The Pure and Applied Mathematics **2** (1995), No 1, pp. 35-41
J. Korea Soc. of Math. Edu. (Series B)

ON A CONDITION OF OSCILLATORY OF 3–ORDER DIFFERENTIAL EQUATION

In Goo Cho

1. Introduction

We consider the linear differential equations

$$y''\,' + P(x)y' + Q(x)y = 0 \tag{1}$$

$$(y'' + P(x)y)' - Q(x)y = 0 \tag{2}$$

Where (2) in the adjoint of (1) and $P(x)$, $Q(x)$ are continuous functions satisfying

$$P(x) \geq 0,\ Q(x) \leq 0,\ P(x) - Q(x) \geq 0 \text{ on } [a, \alpha). \tag{3}$$

In this, we show that a condition a oscillatory of(1).

Definition 1.1. Let $y(x)$ be a solution of (1) Then $y(x)$ is said to be oscillatory if the set of zeros of $y(x)$ is not bounded above. And the solution $y(x)$ is non-oscillatory if it is not oscillatory. The equation (1) is oscillatory if it has at least one oscillatory solution.

If all solutions of (1) are non-oscillatory, then(1) is said to be non-oscillatory.

Definition 1.2. A third orde r linear differential equation is said to be disconjugato on $[a,\ \alpha)$ if no nontrivial solution has three zeros on $[a,\ \alpha)$.

Definition 1.3. Let c be any point on $[a,\ \alpha)$ and let $U_i(x,c),\ i = 1,2$ be the pair of solutions determined by the initial conditions

$U_1(x,c) : y(c) = 0, \quad y'(c) = 1, \quad y''(c) = 0$
$U_2(x,c) : y(c) = 0, \quad y'(c) = 0, \quad y''(c) = 1$

$U_2(x,c)$ and $U_1(x,c)$ are called first and second principal solutions, nespectively at $x = c$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition 1.4. Let $D_2(y) = y'' + P(x)y$. The first and second principal solutions $U_2^*(x,\ c)$ and $U_1^*(x,c)$ of(2) at $x = c,\ c \in [a,\ \theta)$ and determined by the initial conditions:

$$U_1^*(x,c) : z(c) = 0, \quad z'(c) = 1, \quad D_2z(c) = 0$$
$$U_2^*(x,c) : z(c) = 0, \quad z'(c) = 0, \quad D_2z(c) = 1.$$

The Wronskian of any two solutions of (1) is a solution of (2) and the converse holds. Thus

$$U_2^*(x,c) : U_1(x,c)U_2'(x,c) - U_2(x,c)U_1'(x,c)$$
$$U_2(x,c) : U_1^*(x,c)U_2'^*(x,c) - U_2^*(x,c)U_1'^*(x,c)$$

differentiatiry these identities yields

$$U_2^*(x,c) : U_1(x,c)U_2''(x,c) - U_2(x,c)U_1''(x,c)$$
$$D_2U_2^*(x,c) : U_1'(x,c)U_2''(x,c) - U_2'(x,c)U_1''(x,c)$$

2. Preliminaries

Lemma 2.1 [1. Theorem 6]. The differential equation (1) is dis conjugato on $[c,\ \theta)$, $a \le c$, iff there exists a pair of functions $y(x)$ and $z(x)$ such that

(a) $y(c) = z(c) = 0$
(b) $y(x) > 0$ and $z(x) > 0$ on $(c,\ \theta)$
(c) $L_3(y(x)) \le 0$ and $L_3^*(z(x)) \le 0$ on $[c,\ \theta)$

Lemma 2.2 [1. lemma 1]. For each $t \ge a,\ \gamma_{ij}(t) = \gamma_{ji}^*(t)$.

Where, the number $\gamma_{ij}(t)$ $(3 \le i + j \le 4,\ i,\ j$ are positive integer) is the least upper bound of the set of b such that there is a $c \in [t, b)$ with the property that there exists a solution of (1) with a zero of multiplicity at least i at c and a zero of multiplicity at least j at b. The number $\gamma_{ij}^*(b)$ is defined same method for (2).

As a consequence of Lemma 2.1 and 2.2 we have

Lemma 2.3. The differential equation (1) is disconjugate on $[c,\ \theta),\ c \ge a$, if and only if $U_2(x,c),\ U_2^*(x,c)$ are the first principal solutions of (1) and (2) respectively, at $x = c$.

3. Main theorems

We begin by estblishing a disconjugacy criterion for (1) under (3).

Lemma 3.1. If (1) is disconjugate and its coefficients satisfy(3) then $D_2U_2^*(x,c) \geq 0$ on $[c,\ \theta)$, $c \in [a,\ \theta)$.

Proof. Since (1) is disconjugate on $[a,\ \theta)$, we have $U_2(x,c) > 0$ and $U_2^*(x,c) > 0$ on $[c,x)$ by Lemma 2.3.

Suppose $D_2U_2^*(x,c) = U_2^{*\prime\prime}(x,c) + P(x)U_2^*(x,c)$ has a zero at $x = t_1$, $t_1 \in (c.\theta)$. Then $U_2^{*\prime\prime}(x,c) < 0$ on $[t_1,\ \theta)$. Now $p(x) \geq 0$ implies $U_2^{*\prime\prime}(x,c) < 0$ on $[t_1,\ \theta)$. Thus, $U_2^*(x,c)$ is decreasing on $(t_1,\ \theta)$, Suppose $U_2^*(x,c) > 0$ on $[t_1,\ \theta)$,Since $(U_2^{*\prime\prime}(x,c)+P(x)U_2^{*\prime}(x,c))'+(p'(x)-Q(x))U_2^*(x,c) = 0$, we have $U_2^{*\prime\prime\prime}(x,c) < 0$ on $(t_1,\ \theta)$ which implies $u_2^{*\prime}(x,c)$ is eventually negative, contradicting our assumption.

Therefore there exists t_2 such that $U_2^{*\prime}(x,c) < 0$ on $(t_2,\ \theta)$.

Now $U_2^{*\prime\prime}(x,c) < 0$ and $U_2^{*\prime}(x,c) < 0$ on $(t_2,\ \theta)$ implies $U_2^*(x,c)$ is eventually negative, which contradicts the fact that $U_2^*(x,c) > 0$ on $(c,\ \theta)$. Therefore $U_2^{*\prime\prime}(x,c) + P(x)U_2^*(x,c) > 0$ on $[c,\ \theta)$.

Lemma 3.2. If (1) is disconjugate and its cocfficients satisfy (3) then $U_2'(x,c) > 0$ on $(c,\ \theta)$, $c \in [a,\ \theta)$.

Proof. Suppose $U_2'(x,c)$ has a zero and let $x = t_1$ be the first such zero, and $U_2'(x,c)$ has a secon zero at $x = t_2$.

From the identity $U_2^*(x,c)$ in defimition 1.4 we have $U_1'(t_2,c) < 0$.

Let's define $\lambda_1(x) = \frac{U_1'(x,c)}{U_2'(x,c)}$ on (t_1,t_2) then

$\lim \lambda_1(x) = \infty$ on (t_1,t_2) and

$\lambda_1'(x) = \frac{-D_2U_2^{*\prime}(x,c)}{(U_2'(x,c))^2}$

and by Lemma 3.1, $\lambda_1'(x) < 0$ on (t_1,t_2), yieldiny a contradiction.

Therefore, if $U_2'(x,c)$ has a zero at $x = t_1$, then $U_2'(x,c) < 0$ on $(t_1,\ \infty)$. By Rolle's theorem, $U_2''(x,c)$ has a zero at $x = s_1$ ($s_1 \in (c,t_1)$) and $U_2''(x,c) < 0$ at $x = t_1$. Since $U_2''\ '(x,c) = -p(x)U_2'(x,c) - Q(x)U_2(x,c) > 0$ on $(t_1,\ \infty)$, it follows that $U_2''(x,c)$ is increasing on $(t_1,\ \infty)$.

If $U_2''(x,c) < 0$ on $(t_1,\ \infty)$, then $U_2(x,c)$ is eventually negative, it's a contradiction. Therefore, $U_2''(x,c)$ must have a second zero at $x = s_2$ With $U_2''(x,c) > 0$ on $(s_2,\ \infty)$. But $U_2''\ '(x,c) = -p(x)U_2'(x,c) - Q(x)U_2(x,c) > 0$ and $U_2''(x,c) > 0$ on $(s_2,\ \infty)$ implies $U_2'(x,c)$ is eventually positive, contradicting our result above.

Hence $U_2'(x,c) > 0$ on $(c,\ \infty)$.

Lemma 3.3 [7, Theorem 1.2, p5]. Let $(\gamma(x)y')' + g(x)y = 0$ be disconjugate on $[a, \infty)$. If $\int_a^\infty \frac{1}{\gamma(x)}dx = \infty$, $g(x) \geq 0$ with $g(x) \neq 0$ for lasge x, and $y(x)$ is any non-trivial solution of $(\gamma(x)y')' - g(x)y = 0$ with $y(a) = 0$, then $y(x)y'(x) > 0$ on (a, ∞).

Lemma 3.4. If (1) is disconjugate on $[c, \infty)$ and its coefficients satisfy (3), then $D_2U_2(x,c) \geq 1$, $U_2^{*\prime}(x,c) > 0$, $U_2^{*\prime\prime}(x,c) > 0$ and $U_2^{*\prime\prime\prime}(x,c) < 0$ on (c, ∞), $c \in [a, \infty)$.

Proof. since $U_2(x,c)$ is a solution of (1), we have

$U_2''{}'(x,c) + P(x)U_2'(x,c) + Q(x)U_2(x,c) = 0$ and integrating form c to x we obtain $U_2''(x,c) + p(x)U_2(x,c) - 1 - \int_c^x (p(t) - Q(t)U_2(t,c)dt = 0$. Thus

$U_2''(x,c) + p(x)U_2(x,c) \geq 1$ on $[c, \infty)$. Suppose $U_2^{*\prime}(x,c)$ has a first zero at s_1 and a second zero at s_2. The identity

$U_2(x,c) = U_1^*(x,c)U_2^{*\prime}(x,c) - U_2^*(x,c)U_1'(x,c)$ implis $U_2^{*\prime}(s_2,c) < 0$. Define

$$\lambda_1^*(x) = \frac{U_1^{*\prime}(x,c)}{U_2^{*\prime}(x,c)} \text{ on } (s_1, s_2). \qquad \text{Then}$$

$$\lim_{x \to s_2} \lambda_1^*(x) = \infty \text{ on} (s_1, s_2). \qquad \text{And}$$

$$\lambda_1^{*\prime}(x) = \frac{D_2U_2(x,c)}{(U_2^{*\prime}(x,c))^2} < 0 \qquad \text{on } (s_1, s_2),$$

which is a contradiction.

Thus $U_2^{*\prime} < 0$ on (s_1, ∞). Using the Lagrange Identity, $U_2^*(x,c)$, $U_1^*(x,c)$ are solutions of

$$U_2(x,c)y'' - U_2'(x,c)y' + D_2U_2(x,c)y = 0 \tag{4}$$

Also $U_2^*(x,c)$, $U_1^*(x,c)$ are solutions of (2)

$$y'' + (P(x)y' + (P'(x) - Q(x))y = 0.$$

Elimination the y-term from (4) and (2), we have

$$(D_2U_2(x,c))y''{}' - (p'(x) - Q(x))U_2(x,c)y'' + \{(P(x)D_2U_2(x,c) + (P'(x) - Q(x))U_2'(x,c)\}y' = 0.$$

Which can be written

$$y''' - \frac{(p'(x) - Q(x))U_2(x,c)}{D_2U_2(x,c)}y'' + \frac{p(x)D_2U_2(x,c) + (p'(x) - Q(x))U_2'(x,c)}{D_2U_2(x,c)}y' = 0,$$

since $D_2U_2(x,c) \geq 1$. Let $z = U_2^{*\prime}(x,c)$. Then z is a solution of

$$z''' - \frac{(p'(x) - Q(x))U_2(x,c)}{D_2U_2(x,c)}z' + \frac{p(x)D_2U_2(x,c) + (P'(x) - Q(x))U_2'(x,c)}{D_2U_2(x,c)} = 0. \quad (5)$$

Letting $\gamma(x) = e^{-\int_c^x \frac{P(t)-Q(t)}{D_2U_2(t,c)}dt} > 0$
and multipling (5) by $\gamma(x)$, yilds

$$(\gamma(x)y')' + \gamma(x)\frac{p(x)D_2U_2(x,c) + (p'(x) - Q(x))U_2'(x,c)}{D_2U_2(x,c)}y = 0. \quad (6)$$

By Lemma 2.2, we have

$$\frac{p(x)D_2U_2(x,c) + (p'(x) - Q(x))U_2'(x,c)}{D_2U_2(x,c)} > 0$$

and by over result above, $U_2^{*\prime}(x,c) < 0$ on (s_1, ∞).

Thus (6) is disconjugate on (s_1, ∞). Now,

$$\int_{s_1}^{\infty} \gamma(x)^{-1}dx = \int_{s_1}^{\infty} e^{\int_c^x \frac{p'(t)-Q(t)}{D_2U_2(t,c)}dx}dx = \infty \text{ and } z(s_1) = 0.$$

Therefore Lemma 3.3., $z(x) \cdot z'(x) > 0$ on (s_1, ∞) and we have $z'(x) < 0$ on (s_1, ∞).

But $z'(x) = U_2^{*\prime\prime}(x,c) < 0$ and $U_2^{*\prime}(x,c) < 0$ on (s_1, ∞) implies $U_2^*(x,c)$ is evertually negative, contradicting the fact that $U_2^*(x,c) > 0$ on (c, ∞). Finally

$$U_2^{*\prime\prime\prime}(x,c) + p(x)U_2^{*\prime}(x,c) + (P'(x) - Q(x))U_2^*(x,c) = 0$$

implies $U_2^{*\prime\prime\prime}(x,c) < 0$.

By the above lemmas we have following theorem.

Theorem 3.1. If (1) is disconjugate and its coefficients satisfy (3) then

$$U_2'(x,c) > 0,\ U_2''(x,c) > 0,\ U_2^{*\prime}(x,c) > 0,\ u_2^{*\prime\prime}(x,c) > 0,$$

$$D_2U_2(x,c) = U_2''(x,c) + p(x)U_2(x,c) \geq 1,\ D_2U_2^*(x,c) \geq 0$$

and $U_2^{*\prime\prime\prime}(x,c) < 0$ on $(c,\ \infty)$ for each $c \in [a,\ \infty)$,

To make a condition for the oscillation of (1), we need following.

Definition 3.1. Equation (1) is said to be strongly oscillatory if every solution of (1) is oscillatory.

Definition 3.2. If (1) has a non-trivial solution with three zeros on $[t,\ \infty)$, $t \in [a,\ \infty)$, then the first conjugate point $\eta_1(t)$ of $x = t$ is defined by

$$\eta_1(t) = \inf x_3; t \leq x_1 \leq x_2 \leq x_3,\ y(x_i) = 0,\ i = 1,2,3,\ y \neq 0,\ L_3(y) = 0.$$

Lemma 3.5 [3. theorem 3.28]. If each of (1) and (2) is nonoscillatory, then (1) is disconjugate for large x.

Lemma 3.6 [6. theorem 3.3]. If $2Q(x) - p'(x) \leq 0$ and not identically zero in any interval then $y''' + p(x)y' + Q(x)y = 0$ has a solution $U(x)$ for which

$$\begin{aligned} F(U(x)) &= U'(x)^2 - 2U(x)U''(x) - P(x)U^2(x) \\ &= F[U(a)] + \int_a^x (2q(t) - p'(t))U^2(t)dt \end{aligned}$$

is always negative. Consequently $U(x)$ is nonoscillatory.

Theorem 3.2. If the coefficients of (1) satisfyk (3) and $\eta_1(t) < \infty$ for each $t \in [a,\ infty)$, then (1) is oscillatory.

Proof. Assume (1) is non-oscillatory on $[a,\ \infty)$. If (2) is nonoscillatory, then by Lemma 2.5 (1) is disconjugate for large x, which is contradiction to our hypotheses. Therefore (2) is oscillatory. Consider

(2) $\quad y''' + p(x)y' + (p'(x) - Q(x))y = 0,\ p(x) \geq 0,\ p'(x) - Q(x) \geq 0$ and $2(p'(x) - Q(x)) - p'(x) = p'(x) - 2Q(x) \geq p'(x) - Q(x)\ [a,\ \infty)$.

Thus using a result of Lemma 3.6., (2) has a nonoscillatory solution which contradicts (2) being strongly oscillatory.

Therefore (1) is oscillatory.

REFERENCES

1. N.V. Azbelev and Z.B. Caljuk, *On the question of distribution of zeros of solutions of linear differential equations of the therd oredr*, AMS. Transl **42** (1964), 233-245.
2. John H.Barret, *Oscillation Theory of Ordinary linear differential equations*, Advances in Math **3** (1969), 415-509.
3. J.M. Dolan, *Oscillatory lehavior of solutions of linear differential equation of third order, Doctoral dessertation*, Univ. of Tennessee (1967).
4. P. hartman, *Ordinary differential Equations*, John Wiley (1964).
5. W.J. Kim, *Oscillatory properties of linear thisd differential equations*, Proc. Amer. Math.Soc **26** (1970), 286-293.
6. A.c. Lager, *The behavior of solutions of the differential equation* $y'' \ ' + p(x)y' + q(x)y = 0$, Pacific J.Math **17** (1966), 435-466.
7. Robert. Mckelvey, *Lecfures on ordinary differential equations*, Academic press (1970).
8. R. kent Nagle and Edward B.saff, *Fundamentals of Diffential Equations and Boundary Value Problems*, Addison-wesley publishing company (1993).

Junior College of Inchon