geometry"*, Edt. S. Albeverio, G. Casati, U. Cattaneo, D. Merlini, R. Moresi, World Scient., Singapore (1989).

7. Fukushima, *M.*, *"Dirichlet forms and Markov processes, "North Holland and Kodansha.*
8. Johnson, G.W., *Existence Theorems for the Analytic Operator-valued Feynman Integral*, Monograph, U. of Sherbrooke Series **20** (1988).
9. Kato, *T.*, *Perturbation Theory for Linear Operators, 2nd ed.*, Springer Verlag, Berlin (1976).
10. Reed, M. and Simon, *B.*, *Methods of Modern mathematical Physics, Fourier Analysis, Self-Adjointness*, Academic Press, New York **II** (1975).

Departments of Mathematics
Keonyang University
Nonsan Chung-Nam
320-800, Korea