The Pure and Applied Mathematics **2** (1995), No 1, pp. 9-16
J. Korea Soc. of Math. Edu. (Series B)

THE COMPLETION OF SOME METRIC SPACE OF FUZZY NUMBERS

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1. Introduction

D. Dubois and H. Prade introduced the notions of fuzzy numbers and defined its basic operations [3]. R. Goetschel, W. Voxman, A. Kaufmann, M. Gupta and G. Zhang [4, 5, 6, 9] have done much work about fuzzy numbers.

Let $\mathbb{R}$ be the set of all real numbers and $F^*(\mathbb{R})$ all fuzzy subsets defined on $\mathbb{R}$. G. Zhang [8] defined the fuzzy number $\tilde{a} \in F^*(\mathbb{R})$ as follows :

(1) $\tilde{a}$ is normal,

(2) for every $\lambda \in (0, 1]$, $a_\lambda = \{x \mid \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^-, a_\lambda^+]$.

Now, let us denote the set of all fuzzy numbers defined by G. Zhang as $F(\mathbb{R})$.

The purpose of this paper is to prove that the metric space $(F(\mathbb{R}), \delta)$ can be completed by using the equivalence classes of Cauchy sequences, where δ is defined by $\delta(\tilde{a}, \tilde{b}) = \sup_{0<\lambda\leq 1} d(a_\lambda, b_\lambda)$. In section 2, we quote basic definitions and theorems from [1] which will be needed in the proof of main theorem. In section 3, after defining the isometry and the completion concepts, we prove main theorem :

「 The metric space $(F(\mathbb{R}), \delta)$ has a completion $(\hat{F}(\mathbb{R}), \hat{\delta})$ which has a subspace X that is isometric with $F(\mathbb{R})$ and is dense in $\hat{F}(\mathbb{R})$. This space $(\hat{F}(\mathbb{R}), \hat{\delta})$ is unique except for isometries.」

2. Basic definitions and results of fuzzy numbers

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In this section, we quote basic definitions and theorems from [1] which will be needed in the proof of main theorem.

Let $\mathbb{R}$ be the set of all real numbers and $F^*(\mathbb{R})$ all fuzzy subsets defined on $\mathbb{R}$.

Definition 2.1. Let $\tilde{a} \in F^*(\mathbb{R})$. $\tilde{a}$ is called a fuzzy number if $\tilde{a}$ has the properties :

(1) $\tilde{a}$ is normal, i.e., there exists $x \in \mathbb{R}$ such that $\tilde{a}(x) = 1$,

(2) whenever $\lambda \in (0, 1]$, $a_\lambda = \{x \mid \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^-, a_\lambda^+]$.

Let $F(\mathbb{R})$ be the set of all fuzzy numbers on the real line $\mathbb{R}$.

If we define $\tilde{a}(x)$ by

$$\begin{aligned} \tilde{a}(x) &= 1 \quad \text{for } x = k, \\ &= 0 \quad \text{for } x \neq k \quad (k \in \mathbb{R}), \end{aligned}$$

then $\tilde{a} \in F(\mathbb{R})$ and $a_\lambda = [k, k]$. In here, we can see that any real number is a fuzzy numbers.

Definition 2.2. Let $\tilde{a}, \tilde{b}, \tilde{c} \in F(\mathbb{R})$. We say that $\tilde{c} = \tilde{a} + \tilde{b}$ if for every $\lambda \in (0, 1]$, $c_\lambda^- = a_\lambda^- + b_\lambda^-$ and $c_\lambda^+ = a_\lambda^+ + b_\lambda^+$. We say that $\tilde{c} = \tilde{a} - \tilde{b}$ if for every $\lambda \in (0, 1]$, $c_\lambda^- = a_\lambda^- - b_\lambda^+$ and $c_\lambda^+ = a_\lambda^+ - b_\lambda^-$. For every $k \in \mathbb{R}$ and $\tilde{a} \in F(\mathbb{R})$, we say that $\tilde{c} = k\tilde{a}$ if for every $\lambda \in (0, 1]$, $c_\lambda^- = ka_\lambda^-$, $c_\lambda^+ = ka_\lambda^+$ for $k \geq 0$, and $c_\lambda^- = ka_\lambda^+$, $c_\lambda^+ = ka_\lambda^-$ for $k < 0$.

Note that we can find in [7] the definitions of multiplication, division, maximal and minimal operations of the fuzzy numbers.

Definition 2.3. Let $\tilde{a}, \tilde{b} \in F(\mathbb{R})$. We say that $\tilde{a} \leq \tilde{b}$ if for every $\lambda \in (0, 1]$, $a_\lambda^- \leq b_\lambda^-$ and $a_\lambda^+ \leq b_\lambda^+$. We say that $\tilde{a} < \tilde{b}$ if $\tilde{a} \leq \tilde{b}$ and there exists $\lambda \in (0, 1]$ such that $a_\lambda^- < b_\lambda^-$ or $a_\lambda^+ < b_\lambda^+$. We say that $\tilde{a} = \tilde{b}$ if $\tilde{a} \leq \tilde{b}$ and $\tilde{b} \leq \tilde{a}$.

Definition 2.4. For two closed intervals $a_\lambda = [a_\lambda^-, a_\lambda^+]$, $b_\lambda = [b_\lambda^-, b_\lambda^+]$, we define a metric (distance) d of a_λ, b_λ as follows :

$$d(a_\lambda, b_\lambda) = \max(|a_\lambda^- - b_\lambda^-|, |a_\lambda^+ - b_\lambda^+|).$$

Definition 2.5. A metric (distance) δ of $F(\mathbb{R})$ is a function $\delta : F(\mathbb{R}) \times F(\mathbb{R}) \to \mathbb{R}$ with the properties :

(1) $\delta(\tilde{a}, \tilde{b}) \geq 0,\ \tilde{a} = \tilde{b}$ if and only if $\delta(\tilde{a}, \tilde{b}) = 0$,

(2) $\delta(\tilde{a}, \tilde{b}) = \delta(\tilde{b}, \tilde{a})$,

(3) for every $\tilde{c} \in F(\mathbb{R})$, we have $\delta(\tilde{a}, \tilde{b}) \leq \delta(\tilde{a}, \tilde{c}) + \delta(\tilde{c}, \tilde{b})$.

When δ is a metric of $F(\mathbb{R})$, we call $(F(\mathbb{R}), \delta)$ a metric space of $F(\mathbb{R})$ with the metric δ.

We define

$$\delta(\tilde{a}, \tilde{b}) = \sup_{0<\lambda\leq 1} d(a_\lambda, b_\lambda). \tag{$\star$}$$

Theorem 2.6. $\delta(\tilde{a}, \tilde{b})$ defined by the equality $(\star)$ is a metric of $F(\mathbb{R})$.

Theorem 2.7. For every $\tilde{a}, \tilde{b}, \tilde{c} \in F(\mathbb{R})$, $k \in \mathbb{R}$, we have

(1) $\delta\,(\tilde{a} + \tilde{b},\ \tilde{a} + \tilde{c}) = \delta\,(\tilde{b},\ \tilde{c})$,

(2) $\delta\,(\tilde{a} - \tilde{b},\ \tilde{a} - \tilde{c}) = \delta\,(\tilde{b},\ \tilde{c})$,

(3) $\delta\,(k\tilde{a},\ k\tilde{b}) = |k|\ \delta\,(\tilde{a},\ \tilde{b})$,

(4) If $\tilde{a} \leq \tilde{b} \leq \tilde{c}$, then $\delta\,(\tilde{a},\ \tilde{b}) \leq \delta\,(\tilde{a},\ \tilde{c})$, $\quad \delta\,(\tilde{b},\ \tilde{c}) \leq \delta\,(\tilde{a},\ \tilde{c})$.

Definition 2.8. Let $\{\tilde{a}_n\} \subset F(\mathbb{R})$, $\tilde{a} \in F(\mathbb{R})$. A sequence $\{\tilde{a}_n\}$ is said to converge to $\tilde{a}$ in the metric δ, denoted by

$$\lim_{n\to\infty} \tilde{a}_n = \tilde{a} \quad \text{or} \quad \tilde{a}_n \to \tilde{a} \text{ as } n \to \infty,$$

if for any $\varepsilon > 0$ there exists an integer $N > 0$ such that $\delta(\tilde{a}_n, \tilde{a}) < \varepsilon$ for $n \geq N$.

Theorem 2.9. Let $\{\tilde{a}_n\}, \{\tilde{b}_n\} \subset F(\mathbb{R})$, $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, $k \in \mathbb{R}$. If $\lim\limits_{n\to\infty} \tilde{a}_n = \tilde{a}$ and $\lim\limits_{n\to\infty} \tilde{b}_n = \tilde{b}$, then

(1) $\lim\limits_{n\to\infty} (\tilde{a}_n \pm \tilde{b}_n) = \tilde{a} \pm \tilde{b}$ (the same order of sign),

(2) $\lim\limits_{n\to\infty} (k\tilde{a}_n) = k\tilde{a}$.

Theorem 2.10. Let $\lim\limits_{n\to\infty} \tilde{a}_n = \tilde{a}$, $\lim\limits_{n\to\infty} \tilde{b}_n = \tilde{b}$. Then

$$\lim_{n\to\infty} \delta(\tilde{a}_n, \tilde{b}_n) = \delta(\tilde{a}, \tilde{b}).$$

Definition 2.11. A sequence $\{\tilde{a}_n\}$ of $F(\mathbb{R})$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists an integer $N > 0$ such that $\delta(\tilde{a}_n, \tilde{a}_m) < \varepsilon$ for n, $m > N$.

If a metric space has the property that every Cauchy sequence converges, the space is called a complete metric space.

Theorem 2.12. The metric space $(F(\mathbb{R}), \delta)$ is complete.

3. Main theorem

In this section, we prove that the metric space $(F(\mathbb{R}), \delta)$ has a completion $(\hat{F}(\mathbb{R}), \hat{\delta})$.

Definition 3.1. [2] Let $X_1 = (X_1, d_1)$, $X_2 = (X_2, d_2)$ be metric spaces. Then,

(*a*) A mapping f of X_1 into X_2 is said to be isometric or an isometry if f preserves distances, that is, if for all $x, y \in X_1$, $d_2(f(x), f(y)) = d_1(x, y)$, where $f(x)$ and $f(y)$ are the images of x and y respectively.

(*b*) The space X_1 is said to be isometric with the space X_2 if there exists a bijective isometry of X_1 onto X_2. The spaces X_1 and X_2 are then called isometric spaces.

Definition 3.2. [2] The complete metric space (X_1^*, d_1^*) is said to be a completion of the given metric space (X_1, d_1) if

(1) (X_1, d_1) is isometric with a subspace (X_1, d_1^*) of (X_1^*, d_1^*),

(2) X is dense in X_1^*, i.e., the closure of X, $\overline{X} = X_1^*$.

MAIN Theorem. The metric space $(F(\mathbb{R}), \delta)$ has a completion $(\hat{F}(\mathbb{R}), \hat{\delta})$ which has a subspace X that is isometric with $F(\mathbb{R})$ and is dense in $\hat{F}(\mathbb{R})$. This space $(\hat{F}(\mathbb{R}), \hat{\delta})$ is unique except for isometries, that is, if $(\check{F}(\mathbb{R}), \check{\delta})$ is another completion having a dense subspace Y isometric with $F(\mathbb{R})$, then $\hat{F}(\mathbb{R})$ and $\check{F}(\mathbb{R})$ are isometric.

Proof. The proof is somewhat lengthy. We divide it into four steps (a) to (d).
We construct :

(*a*) $\hat{F}(\mathbb{R}) = (\hat{F}(\mathbb{R}), \hat{\delta})$,

(*b*) an isometry f of $F(\mathbb{R})$ onto X, where $\overline{X} = \hat{F}(\mathbb{R})$.

Then we prove :

(c) completeness of $\hat{F}(\mathbb{R})$,

(d) uniqueness of $\hat{F}(\mathbb{R})$ except for isometries.

(a). Construction of $\hat{F}(\mathbb{R}) = (\hat{F}(\mathbb{R}), \hat{\delta})$.

Let $\{\tilde{x}_n\}$ and $\{\tilde{x}'_n\}$ be Cauchy sequences in $F(\mathbb{R})$. Define $\{\tilde{x}_n\}$ to be equivalent to $\{\tilde{x}'_n\}$ written $\{\tilde{x}_n\} \sim \{\tilde{x}'_n\}$, if

$$\lim_{n\to\infty} \delta(\tilde{x}_n, \tilde{x}'_n) = 0. \tag{1}$$

Let $\hat{F}(\mathbb{R})$ be the set of all equivalence classes $\hat{x}, \hat{y}, \cdots$ of Cauchy sequences thus obtained. We write $\{\tilde{x}_n\} \in \hat{x}$ to mean that $\{\tilde{x}_n\}$ is a member of $\hat{x}$ (a representative of the class $\hat{x}$). We now set

$$\hat{\delta}(\hat{x}, \hat{y}) = \lim_{n\to\infty} \delta(\tilde{x}_n, \tilde{y}_n) \tag{2}$$

where $\{\tilde{x}_n\} \in \hat{x}$ and $\{\tilde{y}_n\} \in \hat{y}$. We show that this limit exists. We have

$$\delta(\tilde{x}_n, \tilde{y}_n) \leq \delta(\tilde{x}_n, \tilde{x}_m) + \delta(\tilde{x}_m, \tilde{y}_m) + \delta(\tilde{y}_m, \tilde{y}_n),$$

hence we obtain

$$\delta(\tilde{x}_n, \tilde{y}_n) - \delta(\tilde{x}_m, \tilde{y}_m) \leq \delta(\tilde{x}_n, \tilde{x}_m) + \delta(\tilde{y}_m, \tilde{y}_n)$$

and a similar inequality with m and n interchanged. Together,

$$|\delta(\tilde{x}_n, \tilde{y}_n) - \delta(\tilde{x}_m, \tilde{y}_m)| \leq \delta(\tilde{x}_n, \tilde{x}_m) + \delta(\tilde{y}_m, \tilde{y}_n). \tag{3}$$

Since $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ are Cauchy, we can make the right side as small as we please. This implies that the limit in (2) exists because $(F(\mathbb{R}), \delta)$ is complete.

We must also show that the limit in (2) is independent of the particular choice of representatives. In fact, if $\{\tilde{x}_n\} \sim \{\tilde{x}'_n\}$ and $\{\tilde{y}_n\} \sim \{\tilde{y}'_n\}$, then by (1), (3),

$$|\delta(\tilde{x}_n, \tilde{y}_n) - \delta(\tilde{x}'_n, \tilde{y}'_n)| \leq \delta(\tilde{x}_n, \tilde{x}'_n) + \delta(\tilde{y}_n, \tilde{y}'_n) \to 0$$

as $n \to \infty$, which implies the assertion

$$\lim_{n\to\infty} \delta(\tilde{x}_n, \tilde{y}_n) = \lim_{n\to\infty} \delta(\tilde{x}'_n, \tilde{y}'_n).$$

We prove that $\hat{\delta}$ in (2) is a metric on $\hat{F}(\mathbb{R})$. Obviously, $\hat{\delta}$ satisfies $\hat{\delta}(\hat{x}, \hat{y}) \geq 0$ (see Definition of $\delta(\tilde{x}, \tilde{y})$) as well as $\hat{\delta}(\hat{x}, \hat{x}) = 0$ and $\hat{\delta}(\hat{x}, \hat{y}) = \hat{\delta}(\hat{y}, \hat{x})$. Furthermore,

$$\hat{\delta}(\hat{x}, \hat{y}) = 0 \quad \Rightarrow \quad \{\tilde{x}_n\} \sim \{\tilde{y}_n\} \quad \Rightarrow \quad \hat{x} = \hat{y}$$

gives $\hat{\delta}(\hat{x}, \hat{y}) = 0 \Leftrightarrow \hat{x} = \hat{y}$, and the triangle inequality for $\hat{\delta}$ follows from

$$\delta(\tilde{x}_n, \tilde{y}_n) \leq \delta(\tilde{x}_n, \tilde{z}_n) + \delta(\tilde{z}_n, \tilde{y}_n)$$

by letting $n \to \infty$.

(b). Construction of an isometry $f : F(\mathbb{R}) \to X \subset \hat{F}(\mathbb{R})$.

With each $\tilde{a} \in F(\mathbb{R})$ we associate the class $\hat{a} \in \hat{F}(\mathbb{R})$ which contains the constant Cauchy sequence $\{\tilde{a}, \tilde{a}, \cdots\}$. This defines a mapping $f : F(\mathbb{R}) \to X$ onto the subspace $X = f(F(\mathbb{R})) \subset \hat{F}(\mathbb{R})$. The mapping f is given by $\tilde{a} \mapsto \hat{a} = f(\tilde{a})$, where $\{\tilde{a}, \tilde{a}, \cdots\} \in \hat{a}$. We see that f is an isometry since (2) becomes simply

$$\hat{\delta}(\hat{x}, \hat{y}) = \delta(\tilde{a}, \tilde{b}),$$

here $\hat{b}$ is the class of $\{\tilde{y}_n\}$ where $\tilde{y}_n = \tilde{b}$ for all n. Any isomerty is injective, and $f : F(\mathbb{R}) \to X$ is surjective since $f(F(\mathbb{R})) = X$. Hence X and $F(\mathbb{R})$ are isometric.

We show that X is dense in $\hat{F}(\mathbb{R})$. We consider any $\hat{x} \in \hat{F}(\mathbb{R})$. Let $\{\tilde{x}_n\} \in \hat{x}$. For every $\varepsilon > 0$ there is an integer $N > 0$ such that

$$\delta(\tilde{x}_n, \tilde{x}_N) < \varepsilon/2 \quad \text{for } n > N.$$

Let $\{\tilde{x}_N, \tilde{x}_N, \cdots\} \in \hat{x}_N$. Then $\hat{x}_N \in X$. By (2),

$$\hat{\delta}(\hat{x}, \hat{x}_N) = \lim_{n\to\infty} \delta(\tilde{x}_n, \tilde{x}_N) \leq \varepsilon/2 < \varepsilon.$$

This shows that every ε-neighborhood of the arbitrary $\hat{x} \in \hat{F}(\mathbb{R})$ cotains an element of X. Hence X is dense in $\hat{F}(\mathbb{R})$.

(c). Completeness of $\hat{F}(\mathbb{R})$.

Let $\{\hat{x}_n\}$ be any Cauchy sequence in $\hat{F}(\mathbb{R})$. Since X is dense in $\hat{F}(\mathbb{R})$, for every $\hat{x}_n \in \hat{F}(\mathbb{R})$ there is a $\hat{z}_n \in X$ such that

$$\hat{\delta}(\hat{x}_n, \hat{z}_n) < \frac{1}{n}. \tag{4}$$

Hence, by the triangle inequality,

$$\begin{aligned}\hat{\delta}(\hat{z}_m, \hat{z}_n) &\leq \hat{\delta}(\hat{z}_m, \hat{x}_m) + \hat{\delta}(\hat{x}_m, \hat{x}_n) + \hat{\delta}(\hat{x}_n, \hat{z}_n) \\ &< \frac{1}{m} + \hat{\delta}(\hat{x}_m, \hat{x}_n) + \frac{1}{n}\end{aligned}$$

and this is less than any given $\varepsilon > 0$ for sufficiently large m and n because $\{\hat{x}_n\}$ is Cauchy. Hence $\{\hat{z}_m\}$ is Cauchy. Since $f : F(\mathbb{R}) \to X$ is isometric and $\hat{z}_m \in X$, the sequence $\{\tilde{z}_m\}$, where $\tilde{z}_m = f^{-1}(\hat{z}_m)$, is Cauchy in $F(\mathbb{R})$. Let $\hat{x} \in \hat{F}(\mathbb{R})$ be the class to which $\{\tilde{z}_m\}$ belongs. We show that $\hat{x}$ is the limit of $\{\hat{x}_n\}$. By (4),

$$\begin{aligned}\hat{\delta}(\hat{x}_n, \hat{x}) &\leq \hat{\delta}(\hat{x}_n, \hat{z}_n) + \hat{\delta}(\hat{z}_n, \hat{x}) \\ &< \frac{1}{n} + \hat{\delta}(\hat{z}_n, \hat{x}).\end{aligned} \tag{5}$$

Since $\{\tilde{z}_m\} \in \hat{x} \in \hat{F}(\mathbb{R})$ and $\hat{z}_n \in X$, so that $\{\tilde{z}_n, \tilde{z}_n, \cdots\} \in \hat{z}_n$, the inequality (5) becomes

$$\hat{\delta}(\hat{x}_n, \hat{x}) < \frac{1}{n} + \lim_{m\to\infty} \delta(\tilde{z}_n, \tilde{z}_m)$$

and the right side is smaller than any given $\varepsilon > 0$ for sufficiently large n. Hence the arbitrary Cauchy sequence $\{\hat{x}_n\}$ in $\hat{F}(\mathbb{R})$ has the limit $\hat{x} \in \hat{F}(\mathbb{R})$, and $\hat{F}(\mathbb{R})$ is complete.

(d). Uniqueness of $\hat{F}(\mathbb{R})$ except for isometries.

If $(\check{F}(\mathbb{R}), \check{\delta})$ is another completion with a subspace Y dense in $\check{F}(\mathbb{R})$ and isometric with $F(\mathbb{R})$, then for any $\check{x}, \check{y} \in \check{F}(\mathbb{R})$ we have sequences $\{\check{x}_n\}$, $\{\check{y}_n\}$ in Y such that $\check{x}_n \to \check{x}$ and $\check{y}_n \to \check{y}$. Hence we have

$$\check{\delta}(\check{x}, \check{y}) \leq \check{\delta}(\check{x}, \check{x}_n) + \delta(\tilde{x}_n, \tilde{y}_n) + \check{\delta}(\check{y}_n, \check{y})$$

for every n, where $\{\tilde{x}_n, \tilde{x}_n, \cdots\} \in \check{x}_n$ and $\{\tilde{y}_n, \tilde{y}_n, \cdots\} \in \check{y}_n$. Since it is true for every n, it is true in the limit as n becomes infinite, which yields

$$\check{\delta}(\check{x}, \check{y}) \leq \lim_{n\to\infty} \delta(\tilde{x}_n, \tilde{y}_n).$$

But
$$\delta(\tilde{x}_n, \tilde{y}_n) \leq \check{\delta}(\check{x}_n, \check{x}) + \check{\delta}(\check{x}, \check{y}) + \check{\delta}(\check{y}, \check{y}_n)$$

which yields the reverse inequality. Hence

$$\check{\delta}(\check{x}, \check{y}) = \lim_{n\to\infty} \delta(\tilde{x}_n, \tilde{y}_n).$$

In a completly analogous manner, we can also show that

$$\hat{\delta}(\hat{x}, \hat{y}) = \lim_{n\to\infty} \delta(\tilde{x}_n, \tilde{y}_n).$$

Consequently,

$$\hat{\delta}(\hat{x}, \hat{y}) = \check{\delta}(\check{x}, \check{y}),$$

that is, the distance on $\check{F}(\mathbb{R})$ and $\hat{F}(\mathbb{R})$ must be the same. Hence $\check{F}(\mathbb{R})$ and $\hat{F}(\mathbb{R})$ are isometric. □

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