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A NOTE ON NOETHERIAN AND ARTINIAN BCK–ALGEBRAS

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ABSTRACT. In this paper we introduce the notion of Artinian and obtain some properties of Artinian and Noetherian BCK-algebras.

1. Introduction.

The notion of ideals in BCK-algebras was introduced by K. Iséki ([3]) in 1975. The ideal theory plays a fundamental role for the general development of BCK-algebras ([2,4]). The concepts of ideals, quotient algebras, and homomorphisms are all closely related to each other. In 1977, J.Ahsan ([1]) initiated to study the decomposition properties of BCK-algebras, and some further results were obtained by M. Palasinski ([7]) in 1982. Let us recall definitions and theorems. We mainly refer to the first book on BCK-algebras ([6]).

Definition 1.1. Let $(X;*,0)$ be a BCK-algebra and I be a non-empty subset of X. Then I is called an *ideal* of X if, for all x, y in X,
(a) $0 \in I$
(b) $x * y \in I$ and $y \in I$ imply $x \in I$. Obviously, $\{0\}$ and X are ideals of X. We say X a trivial ideal. An ideal I is *proper* if $I \neq X$.

Theorem 1.2. Any ideal of a BCK-algebra X is a subalgebra of X.

Theorem 1.3. If I and J are ideals of a BCK-algebra X and $I \subset J$, then
(a) I is also an ideal of the subalgebra J,
(b) J/I as the quotient of the subalgebra J via the ideal I is an ideal of X/I.

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The set of all ideals on X is denoted by $\mathcal{I}(X)$ and the set of all ideals containing I on X is denoted by $\mathcal{I}(X, I)$. A mapping f from $\mathcal{I}(X, I)$ to $\mathcal{I}(X/I)$ is defined by, for any $J \in \mathcal{I}(X, I)$, $f(J) = J/I$.

Theorem 1.4. If I is an ideal of a BCK-algebra X, then there is a bijection from $\mathcal{I}(X, I)$ onto $\mathcal{I}(X/I)$.

Suppose I is an ideal of X. For any $x, y \in X$, we define $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. Then it is easy to show $\sim$ is an equivalence relation on X. We denote the equivalence class containing x by C_x. The mapping ν from X to X/I is defined by $\nu(x) = C_x$ for all x in X, obviously $\nu(x * y) = \nu(x) * \nu(y)$. This says ν is a homomorphism, called the *natural* homomorphism.

Theorem 1.5. If A is an ideal of a BCK-algebra X/I, then $\nu^{-1}(A)$ is an ideal of X and $I \subseteq \nu^{-1}(A)$.

Definition 1.6. Given a BCK-algebra X, we say that X satisfies the *maximal condition* if each non-empty subset of $\mathcal{I}(X)$ contains at least one maximal member with respect to the set theoretic inclusion $\subseteq$. We say X satisfies the *ascending chain condition*, abbreviated by ACC, if there does not exist an infinite properly ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ in $\mathcal{I}(X)$.

In an entirely analogous way the minimal condition and the descending chain condition (abbreviated by DCC) are defined.

Theorem 1.7. Let X be a BCK-algebra. Then
(a) X satisfies the maximal condition if and only if X satisfies ACC,
(b) X satisfies the minimal condition if and only if X satisfies DCC.

Theorem 1.8. Suppose I is an ideal of a BCK-algebra X. Then X satisfies ACC if and only if the quotient algebra X/I and I satisfy ACC.

Definition 1.9. A BCK-algebra X is said to be *Noetherian* if each ideal of X is finitely generated.

In the following theorem we give some characterizations of Noetherian algebras.

Theorem 1.10. In a BCK-algebra X, the following are equivalent :
(a) X is Noetherian,
(b) X satisfies ACC,
(c) X satisfies the maximal condition.

Definition 1.11. A BCK-algebra X is said to be *Artinian* if X satisfies DCC.

Corollary 1.12. In a BCK-algebra X, the following are equivalent :
(a) X is Artinian,
(b) X satisfies the minimal condition.

Proof. This is immediate from Theorem 1.7 and the definition of Artinian.

2. Main Results.

In this section we obtain an exact anolog of ACC and study Noetherian BCK-algebras with related to principal ideal.

Theorem 2.1. Suppose I is an ideal of a BCK-algebra X. Then X satisfies DCC if and only if the quotient algebra X/I and I satisfy DCC.

Proof. Let $\nu : X \to X/I$ be the natural homomorphism of BCK-algebras. If $I_1 \supseteq I_2 \supseteq \cdots$ is any descending chain $\mathcal{I}(X)$, then $I_1 \cap I \supseteq I_2 \cap I \supseteq \cdots$ and $\nu(I_1) \supseteq \nu(I_2) \supseteq \cdots$ are descending chains in $\mathcal{I}(X)$ and $\mathcal{I}(X/I)$ respectively. Hence there exist natural numbers m_1 and m_2 such that $I_{m_1} \cap I = I_i \cap I$ and $\nu(I_{m_2}) = \nu(I_j)$ whenever $m_1 \leq i$ and $m_2 \leq j$ by Theorem 1.7(b). Assume $m_0 := max\{m_1, m_2\}$. For $i \geq m_0$ and $x \in I_{m_0}$, we have

$$C_x = \nu(x) \in \nu(I_{m_0}) = \nu(I_i).$$

This means that there is $y \in I_i$ such that $C_x = C_y$, it follows that $C_x * C_y = C_0$, i.e., $x * y \in I$. Since $x \in I_{m_0}$ and $x * y \leq x$, we have $x * y \in I_{m_0}$. Hence $x * y \in I_{m_0} \cap I$, and so $x * y \in I_i$. Combining $y \in I_i$ we obtain $x \in I_i$. This implies $I_{m_0} \subseteq I_i$. The opposite inclusion is trivial. Consequently $I_{m_0} = I_i$, and hence X satisfies DCC.

Conversely, if $I_1 \supseteq I_2 \supseteq \cdots$ is a descending chain in $\mathcal{I}(X/I)$, then $\nu^{-1}(I_1) \supseteq \nu^{-2}(I_2) \supseteq \cdots$ is a descending chain in $\mathcal{I}(X)$. Since X satisfies DCC, there is a natural number n_0 such that $\nu^{-1}(I_{n_0}) = \nu^{-1}(I_i)$ whenever $i \geq n_0$. Hence we have $I_{n_0} = I_i$ whenever $i \geq n_0$. This means that X/I satisfies DCC. It is easy to check that I satisfies DCC. This proves the theorem.

Proposition 2.2. Given two BCK-algebras X, Y, if $f : X \to Y$ is an epimorphism and X is Noetherian(Artinian), then so is Y.

Proof. By Homomorphism Theorem (see [6, p.122]), $X/Ker(f) \cong Y$. By Theorem 1.3, every ideal of $X/Ker(f)$ is of the form $I/Ker(f)$ where I is an ideal of X with $Ker(f) \subseteq I$. Take any ascending chain of ideals in $Y \cong X/Ker(f)$ as follows:

$$I_0/Ker(f) \subseteq I_1/Ker(f) \subseteq \cdots$$

Then $Ker(f) \subseteq I_0 \subseteq I_1 \subseteq \cdots$ is an ascending chain of ideals in X. Since X is Noetherian, we have $I_n = I_{n+1} = \cdots$ for some natural number n. Hence we obtain $I_n/Ker(f) = I_{n+1}/Ker(f) = \cdots$. Therefore $X/Ker(f) \cong Y$ is Noetherian. Similar arguments can be applied to the Artinian case.

Proposition 2.3. If a BCK-algebra X is Noetherian(Artinian), then any subalgebra S of X is also Noetherian(Artinian).

Definition 2.4. Suppose that X is a BCK-algebra. An ideal I is said to be *principal* if there exists $a \in X$ such that $I = \{x \in X : x \leq a\}$. The set of all principal ideals is denoted by $\mathcal{PI}(X)$.

Definition 2.5. A BCK-algebra X is *principal* if every ideal of X is principal.

Proposition 2.6. If a BCK-algebra X is principal, then X is Noetherian.

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be any ascending chain of $\mathcal{I}(X)$. Then $I := \cup\{I_i\}$ is an ideal of X. Since X is principal, there exists $a \in X$ such that $I = \{x \in X : x \leq a\} = (a]$. Thus $a \in I = \cup I_i$ and so $a \in I_n$ for some n. Hence $I \subseteq I_n$. For any natural number $j \geq n$, we have

$$(a] \subseteq I_n \subseteq I_j \subset I$$

Therefore $I_j = I_n$ for any natural number $j \geq n$. This complets the proof.

A proper ideal I of a BCK-algebra X is said to be *irreducible* if $I = A \cap B$ for some $A, B \in \mathcal{I}(X)$ implies $I = A$ or $I = B$. J. Ahsan ([1]) introduced the notion of decomposition properties, and M Palasinski ([7]) obtained further results.

Definition 2.7. An ideal I of a BCK-algebra X has an *irreducible decomposition* if I can be represented as an intersection of a finite number of irreducible ideals of X.

Lemma 2.8. If a BCK-algebra X is Noetherian, then each of ideal of X has an irreducible decomposition.

By applying Proposition 2.6 we obtain the following theorem:

Theorem 2.9. If a BCK-algebra X is principal, then each ideal of X has an irreducible decomposition.

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