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Note on Calculation of Cnoidal Wave Parameters
크노이드波의 媒介變數 算定

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Abstract □ A new evaluation procedure for calculating the Jacobian elliptic parameter is presented. This procedure is useful in calculating the trajectory for cnoidal wave generation. Upon specification of water depth, the wave height and either the wave period or the wavelength, the presented algorithm uses the Newton-Raphson method and the arithmetic and geometric-mean scales to calculate the profile directly, without trial and error procedures or look-up in tables. It is shown that the algorithm provides equally accurate result as the ad hoc methods previously used.

要 旨：크노이드波의 發生에 필요한 Jacobian 媒介變數 算定에 관한 새로운 방법이 제안되었다. 提案된 方法에서는 水深, 波高 및 波의 週期 또는 波長이 주어지면, Newton-Raphson 方法과 算術 및 幾何平均方法을 이용하여 既存의 試行錯誤法 또는 圖表에 의한 방법을 사용하지 않고 直接 크노이드波의 媒介變數를 계산한다. 제안된 방법은 또한 기존의 施行錯誤法과 비교하여 過程은 더욱 單純하면서도 同 ·한 精確度를 갖는 結果를 提供한다.

1. INTRODUCTION

The evaluation of the parameters and of the trajectory for the cnoidal wave generation in shallow water is complex. The calculation of the surface profile involves the solution of four implicit simultaneous equations. This difficulty has led most previous applications (e.g., Wiegel, 1960; Svendsen and Hansen, 1977; Goring, 1978; Fenton, 1979; Sobey *et al.*, 1987) to either use trial and error type methods or to find the parameters from tables.

When we want to generate a kind of shallow water waves in numerical or in laboratory wave tank, especially a train of cnoidal waves, it is important to evaluate accurately the related parameters. In this note we introduce a direct and elegant procedure for evaluating the elliptic parameter using a Newton-Raphson method and we will show that the results are correct to the same order of accuracy as the previous methods.

2. GENERATION EQUATION

The specification of the horizontal velocity or the displacement of a wavemeker as a function of time is mathematically equivalent. For simplicity only refers to the horizontal displacement. The relationship between the free surface profile of generated long waves and the horizontal velocity of wavemaker is given by a trajectory equation derived from the continuity equation, i.e.,

$$\frac{d\xi}{dt}=u(\xi,\ t)=\frac{c\zeta}{h+\zeta} \tag{1}$$

in which ξ is the displacement of wavemaker, u the horizontal velocity component of wavemaker, c the phase speed, h the undisturbed water depth and ζ the free surface displacement of long wave.

Employing h as the characteristic length scale and $\sqrt{h/g}$ as the characteristic time scale, we may introduce the following dimensionless variables:

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$$t'=t\sqrt{g/h},\ (x',\ z')=(x,\ z)\frac{1}{h},\ (\zeta',\ H')=(\zeta,\ H)\frac{1}{h},$$

$$u'=\frac{u}{\sqrt{gh}}. \tag{2}$$

The primes will be dropped for simplicity. Using these dimensionless variables, the free surface profile of cnoidal wave can be written in the following dimensionless form (Goring, 1978; Liu and Cho, 1994):

$$\zeta(x,\ t)=z_t-1+H\ \mathrm{cn}^2\left[2K\left(\frac{x}{L}-\frac{t}{T}\right)\middle| m\right] \tag{3}$$

in which L denotes the wavelength, T the wave period, H the wave height and z_t the height of trough above the bottom. And cn is the Jacobian elliptic function, K the complete elliptic integral of the first kind and m the elliptic parameter. Substituting Eq. (3) into the dimensionless form of Eq. (1), we obtain the generation equation for cnoidal wave as:

$$\frac{d\xi}{dt}=c\left[1-\frac{1}{z_t+H\ \mathrm{cn}^2\left[2K\left(\frac{\xi}{L}-\frac{t}{T}\right)\middle| m\right]}\right]. \tag{4}$$

Eq. (4) is a first-order ordinary differential equation and can be integrated numerically using a fourth-order Runge-Kutta method (Synolakis, 1990; Liu and Cho, 1994).

The relationships among cnoidal wave parameters in dimensionless form can be written as (e.g., Mei, 1989):

$$z_t=\frac{H}{mK}(K-E)+1-H \tag{5}$$

$$HL^2=\frac{16}{3}mK^2 \tag{6}$$

$$L^2=T^2\left[1+\frac{H}{m}\left(2-m-3\frac{E}{K}\right)\right] \tag{7}$$

in which E represents the complete elliptic integral of the second kind. It is noted that the expression of (A.4) in Goring (1978) corresponding to equation (7) is not correct. There are several notational conventions in use for the Jacobian elliptic parameter, integrals and functions. We follow the notations used by Abramowitz and Stegun (1972) throughout the study. It is remarked here again that the free surface profile of a train of cnoidal waves can be determined once a water depth, a wave height and either a wave period or a wavelength are given.

3. EVALUATION OF ELLIPTIC PARAMETER

One of the difficulties in cnoidal wave problem is to evaluate the Jacobian elliptic parameter m, elliptic integrals K and E and function cn. We present an evaluation procedure for the Jacobian elliptic parameter in this section and elliptic integrals and elliptic function in the following section. Probably the most important task is evaluating m accurately because K, E and cn are all functions of m. Since in laboratory applications, the specification of the wave period is more practical than specifying the wavelength, the proposed algorithm calculates the surface profile from the wave period, the undisturbed water depth and the wave height. Substituting Eq. (7) into Eq. (6), we obtain an equation for the wave period T given as

$$T^2=\frac{16m^2K^2}{3H\left[m+H\left(2-m-3\frac{E}{K}\right)\right]}. \tag{8}$$

Because Eq. (8) is an implicit form for m, it should be solved by either an iterative numerical scheme or a trial and error type method. A trial and error type method is a plausible choice (Goring, 1978; Fenton, 1979). In this note we compute m with both a trial and error method (hereinafter TE) and the Newton-Raphson method (hereinafter NR). As pointed out by Sobey *et al.* (1987), m should be determined significantly accurate since the surface profile of cnoidal wave is highly sensitive to m. The details of TE can be found in Goring (1978) and will not be repeated here.

We describe the evaluation procedure of m with NR. We should know the first derivatives of K and E with respect to m to use NR. The relationships between the elliptic integrals and their derivatives with respect to m are given as (Jahnke and Emde, 1943; Arfken, 1985):

$$\frac{dK}{dm}=\frac{1}{2mm'}(E-m'K),\ \frac{dE}{dm}=\frac{1}{2m}(E-K) \tag{9}$$

in which m' is the complementary elliptic parameter, i.e. $m'=1-m$. Eq. (8) can be rewritten as

$$Y(m)=m+2H-mH-3H\frac{E}{K}-\frac{16m^2K^2}{3HT^2}. \tag{10}$$

Taking derivative of $Y(m)$ with respect to m and using Eq. (9), we obtain

$$\frac{dY}{dm}=1-H+\frac{3H}{2mm'K^2}(m'K^2+E^2-2m'KE)$$
$$-\frac{16mK}{3m'HT^2}(m'K+E). \tag{11}$$

Then, we can compute m using Eqs. (10) and (11) with NR. The procedure for NR is much simpler and more refined than that for TE. We also need to evaluate K and E to compute m using either TE or NR. The arithmetic and geometric mean scales (hereinafter AGM), the most efficient and accurate method (Goring, 1978), are used to compute K and E. The details of AGM will be described in the next section. If a wavelength is given instead of a wave period, Eq. (6) can be used directly and the basic principle remains the same. Computation of other parameters is straightforward if m is given.

4. EVALUATION OF ELLIPTIC INTEGRALS AND FUNCTION

4.1 Elliptic Integrals

We briefly describe two ways of evaluating the elliptic integrals, K and E (Abramowitz and Stegun, 1972): (i) AGM, (ii) the infinite series expansion. First, we can use AGM to evaluate the elliptic integrals. The calculation of elliptic integrals using AGM begins with a set of given numbers (a_o, b_o, c_o). Then, we proceed to determine successive sets of numbers: (a_1, b_1, c_1), (a_2, b_2, c_2),$\cdots$, (a_n, b_n, c_n) through the arithmetic and geometric means. That is,

step		
	a_o	b_o
1	$a_1=\frac{1}{2}(a_o+b_o)$	$b_1=(a_o\cdot b_o)^{1/2}$
2	$a_2=\frac{1}{2}(a_1+b_1)$	$b_2=(a_1\cdot b_1)^{1/2}$
⋮	⋮	⋮
n	$a_n=\frac{1}{2}(a_{n-1}+b_{n-1})$	$b_n=(a_{n-1}\cdot b_{n-1})^{1/2}$

$$c_o$$
$$c_1=\frac{1}{2}(a_o-b_o)$$
$$c_2=\frac{1}{2}(a_1-b_1)$$
$$\vdots$$
$$c_n=\frac{1}{2}(a_{n-1}-b_{n-1})$$

The calculation stops at nth step where $a_n=b_n$, i.e., $c_n=0$. The values of a_o, b_o and c_o are 1, $\sqrt{m'}$ and $\sqrt{m}$, respectively, to compute K and E. Finally, K and E can be computed as

$$K(m)=\frac{\pi}{2a_n},\ E(m)=K(m)\left[1-\sum_{k=0}^{n}2^{k-1}c_k^2\right]. \tag{12}$$

It is found that c_n is generally less than 10^{-17} with $n\leq 8$. To determine the incomplete elliptic integrals we can start with $a_o'=1$, $b_o'=\sqrt{m}$ and $c_o'=\sqrt{m'}$. After the same procedure the incomplete elliptic integrals of the first and second kind, K' and E', are given as

$$K'(m)=\frac{\pi}{2a_n'},\ E'(m)=K'(m)\left[1-\sum_{k=0}^{n}2^{k-1}(c_k')^2\right]. \tag{13}$$

Second, we can use the series expansion to compute K and E. For the range of $0<m<1$, the complete elliptic integrals can be expressed as

$$K(m)=\frac{\pi}{2}+\frac{\pi}{2}\sum_{n=1}^{\infty}\left[\frac{1}{2^{2n-1}}\frac{(2n-1)!}{(n-1)!n!}\right]^2 m^n \tag{14}$$

$$E(m)=\frac{\pi}{2}-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[\frac{1}{2^{2n-1}}\frac{(2n-1)!}{(n-1)!n!}\right]^2 m^n \tag{15}$$

in which the upper bound of the series could be truncated according to the degree of required accuracy. It is known that the series for K and E converge very slowly as m approaches to 1 (Arfken, 1985).

More rapidly converging series for K and E as m approaches to 1 are given in Jahnke and Emde (1943) and Dwight (1961) as:

$$K(m)=\Lambda+\frac{1}{4}\{\Lambda-1\}m'+\frac{9}{64}\left\{\infty-\frac{7}{6}\right\}(m')^2+\frac{25}{256}\left\{\Lambda-\frac{37}{30}\right\}(m')^3+\cdots \quad (16)$$

$$E(m)=1+\frac{1}{2}\left\{\Lambda-\frac{1}{2}\right\}m'+\frac{3}{16}\left\{\Lambda-\frac{13}{12}\right\}(m')^2+\frac{15}{128}\left\{\Lambda-\frac{6}{5}\right\}(m')^3+\cdots \quad (17)$$

in which $\Lambda=\ln 4-\ln\sqrt{m'}$ and Eqs. (16) and (17) are valid only for $m'\ll 1$. Wiegel (1960) used Eqs. (16) and (17) to calculate K and E when m approaches to 1, but there is a typographical error in expression of K in Wiegel (1960). In general, AGM can apply to any range of m, while a series expansion can apply to a limited range of m.

4.2 Elliptic Function

Two procedures are also introduced to estimate the Jacobian elliptic functioncn. First, the elliptic function, cn (θ), can be evaluated using AGM as:

1. Compute $\phi_n=2^n a_n\theta$ in radians with a given θ.
2. Calculate successively ϕ_{n-1}, ϕ_{n-2},$\cdots$,ϕ_1, ϕ_o from the recursive relation given as

$$\phi_{k-1}=\frac{1}{2}\left[\phi_k+\sin^{-1}\left(\frac{c_k}{a_k}\sin\phi_k\right)\right].$$

3. Finally evaluate cn $(\theta)=\cos\phi_o$.

in which n is the same as that used in Eq. (12). Goring (1978) also used AGM to evaluate of elliptic integrals and function.

Second, the Jacobian elliptic function cn (θ) can also be calculated by using the series expansion. Then, we need to know the incomplete elliptic integral of the first kind, K'. From Eqs. (12) and (13) the relationships between the complete integrals and incomplete integrals are

$$K'=K(m'),\ E'=E(m'). \quad (18)$$

Elliptic function cn ($\theta|m$) can be expressed in the series expansion as

$$\mathrm{cn}(\theta|m)=\frac{2\pi}{\sqrt{mK}}\sum_{k=0}^{\infty}\frac{q^{k+0.5}}{1+q^{2k+1}}\cos\left[(2k+1)\frac{\pi\theta}{2K}\right] \quad (19)$$

in which $q=\exp(-\pi K'/K)$. Synolakis *et al.* (1988) used Eq. (19) to study the run-up of cnoidal wave. It is noted that since both AGM and the series expansions for K and E include π implicitly or explicitly, the use of accurate π is essential.

5. NUMERICAL RESULTS

We apply the procedure proposed in previous sections to calculation of cnoidal wave parameters. Table 1 shows the elliptic parameters calculated using both a trial and error method and the proposed Newton-Raphson method. The calculated elliptic parameters agree each other up to 15 significant figures.

Fig. 1 shows the variation of elliptic integrals with respect to m. Both K and E approach $\pi/2$ as m goes 0, while K and E approach the infinity and the unity, respectively, as m goes to 1. The Jacobian elliptic integrals of the first kind for variable complementary elliptic parameters are plotted in Figs. 2 and 3, respectively. The series expansion (14) converges very slowly (500 terms are used for the series), while the series expansion (16) yields nearly same result as AGM for $m'\ll 1$ as shown in Fig. 2. However, in Fig. 3 the series expansion (15) produces nearly same result as AGM, while the series expansion (17) underestimates K for relatively larger m'. In conclusion, the series expansions given in (14), (15) and (16), (17) may not be used to compute K and E for the entire range of m'. In Fig. 2 and 3, the results of series expansions ((16) and (15) concide with those of AGM, respectively. The cnoidal wave profiles are shown in Fig. 4 with different m ranging from 0 to $1\text{-}10^{-12}$. The profile is very sensitive even for tiny change in m near to the unity. The profile can be calculated by either the

Table 1. The calculated elliptic parameter m (H=0.1)

T	trial and error	proposed
20.0	0.9391229648398143	0.9391229648398142
40.0	0.9997826914777293	0.9997826914777293
60.0	0.9999993065683998	0.9999993065683997
80.0	0.9999999977847914	0.9999999977847913
100.0	0.9999999999929184	0.9999999999929183

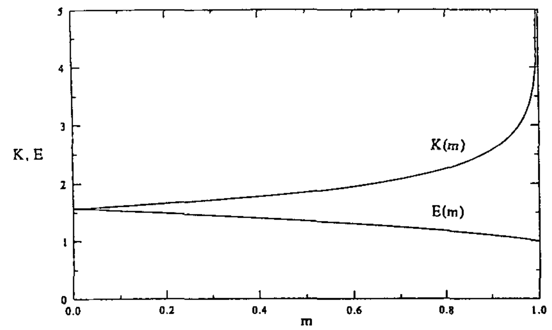

Fig. 1. The variation of Jacobian elliptic integrals of the first and second kinds.

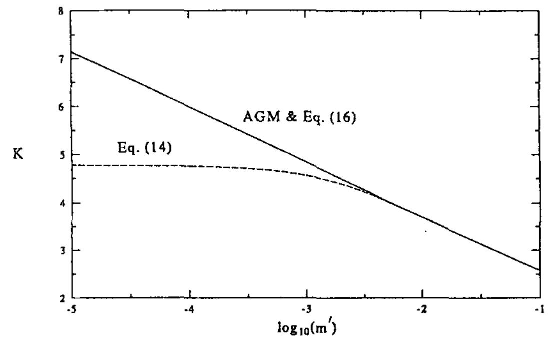

Fig. 2. The variation of Jacobian elliptic integral of the first kind.

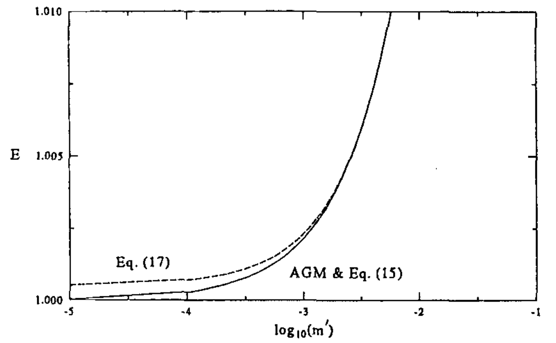

Fig. 3. The variation of Jacobian elliptic integral of the second kind.

arithmetic and geometric mean scales or Eq. (19). It is found that at least 120 terms, when Eq. (19) is used, should be used to obtain the same accuracy as the arithmetic and geometric mean scales for $m=1\text{-}10^{-12}$.

In this study we present a simple but accurate algorithm to calculate the Jacobian elliptic parameter. The algorithm consists of the Newton-Raphson method and the arithmetic and geometric mean scales. The presented algorithm can be efficiently and easily applied to calculation of a train of cnoidal waves either in numerical analysis or in laboratory experiment.

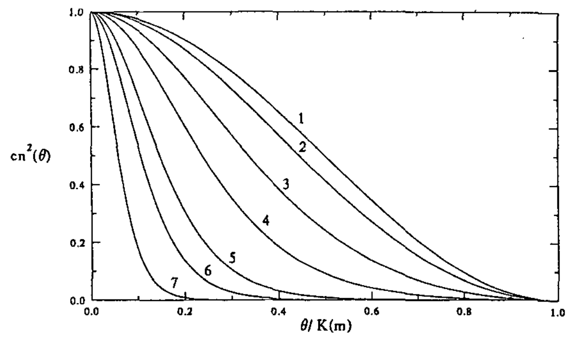

Fig. 4. The Jacobian elliptic function, $\mathrm{cn}^2(\theta)$, profiles: 1; m=0.0, 2; m=0.5, 3; m=0.9, 4; $m=1\text{-}10^{-2}$, 5; $m=1\text{-}10^{-4}$, 6; $m=1\text{-}10^{-6}$, 7; $m=1\text{-}10^{-12}$

Table 2. Limiting values as $m\to 0$ and 1

limit	$K(m)$	$E(m)$	z_t	L	cn(θ)
$m\to 0$	π/2	π/2	$h\text{-}H$	0	cos(θ)
$m\to 1$	∞	1	h	∞	sech(θ)

APPENDIX

It is well known that cnoidal wave has two limiting cases: sinusoidal wave and solitary wave. In this Appendix, We briefly show the limiting values of parameters and the phase speed, and thus limting cases of cnoidal wave, too. From the series equations K, E and cn have two limiting values as $m\to 0$ and 1, respectively, as listed in Table 2.

The phase speed of cnoidal wave (in dimensional form) approaches asymptotically to the phase speed of of sinusoidal and solitary waves as m goes 0 and 1, respectively. As m goes to 0, from Eq. (5)

$$c=\lim_{m\to 0}\sqrt{gh\left[1+\frac{H}{mh}\left(2-m-3\frac{E}{K}\right)\right]}$$
$$=\sqrt{gh\left(1-\frac{H}{mh}\right)}$$

$$=\sqrt{gh\left[1-\frac{1}{3}\left(\frac{2\pi}{L}\right)^2h^2\right]}$$
$$=\sqrt{gh\left[1-\frac{1}{3}(kh)^2\right]}$$
$$=\sqrt{\frac{g}{h}\tanh kh} \tag{20}$$

in which Eq. (4) has been used to replace m. As m goes to 1

$$c=\lim_{m\to 1}\sqrt{gh\left[1+\frac{H}{mh}\left(2-m-3\frac{E}{K}\right)\right]}$$
$$=\sqrt{gh\left(1+\frac{H}{h}\right)}. \tag{21}$$

Eqs. (20) and (21) represent phase speeds of sinusoidal and solitary waves, respectively. The free surface profile of cnoidal wave also approaches asymptotically to the free surface profiles of sinusoidal and solitary waves as m goes 0 and 1, respectively, that is

$$\zeta(x,\ t)=\lim_{m\to 0}\left[z_t-h+H\ \mathrm{cn}^2\left\{2K\left(\frac{x}{L}-\frac{t}{T}\right)|m\right\}\right]$$
$$=-H+H\cos^2\left[\pi\left(\frac{x}{L}-\frac{t}{T}\right)\right]$$
$$=\frac{H}{2}\cos\left[2\pi\left(\frac{x}{L}-\frac{t}{T}\right)\right] \tag{22}$$

$$\zeta(x,\ t)=\lim_{m\to 1}\left[z_t-h+H\ \mathrm{cn}^2\left\{2K\left(\frac{x}{L}-\frac{t}{T}\right)|m\right\}\right]$$
$$=h-h+H\mathrm{sech}^2\left[2K\left(\frac{x}{L}-\frac{t}{T}\right)\right]$$
$$=H\mathrm{sech}^2\left[\sqrt{\frac{3H}{4h^3}}(x-ct)\right] \tag{23}$$

in which $K/L=\sqrt{3H/(16h^3)}$ and $c=L/T$ have been used. Eqs. (22) and (23) denote surface profiles of sinusoidal and solitary waves, respectively.

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