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A NOTE ON THE OSCILLATION CRITERIA OF SOLUTIONS TO SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

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1.Introduction

Consider a solution $y(t)$ of the nonlinear equation

$$(E)\ y'' + f(t,y) = 0.$$

A solution $y(t)$ is said to be oscillatory if for every $T > 0$ there exists $t_0 > T$ such that $y(t_0) = 0$. Let F be the class of solutions of (E) which are indefinitely continuable to the right, i.e. $y \in F$ implies $y(t)$ exists as a solution to (E) on some interval of the form $[t_y, \infty)$.

Equation (E) is said to be oscillatory if each solution from F is oscillatory. If no solutoin in F is oscillatory, equation (E) is said to be nonoscillatory. In [2], F.V.Atkinson treated the case $f(t,x) = a(t)x^{2n-1}$, with $a(t)$ continuous and positive, and $n > 1$. For this particular equation, he showed that (E) is oscillatory if and only if

$$\int_0^\infty ta(t)dt = \infty.$$

A number of authers have related the oscillation properties of second-order differntial equation to the oscillation of solutions of third -order differential equation.

We extend the results of Atkinson, also we are concerned with the nonoscillation of solution of (E).

2. The case for the oscillation

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Theorem 2.1. Let $f(t,x)$ be continuous on $S=[0,\infty)\times(-\infty,\infty)$, with $b(t)\psi(x)\geq f(t,x)\geq a(t)\phi(x)$ for (t,x) in S, where

a) a(t) and b(t) are nonnegative locally integrable function,

b) $\phi(x)$ and $\psi(x)$ are nondecreasing, with $x\phi(x)>0$ and $x\psi(x)>0$ for $x\neq 0$, on $(-\infty,\infty)$,

and for some $\alpha\geq 0$,

$$\int_a^\infty [\phi(\mu)]^{-1}du<\infty\ ,\ \int_{-a}^{-\infty}[\psi(\mu)]^{-1}du<\infty.$$

Then equation (E) is oscillatory if and only if

$$\int_0^\infty ta(t)dt=\int_0^\infty tb(t)dt=\infty.$$

Proof. Suppose that (E) has a nonoscillatory solution $y(t)$, from the class F, say $y(t)>0$ for $t>T$. Then, for$T\leq s\leq t$,

$$y'(t)-y'(s)=-\int_s^t f(u,y(u))du\leq 0 \tag{1}$$

and so $y'(t)$ is nonincreasing. This, and the fact that $y(t)>0$, implies that $\lim_{t\to\infty}y'(t)=L$ exists, where $0\leq L\leq\infty$. Let $t\to\infty$ in (1) to get, for $s\geq T$,

$$y'(s)=L+\int_s^\infty f(u,y(u))du\geq\int_s^\infty f(u,y(u))du\geq 0.$$

Integrating from s=T to s=t, we obtain

$$y(t)\geq y(t)-y(T)\geq\int_T^s\int_s^\infty f(u,y(u))duds\geq\int_T^s(u-T)f(u,y(u))du,$$

which implies that

$$y(t)\geq\int_T^t(u-T)\alpha(u)\phi(y(u))du. \tag{2}$$

From the monotonicity of ϕ, we have

$$\phi(y(t))[\phi(\int_T^s (u-T)a(u)\phi(y(u))du]^{-1} \geq 1,$$

and if we multipliy by $(t-T)a(t)$ and integrate from γ to s, we get, after a change of variable on the left,

$$\int_L^U [\phi(u)^{-1}]du \geq \int_r^s (t-T)a(t)dt \tag{3}$$

where $L = \int_T^r (u-T)a(u)\phi(y(u))du, \ U = \int_T^s (u-T)a(u)\phi(y(u))du.$

If, by an appropriate choice of γ , we can make $L \geq \alpha$, then the left hand side of (3) is bounded above for all $s > \gamma$, hence $\int_0^\infty ta(t)dt \geq \infty$.

If this is not possible, then for all $\gamma \geq T$,

$$\alpha > \int_T^\gamma (u-T)a(u)\phi(y(u))du \geq \phi(y(T)) \int_T^\gamma (u-T)a(u)du,$$

and the result again follows.

For the case when $y(t) < 0$ for $t \geq T$, the procedure is the same, the charges in detail being that $y'(t)$ goes to a finite nonpositive limit, the inequalities (1) through (2) are reserved while the inequality (3) is in the same direction, and $a(t)$ and $\phi(x)$ are replaced in (2) and (3) by $b(t)$ and $\psi(x)$.

To prove the other half of the theorem, we must show that if either $\int_0^\infty ta(t)dt < \infty$ or $\int_0^\infty tb(t)dt < \infty$, then (E) has a nonoscillatory solution. Suppose $\int_0^\infty tb(t)dt < \infty$,and consider the equation

$$y(t) = 1 - \int_t^\infty (s-t)f(s, y(s))ds. \tag{4}$$

If (4) has a nonnegative continuous solution on some interval $[T, \infty)$, it is clear that

$$y''(t) + f(t, y(t)) = 0 \ \textit{ on this interval.}$$

Also, since the improper integral in (4) would converge, we would $\lim_{t\to\infty} y(t) = 1$, that is, $y(t)$ would be nonoscillatory.

Let a positive integer T be chosen such that $\psi(1)\int_T^\infty sb(s)ds \geq \frac{1}{2}$.

We define, for N a positive integer, $N \geq T$, $y_N(t)$ =1 for $t \geq N$

$$y_N(t) = 1 - \int_{t+\frac{1}{N}}^{\infty} (s - t - \frac{1}{N})f(s, y_{N(s)})ds \text{ for } T \leq t \leq N.$$

This formula defines $y_N(t)$ succesivley on the intervals $[N - k/N, N - (k-1)/N]$ for $k = 1, 2, \ldots, N(N-T)$; hence $y_N(t)$ is defined on $[T, \infty)$. For $N - 1/N \leq t \leq \infty$, we have

$$0 \leq \int_{t+\frac{1}{N}}^{\infty} (s - t - 1/N)f(s, y_N(s))ds \leq \phi(1)\int_T^\infty sb(s)ds \leq \frac{1}{2}$$

hence $1/2 \leq y_N(t) \leq 1$ on this interval. Any easy induction then shows that $1/2 \leq y'_N(t) \leq 1$ on the interval $[T, \infty)$.

Consequently, for $t \geq T$,

$$|y'_N(t)| = \int_{t+\frac{1}{N}}^{\infty} f(s, y_N(s))ds \leq \phi(1)\int_{t+\frac{1}{N}}^{\infty} b(s)ds \leq \frac{1}{2}.$$

Since the family $\{y_N(t)\}$ is equicontinuous and uniformly bounded on $[T, \infty)$, we extract an uniformly convergent subsequence $\{y_k(t)\}$, $\lim_{k\to\infty} y_k(t) = y(t)$.

We now choose any large real number B, and write

$$y_k(t) = 1 - \int_{t+(\frac{1}{N(k)})}^{B} \left(s - t - \frac{1}{N(k)}\right) f(s, y_k(s))ds + \varepsilon(k, B),$$

where $|\varepsilon(k, B)| \leq \phi(1)\int_B^\infty sb(s)ds$.

If we let $k \to \infty$, we have

$$\lim_{k\to\infty} inf\ \varepsilon(k, B) \leq y(t) - 1 + \int_t^B (s-t)f(s, y(s))ds \leq \lim_{k\to\infty} sup\ \varepsilon(k, B).$$

If we now let $B \to \infty$, it is clear from the above bound on $\varepsilon(k, B)$ that the liminf and limsup terms go to zero, and so $y(t)$ satisfies equation(4).

For the case, $\int_0^\infty ta(t)dt \leq \infty$, we consider the integral equation

$$y(t) = -1 - \int_t^\infty (s-t)f(s, y(s))ds,$$

and the procedure is the same, except that $-1 \leq y_k(t) \leq 0$, and $a(t)\phi(x)$ replaces $b(t)\psi(x)$.

Corollary 2.2. Let $f(t,x)$ be continuous on S, with $xf(t,x) \geq 0$ for $x \neq 0$.

If $|f(t,x)| \geq a(t)|\phi(x)|$ for $(t,x) \in S$, where $a(t)$ is locally integrable and continuous on $[0,\infty)$, while $\phi(x)$ is nondecreasing , $x\phi(x) > 0$ for $x \neq 0$, on $(-\infty,\infty)$, and

$$\int_{\alpha}^{\infty} [\phi(u)]^{-1} < \infty, \int_{-\alpha}^{-\infty} [\phi(u)]^{-1} du < \infty, \textit{for some } \alpha \geq 0,$$

then $\int_0^\infty ta(t)dt = \infty \Rightarrow (E)$ is oscillatory.

Corollary 2.3. Let $f(t,x)$ be continuous on S, with $xf(t,x) > 0$ for $x \neq 0$.

If $|f(t,x)| \leq a(t)|\phi(x)|$ for $(t,x) \in S$, where $a(t)$ is locally integrable and continuous on $[0,\infty)$, while $\phi(x)$ is nondecreasing and $x\phi(x) > 0$ for $x \neq 0$, on $(-\infty,\infty)$, then (E) is oscillatory $\Rightarrow \int_0^\infty ta(t)dt = \infty$.

These corollaries are obtained by closely examining which condition on $f(t,x)$ are used in the two halves of Theorem 2.1. The importance of the theorem and the corollries lies in the fact that they show it is the global behavior of $f(t,x)$, rather than its local behavior, which determines the oscillation properties of (E).

3. The case for the nonoscillation

We now establish a sufficient condition for the nonoscillation of (E) our restrictions on $f(t,x)$ are more severe than in Theorem 2.1, though they are still gloval rather than local.

Theorem 3.1. Let $f(t,x)$ be continuous on S, with $f_t(t,x)$ defined and continuous on S, and such that $f(t,0) = 0$, $xf_t(t,x) \leq 0$ and $xf(t,x) > 0$ for $x \neq 0$. Assume that $y(t) \equiv 0$ is the only solution of (E) in the class F such that $y(\tau) = y'(\tau) = 0$ for any $\tau \in [0,\infty)$. Furthermore, assume that for $0 \leq t < \infty$, $0 \leq x < \infty$, we have $f(t,x) \leq a(t)\phi(x)$, where $a(t)$ is locally integrable, $\phi(x)$ is nondecreasing and such that, for some $B \geq 0$,

we have $\phi(xy) \leq H(x)\phi(y)$ for $0 < x < \infty$, $B < y < \infty$ with $\lim_{n\to 0^+} sup\ x^{-1}H(x) < \infty$.

Then

$$\int_0^\infty \phi(t)a(t)dt < \infty \Rightarrow (E) \textit{ is nonoscillatory.}$$

Proof. For any solution from the class F, $y(t)$, defined on some interval $[T,\infty)$, we define

$$V(t) = y^2 + 2\int_0^{y(t)} f(t,u)du \geq 0 \text{ on } [T,\infty).$$

Then

$$V'(t) = 2\int_0^{y(t)} f_t(t,u)du \geq 0 \text{ on } [T,\infty).$$

Thus $V(t)$ is bounded above, and hence so is $|y'(t)|$, say $|y'(t)| \leq M$ for $t \in [T,\infty)$. Suppose $y(t)$ is a solution from F that oscillates at $t = \infty$, and select a sequence of points $s_k \to \infty$ at which $y(s_k) = 0$, $y'(s_k) > 0$. This is possible because no zero of $y(t)$ can be a zero of $y'(t)$, hence one of tow consecutive zeros must be of the type desired.

Let t_k be the first zero of $y'(t)$ on $t \geq s_k$, and note that $y(t)$ is positive and increasing while $y'(t)$ is positive and decreasing on (s_k, t_k).

Since

$$0 \leq y'(s_k) = \int_{s_k}^{t_k} f(u, y(u))du \leq \int_{s_k}^{t_k} a(u)\phi(y(u))du,$$

and

$$0 \leq y(t) = \int_{s_k}^{t_k} y'(u))du \leq y'(s_k)t, \text{ for } s_k \leq t \leq t_k.$$

We have, from the monotonicity of ϕ ,

$$\begin{aligned} 0 \leq y'(s_k) &\leq \int_{s_k}^{t_k} a(u)\phi(y'(s_k)u)du \\ &\leq \int_{s_k}^{t_k} a(u)\phi(Mu)du \leq H(M)\int_{s_k}^{t_k} a(u)\phi(u)du \end{aligned} \tag{5}$$

for k large enough to make $s_k \geq B$. Since, by hypothesis,

$$\int_0^{\infty} a(u)\phi(u)du < \infty,$$

the upper bound in (5) goes to zero as $k \to \infty$, hence $y'(s_k) \to 0$.

From (5) it also follows that

$$0 \leq y'(s_k) \leq H(y'(s_k)) \int_{s_k}^{t_k} a(u)\phi(u)du,$$

thus

$$1 \leq [y'(s_k)]^{-1} \leq H(y'(s_k)) \int_{s_k}^{t_k} a(u)\phi(u)du,$$

which yields the contradiction.

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