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ON SOME PROPERTIES OF THE BLASS TOPOS

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1. Introduction

The topos constructed in [6] is a set-like category that includes among its axioms an axiom of infinity and an axiom of choice. In its final form a topos is free from any such axioms. Set^G is a topos whose object are G-set $\psi_s : G \times S \to S$ and morphism $f : S \to T$ is an equivariants map. We already known that Set^G satisfies the weak form of the axiom of choice but it does not satisfies the axiom of the choice. The Blass topos is the topos whose object are the H-set $\psi_s : H \times S \to S$ where fix group is exorbitant and $Ker(S)$ is large, morphism $f : S \to T$ act on the underlying set S and T which $Eqv(f)$ is exorbitant. In this paper, we show that the Blass topos satisfies the supports split (SS) and we investigate the axiom of choice (AC) in the Blass topos. Additionally the category of the R-module in the Blass topos, the existance of a insertion is not provable.

2. Preliminaries

Definition 2.1. *A topos is a category E that satisfies the following three axiom :*
E1 : All finite limits exist (terminal object 1 and pullback exist)
E2 : An object Ω exists, together with a map true : $1 \to \Omega$ such that for any monomorphism $f : A' \to A$, there is precisely one map $A \to \Omega$, called the characteristic map of f that A' is the finite limit of the true : $1 \to \Omega$ and $\chi_f : A \to \Omega$
E3 : For every object A in E, there exists an object Ω^A, and a map $A \times \Omega^A \to \Omega$, called the evaluation map of A, such that for any Y and f in E, there exists a unique $f' : Y \to \Omega^A$ with $(1_A \times f')ev = f$

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Definition 2.2. *The Blass topos may be described as follows : For an uncountable index set I, let G be the additive group of all integer-valued function on I, and for any subset* $E \subseteq I$ *take* $Z(E)$ *to be the subgroup of all* $s \in G$ *for which* $s(i) = 0$ *for all* $i \in E$. *Then, call a subgroup H of G **exorbitant** iff* $Z(E) \subseteq H$ *for some finite* $E \subseteq I$, *and **large** iff* $Z(E) \subseteq H$ *for countable* $E \subseteq I$. *With this, the object of the Blass topos are those H-set A, for some exorbitant subgroup H of G, such that fix group* $Fix(a) = \{s | s \in H, sa = a\}$ *is exorbitant for each* $a \in A$, *and* $Ker(A) = \cap Fix(a)$ *is large. For any such A and B, a map* $f : A \to B$ *acts on the underlying set of A and B such that its equivariance group* $Eqv(f) = \{s | s \in H \cap K, f(sa) = sf(a)$ *for all* $a \in A\}$ *is exorbitant.*

Definition 2.3. *We say that **supports split** (SS) in E if, for every* $X \in E$, *the canonical epimorphism* $X \to \sigma_1(X)$ *is split.*

Definition 2.4. *We say that E satisfies **axiom of choice** (AC) if supports split in* E/X *for every X.*

Proposition 2.5. *The following conditions are equivalent:*

1) *Every object of E is internally projective.*
2) *Every epi in E is locally split.*
3) *If* $f : X \to Y$ *is an epi in E, then* $\Pi_Y(f)$ *has global support.*

Proof. See Ref. [5]

Definition 2.6. *We say E satisfies **implicit axiom of choice** (IC) if the above condition (2.5) hold.*

Lemma 2.7. *(AC) is equivalent to the conjunction of (SS) and (IC).*

Proof. See Ref. [5]

3. Main part

We study some properties of the Blass topos which is a subtopos of the Fraenkel-Mostowski topos. Blass showed that the existance of enough injective abelian group is a very weak form of the axiom of choice in the Zermelo-Fraenkel set theory. He also showed that the category of abelian group in the Blass topos has no non-zero injective. We investigate some properties for (AC) of the Blass topos.

Theorem 3.1. *The Blass topos satisfies SS (supports split)*

Proof. The terminal object is $\psi : G \times \{*\} \to \{*\}$ with trivial action since fix group $Fix(*) = G$ is exorbitant and $Ker(\{*\})$ is large. The subobject of the terminal object are $\psi : G \times \{*\} \to \{*\}$ and $\phi : G \times \{\} \to \{\}$. First we show that $\psi : G \times \{*\} \to \{*\}$ is a projective object in the Blass topos. Let $\psi_A : G_A \times A \to A$ and $\psi_B : G_B \times B \to B$ are two object in the Blass topos, equivariant group of the epimorphism $e : A \to B$ is exorbitant. For any $f : \{*\} \to B$ in the Blass topos, $f(*)$ is in B. Since epimorphism is a surjective on the underlying set in this situation, there exists an $a \in A$ such that $e(a) = f(*)$. Construct $h : \{*\} \to A$ such that $h(*) = a$, then $Eqv(h)$ is exorbitant since $h(*) \in A$ and for all $a \in A$, $Fix(a)$ is exorbitant, $h(*) = \psi_A(g, h(*))$ implies $h(\psi(g,*)) = \psi_A(g, h(*))$ and $e \circ h = f$, hence $\psi : G \times \{*\} \to \{*\}$ is a projective

object. Secondly $\phi : G \times \{\} \to \{\}$ is an object of the Blass topos since $Fix() = G$ and $Ker(\{\}) = G$, trivially it is a projective object in the Blass topos.

Proposition 3.2. *If a morphism f and a object X are in the Blass topos, then f^X is a morphism in the Blass topos.*

Proof. Let A, B and X are the object of the Blass topos, that is, $\psi_A : G_A \times A \to A$, $\psi_B : G_B \times B \to B$ and $\psi_X : G_X \times X \to X$, each fix group is exorbitant and Ker is large. Let $f : A \to B$ be a morphism in the Blass topos, we show that $f^X : A^X \to B^X$ is a morphism in the Blass topos. Define $f^X(t) = f \circ t$ for all $t : X \to A$, then $Eqv(f^X)$ is exorbitant, since $f^X(\psi(g,t)) = \phi(g, f^X(t))$ iff $f(\psi(g,t)) = \phi(g, ft)$, for all $x \in X$, $f(\psi(g,t))(x) = \phi(g, ft)(x)$ iff $f(\psi_A(g, t\psi_X(g^-1, x))) = \psi_B(g, f(t(\psi_X(g^-1, x))))$, $t(\psi_X(g^-1, x)) \in A$ and $Eqv(f)$ is exorbitant.

Proposition 3.3. *There exists a topos in which SS (supports split) is fail*

Proof. Consider the topos Set^G where $G = \{1, e\}$ is a group defined by $e \cdot e = 1 = 1 \cdot 1$, $e \cdot 1 = e = 1 \cdot e$. We show that the terminal object $\psi : G \times \{*\} \to \{*\}$ with trivial action is not a projective object in Set^G. Let $h : A \to B$ be an epimorphism in Set^G such that $A = \{a, b\}$ with action $\phi : G \times A \to A$, with $\phi(1,a) = a$, $\phi(e,a) = b$, $\phi(1,b) = b$, $\phi(e,b) = a$. For any $f : \{*\} \to B$, there exists a morphism $g : \{*\} \to A$ such that $h \circ g = f$. But g is not equivariant, since $g(*) \neq \phi(e, g(*))$ by the action of ϕ implies $g(\psi(e,*)) \neq \phi(e, g(*))$.

Proposition 3.4. *The axiom of choice (AC) fails in the Blass topos*

Proof. Let A, B are the object of the Blass topos, that is, $\psi_A : G_A \times A \rightarrow A$,$\psi_B : G_B \times B \rightarrow B$, each fix group is exorbitant and Ker is large. Assume the Blass topos satisfies (AC), then for any epimorphism $e : A \rightarrow B$, there exists a morphism $s : B \rightarrow A$ such that $e \circ s = I_B : B \rightarrow B$. Thus for any $b \in B$, $(e \circ s)(b) = I_B(b)$, and for any $x \in G_B$, $x((es)(b)) = x(I_B(b))$. But $x(I_B(b)) = xb$, since $(xg)(y) = x(g(x^{-1}y))$ in $G \times X^Y \rightarrow X^Y$, $x((es)(b)) = x(es)x(b) = x(xb)$. If $xb = b$, that is, ψ_B is a trivial action, then $x((es)(b)) = x(I_B(b))$. But if $xb = c \neq b, xc \neq c$, then $e \circ s \neq I_B$.

Corollary 3.5. *The implicity axiom of choice (IC) fails in the Blass topos.*

We investigate an insertion of the category of R-module in the Blass topos.

Theorem 3.6. *For any unitary ring R in which $Ch(R) = 0$, let F be a free object over X of the category of R-module in the Blass topos. The existance of a insertion $i : X \rightarrow U(F)$ is not provable in the Blass topos.*

Proof. Let F be the free R-module with basis $X = \{C \cup \{x\}\}$ together with $E \subseteq C \subseteq I$ a countable subset of I for which $C - E$ is infinite and x is a element not in C. Define action $\psi_F : G_F \times F \rightarrow F$ such that $\psi_F(s, x) = x$, $\psi_F(s, c) = c + j(s(c))x$ for each $s \in H_F$ and $c \in C$ where $j : Z \rightarrow R$ is the unique ring homomorphism. Then for any $y \in F$, $Fix(y)$ is exorbitant and $Ker(F)$ is large. Define action $\psi_X : G_X \times X \rightarrow X$ such that $\psi_X(s, c) = c$, $\psi_X(s, x) = x$, then $Fix(x) = G$ is exorbitant and $Ker(X)$ is large. Let $i : X \rightarrow U(X)$, then $Eqv(i) = \{s | i(\psi_X(s, c)) = \psi_F(s, i(c))$ for all $c \in C$ $\} = \{s | i(c) = \psi_F(s, c)\} = \{s | c = c + j(s(c))x\}$. We have $Z(C) \subseteq Eqv(i)$ since $s \in Z(C)$ implies $s(c) = 0$ for all countable c, that is, $s \in Eqv(i)$. And $Eqv(i) \subseteq Z(C)$ since $s \in Eqv(i)$ implies $c = c + j(s(c))x$, we have a fact that $j(s(c))x$ is a zero in F. This implies $j(s(c))$ is a zero in R. Thus $s(c)$ is contained in the kernel of $j : Z \rightarrow R$. Since $Ch(R)$ is a zero, $s(c) = 0$ for all countable C. Hence $s \in Z(C)$. Therefore $Eqv(i)$ does not contain $Z(E)$ for some finite E.

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