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Numerical Simulation of Wave Motions in Ideal Fluid of a Finite Depth

by

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유한수심인 이상유체에서의 자유표면파의 수치모사

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Abstract

The present paper is devoted to constructing a numerical algorithm for solving unsteady problems on generation, propagation and interaction of nonlinear waves at a surface of ideal fluid, within the framework of the potential-flow model. The numerical scheme is implicit, with non-linearity iteration at every step of time. the finite-difference method with boundary-fitted coordinates are presented in favor for validity and high efficiency of the numerical model developed. Among these arguments, there are the results of calculations of two test problems-on stretching of a liquid ellipse and on wave generation by lifting a portion of a bottom.

요 약

본 논문에서는 이상유체모델에서의 비선형 자유표면파의 발생, 전파 및 상호간섭에 대한 비정상 문제의 수치해법을 개발하였다. 본 수치해법은 매 시간 스텝에서 비선형 축차해법을 이용한 음함수적(implicit)방법이다. 속도장함수를 구하기 위하여 경계접합좌표계를 도입한 유한차분법을 이용하였다. 본 수치해법의 유효성과 효율성의 검증을 위하여 타원형 유체의 변형과 바닥의 일부분이 올라옴으로서 발생하는 자유표면파의 생성에 대한 두가지 계산결과를 보여준다.

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1. Introduction

The problem on unsteady motion of incompressible, inviscid fluid with a free surface is very complicated because of necessity to satisfy the couple of nonlinear conditions on some boundary parts unknown a priori (free boundaries). As a result, rigorous theoretical analysis of the problem is very hard and numerical methods are used intensively to calculate the problem for a wide variety of particular initial and boundary data. A review of these methods can be found in Young [14] as well as in Fletcher [4,5]. Most of the methods have various advantages. But since every numerical method has some disadvantages, new, more developed techniques continue to appear.

In constructing a numerical algorithm most of difficulties are associated either with treatment of evolutionary equations (the kinematic and dynamic conditions at a free surface) or calculating the potential field in the flow domain at a fixed moment of time . To compute the evolutionary equations, here the implicit Crank-Nicholson scheme with non-linearity iteration is applied. Such kind of scheme provides a high reserve of stability, second-order accuracy and is rather popular (Young [14], Haussling [7], Asaithambi [1]). The peculiarities of the present application include a conservative form of the kinematic and dynamic conditions and the optimal co-location of grid nodes for different function involved in these conditions. No smoothing procedure is used.

To calculate the potential field, i. e. to solve the elliptic problem, the finite-difference method, with boundary-fitted coordinates, is chosen. This technique is described in detail by Thompson et al. [11] and successfully applied by many authors, in particular, by Haussling [7], Asaithambi [1], Daiguji & Shin [2], Young & Vaidhyanathan [15] and so on. The finite-difference method is more simple to be realized and usually leads to linear-equation systems of simple structure (with tri-diagonal matrices), contrary to other alternative methods- the finite element and the boundary integral equation ones (see Young [14]). Some difficulties associated with governing equations getting worse as a consequence of variable exchange seem to be not so principle and can be successfully overcome what has been done here. The primary ideas introduced are the reformulation of the elliptic problem in terms of three unknown functions, the most optimal co-location of grid nodes for different quantities to be found and the application of the economic method of fractional steps. As a result, the boundary conditions are implemented exactly and the finite-difference scheme becomes homogeneous: symmetric approximations are exploited only, no one-sided differences are used to represent derivatives near domain boundaries.

The high efficiency of the numerical algorithm developed is confirmed by the results of test calculations.

2. Problem formulation

Let the domain occupied by fluid be, at every moment of time t, curvilinear rectangle $\Omega(t)=\{-l_1 \le x \le l_2,\ -h(x,t) \le y \le \eta(x,t)\}$ bounded below by uneven movable bottom $y=-h(x,t)$, above - by free surface $y=\eta(x,t)$, from the left and fight sides - by vertical boundaries $x=l_1$ and $x=l_1$, penetrable in a general case. Here the right-hand coordinate system is chosen with axis Ox directed horizontally from left to right and axis Oy pointes vertically upwards. Fluid motion is assumed to be two-dimensional. Such restriction has been accepted to save computer resources, however, the mathematical formulation of problem and the method of its solution can be easily generalized to treat the three-dimensional case.

Fluid is supposed to be homogeneous, incompressible, inviscid. Its motion occurs under gravitation. Surface tension is not taken into account as negligibly small in the problems to be considered. The following factors are permissible as the reasons of water motion(in any combination): initial non-equlibriving condition of water, action of water pressure, movement of a bottom, given water flows through the side boundaries. It is required to determine water motion, i. e. to find free-sur-face configuration and distribution of velocity and pressure in the flow domain.

In assumption of the absence of initial rotation and with regard to the fluid properties and to the set of forces taken into consideration, fluid motion is to be potential

$$\vec{U} = (U, V) = \nabla\Phi(x, y, t) = (\Phi_x, \Phi_y) \qquad (2.1)$$

Here and henceforth, letter subscripts denote partial differenttiation with respect to corresponding variables.

Potential Φ is known to be a harmonic function

$$\Phi_{xx} + \Phi_{yy} = 0 \qquad \text{in} \qquad \Omega(t) \qquad (2.2)$$

and to satisfy the second-type conditions on the domian boundary

$$\Phi_y - \eta_x\Phi_x = \eta_t \qquad \text{on} \quad y = \eta(x,t), \qquad (2.3)$$

$$\Phi_y - h_x\Phi_x = -h_t \qquad \text{on} \quad y = -h(x,t), \qquad (2.4)$$

$$\Phi_x = U_1(y,t) \qquad \text{on} \quad x = -l_1, \qquad (2.5)$$

$$\Phi_x = U_2(y,t) \qquad \text{on} \quad x = l_2. \qquad (2.6)$$

Equation (2.2) expresses the incompressibility property of fluid, (2.3), (2.4), are the conditions of its inability to pass through the surface and bottom. In contrast, the left- and right-side boundaries are considered to be penetrable, the fluid flows through them are defined by relations (2.5), (2.6). Since function $\eta(x,t)$is unknown a priori, condition (2.3) is not sufficient and must be completed by another - the dynamic one expressing the pressure continuity in passing through the surface

$$P(x, \eta(x,t), t) = p(x,t) \qquad (2.7)$$

Here $P(x,t)$is the given outer pressure, $P(x,y,t)$is the pressure inside fluid, defined, as known, by the Cauchy-Lagrange integral (an absolute value of gravitational acceleration as well as fluid density have been equaled to unity)

$$P(x,y,t) = -(\Phi_t + \frac{1}{2}(\Phi_x^2 + \Phi_y^2) + y) \qquad (2.8)$$

Equations (2.1)-(2.8) allow the fluid motion to be determined (i. e. domain $\Omega(t)$ and the velocity-pressure field to be found) from some initial condition. This formulation is quite traditional but seems to be convenient rather for theoretical analysis, when it is desirable to make a number of unknown functions as small as possible. But in order to construct an efficient numerical scheme, some reductions of the formulation should be done. To do so, we introduce the following additional notations

$$d(x,t) = \eta + h, \qquad (2.9)$$

$$q(x,t) = \int_{-h}^{\eta} \Phi_x dy, \qquad (2.10)$$

$$u(x,t) = \left[\Phi_x + \eta_x\Phi_y\right]_{y=\eta}, \qquad (2.11)$$

$$v(x,t) = \left[-\Phi_y + \eta_x\Phi_x\right]_{y=\eta}. \qquad (2.12)$$

Here $d(x,t)$is the water depth, $q(x,t)$ is the fluid flow through a vertical cross-section. Functions $u(x,t)$ and $v(x,t)$ mean the tangential and normal velocity components at a free surface (with accuracy to a sign and normalizing multiplier).

Now we can write the governing equations as follows

$$d_t + q_x = 0, \qquad (2.13)$$

$$u_t + \delta_x = 0, \qquad (2.14)$$

$$q = Q(\eta)u, \qquad (2.15)$$

$$\delta(x,t) = \eta + p + \frac{1}{1+\eta_x^2}\left[\frac{1}{2}(u^2 - v^2) + \eta_x uv\right] \qquad (2.16)$$

$$\eta = d - h, \tag{2.17}$$

$$v = h_t + q_x. \tag{2.18}$$

Equation (2.13) is obtained by integrating the Laplace equation with respect to vertical coordinate y and implementing boundary conditions (2.3), (2.4). It expresses the mass conservation law for an infinitely thin column of water. Being equivalent to the kinematic condition (2.3), it is, however, more preferable because makes it possible to achieve exact conservation of mass by applying some conservative numerical scheme.

Equation (2.14), with function $\delta(x,t)$ being defined by (2.16), is the projection of the momentum equation onto a free surface written in terms of notations (2.11), (2.12) and with account of dynamic condition (2.7). Equation (2.14) can be integrated with respect to x yielding the Cauchy-Lagrange integral (relations (2.7), (2.8)) explored usually. But such integration seems to be fruitless for constructing a numerical algorithm because actually keeps space differentiation. And as previously, due to divergent type of the equation, it is sufficient to apply some conservative scheme to provide exact conservation of momentum.

Relations (2.17), (2.18) are simple consequences of (2.9) and (2.12), (2.3), (2.9), (2.13), correspondingly. As far as water flow q is concerned, it depends upon tangential velocity u and free-surface elevation η. Unfortunately, this dependence cannot be expressed explicitly and is written in the form (2.15), with help of operator Q(linear with respect to u), action of which will be defined later.

As it can be seen, the problem on fluid motion with a free surface is reduced to the typical wave problem (see Whitham [12]): it is required to find a couple of functions - water depth $d(x,t)$and tangential velocity component $u(x, t)$, evolution of which, from some initial conditions

$$d = d_o(x) \qquad \text{at} \quad t = 0, \tag{2.19}$$

$$u = u_o(x) \qquad \text{at} \quad t = 0, \tag{2.20}$$

is governed by equations (2.13), (2.14) of mass and momentum conservation, with additional relations (2.14)-(2.18), completing the system. The problem, being one dimensional (all quantities involved are functions of x, t only, dependence on y is "hidden" in operator $Q(\eta)$), is, however, quite difficult because of nonlinearity of (2.15), (2.16) (with respect to aggregate of d, u and absence of an explicit expression for operator $Q(\eta)$ Action of this operator is defined by relation (2.10), where $\Phi(x,y,t)$ is the solution of elliptic problem (2.2), (2.4)-(2.6), (2.11). Condition (2.11) is helpful to be integrated along boundary $y=\eta(x,t)$ to yield

$$\Phi = \phi \qquad on \quad y = \eta(x,t) \tag{2.21}$$

where

$$\phi(x,t) = \phi_0(t) + \int_0^x u(\xi,t)d\xi \tag{2.22}$$

with arbitrary function $\phi_o(t)$.

Thus, the problem on fluid motion with a free surface is formulated with distinguishing the evolutionary part (eqs. (2.13)-(2.20)) and the elliptic one (eqs. (2.2), (2.4)-(2.6), (2.21)), connected with each other by means of relations (2.10), (2.22). Variable y is absent in the evolutionary problem, similar, time t takes part in the elliptic one only as a parameter. Such decomposition of the general problem on two correlated subproblems, in every of which a number of independent variables is reduced on unity, seems to be helpful for constructing a numerical algorithm.

3. Numerical scheme for calculating the elliptic problem

To develop a numerical scheme for solving the elliptic problem(2.2) (2.4)-(2.6), (2.21), one must not only take into consideration common requirements to guarantee the solution of discrete problem to converge to that of differential one, but also pay much attention to the efficiency of algorithm because of necessity to solve the problem many times repeatedly (at every time step). Hence, there must be efficient enough the procedure of constructing the coefficient matrix for the linear-equation system(discrete analogy of the problem) as well as the procedure of solving this system.

A finite-difference scheme for calculating an elliptic problem similar to the present one can be developed with one of three basic methods: of finite differences(FD), of finite elements(FE) and of boundary integral equations(BIE) (see Yeung[14]). In "good" domains(with boundaries coinciding with coordinate lines)the finite-difference method is thought to satisfy better the efficiency requirements mentioned above, because this method is more simple to be realized. However, in case when a boundary contains, like in this problem, curvilinear and, moreover, movable parts, it becomes problematic to satisfy the conditions on this parts without noticeable decrease of method efficiency - accuracy and economic feasibility. Accuracy of satisfaction of boundary conditions can be improved by fitting the computational grid to domain boundaries (with help of suitable exchange of variables) that usually results in decreasing the efficiency of the FD method making it comparable with the methods FE and BIE less dependent on flow geometry.

In this problem, however, it occurs possible to adapt the computational grid without serious consequences, i. e. without making more complicated both the procedure of constructing coefficient matrices for linear-equation systems and the procedure of solving these systems, due to simplicity of a variable transformation and possibility to apply the economic methods of fractional steps(Yanenko [13]), correspondingly. This has predetermined the choice of the finite-difference method to develop an algorithm for calculating the problem.

Let's consider variable exchange $(x, y) \to (\xi, \zeta)$, which is supposed to be not degenerate ($J = \det[\partial(x, y)/\partial(\xi, \zeta) \neq 0$)and transforms flow domain $\Omega(t)$ into rectangle $\pi = \{\xi_0 \ll \xi \ll \xi_\infty \ll \zeta_0 \ll \zeta \ll \zeta_\infty\}$(time moment t will be fixed and within this Paragraph time-dependence of all functions under consideration will not be indicated). In terms of new variables the Laplace equation takes the form

$$(\alpha\Phi_\xi + \beta\Phi_\zeta)_\xi + (\beta\Phi_\xi + \gamma\Phi_\zeta)_\zeta = 0, \quad \text{in } \pi, \tag{3. 1}$$

where coefficients α, β, γ are defined as follows

$$\alpha(\xi,\zeta) = (x_\zeta^2 + y_\zeta^2)/J, \tag{3.2}$$

$$\beta(\xi,\zeta) = -(x_\xi x_\xi + y_\xi y_\xi)/J, \tag{3.3}$$

$$\gamma(\xi,\zeta) = (x_\zeta^2 + y_\zeta^2)/J, \tag{3.4}$$

$$J(\xi,\zeta) = x_\xi y_\zeta - x_\zeta y_\xi. \tag{3.5}$$

The following quantity remains invariant

$$\alpha\gamma - \beta^2 = 1 \tag{3.6}$$

It seems useful to introduce notations

$$U(\xi,\zeta) = \alpha\Phi_\xi + \beta\Phi_\zeta, \quad V(\xi,\zeta) = \beta\Phi_\xi + \gamma\Phi_\zeta, \tag{3.7}$$

in terms of which(3. 1) takes the form of continuity equation

$$U_\xi + V_\zeta = 0 \quad \text{in } \pi, \tag{3.8}$$

and boundary conditions (2.4)-(2.6), (2.21) are rewritten as follows

$$V = -x_\xi h_t \quad \text{on } \zeta = \zeta_O \tag{3.9}$$

$$U = y_\xi U_2 \quad \text{on } \xi = \xi_O \tag{3.10}$$

$$U = y_\xi U_2 \quad \text{on } \xi = \xi_K \tag{3.11}$$

$$\Phi = \phi \quad \text{on } \zeta = \zeta_K \tag{3.12}$$

In perspective, the following dependencies, resulting from (3.6), (3.7), will occur helpful

$$\Phi_\xi = \gamma U - \beta V, \quad \Phi_\zeta = \alpha V - \beta U \qquad (3.13)$$

To calculate elliptic equation (3.1), the iterative scheme with stabilizing correction is applied (Douglas & Rachford [3], Yanenko [13]), which belongs to economic methods of fractional steps. In doing so, splitting is made with conserving the divergent type of the equation

$$\frac{\Phi^{k+1/2}-\Phi^k}{\omega} = U_\xi^{k+1/2} + V_\zeta^k, \quad \frac{\Phi^{k+1}-\Phi^{k+1/2}}{\omega} = V_\zeta^{k+1} - V_\zeta^k \qquad (3.14)$$

Here the superscripts denote an iteration number, ω is an iteration parameter, a sign of which must be agreed with that of Jacobian : $\omega J > 0$.

The system (3.14) may be completed with help of

$$U^{k+1/2} = \alpha\Phi_\xi^{k+1/2} + \beta\Phi_\zeta^k, \quad V^{k+1} = \beta\Phi_\xi^{k+1/2} + \gamma\Phi_\zeta^{k+1} \qquad (3.15)$$

But in order to avoid some difficulties associated with approximation of mixed derivatives (i. e. Φ_ζ in the expression for U, Φ_ξ - for V) near the domain boundaries, instead of (3.15) it is more convenient to explore relations

$$U^{k+1/2} = (\Phi_\xi^{k+1/2} + \beta_\zeta^k)/\gamma, \quad V^{k+1} = (\Phi_\zeta^{k+1} + \beta U^{k+1/2})/\alpha \qquad (3.16)$$

based on (3.13). Finally, by adding the boundary conditions to (3.14), (3.16), we put the finite-difference scheme to its complete form

$$1. \quad \Phi^{k+1/2} - \omega U_\xi^{k+1/2} = \Phi^k + \omega V_\zeta^k \qquad (3.17)$$

$$U^{k+1/2} = (\Phi_\xi^{k+1/2} + \beta V^k)/\gamma \qquad (3.18)$$

$$U^{k+1/2} = y_\zeta U_1 \quad \text{on} \quad \xi = \xi_0 \qquad (3.19)$$

$$U^{k+1/2} = y_\zeta U_2 \quad \text{on} \quad \xi = \xi_K; \qquad (3.20)$$

$$2. \quad \Phi^{k+1/2} - \omega V_\zeta^{k+1/2} = \Phi^k - \omega V_\zeta^k \qquad (3.21)$$

$$V^{k+1} = (\Phi_\zeta^{k+1} + \beta U^{k+1/2})/\alpha \qquad (3.22)$$

$$V^{k+1} = -x_\xi h_t \quad \text{on} \quad \zeta = \zeta_0 \qquad (3.23)$$

$$\Phi^{k+1} = \phi \quad \text{on} \quad \zeta = \zeta_K; \qquad (3.24)$$

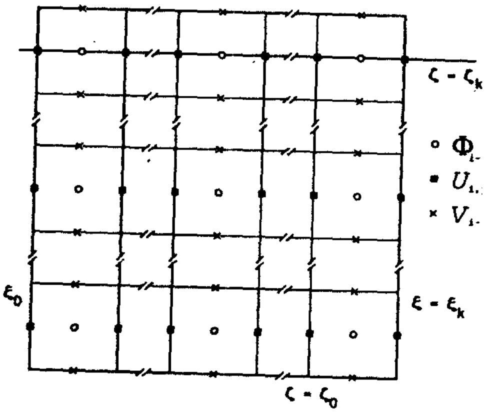


Fig. 1 Grid system for the sliptic problem

For treatment of scheme (3.17)-(3.24) the grid with nodes placed in "chess" order seems to be the most suitable (see Fig. 1). This grid is widely used for solving hydrodynamic problems formulated in terms of "velocity-pressure" (here the potential plays a role of pressure). Rectangle π is covered by the grid with square cells of size æ× æ. Nodes $\Phi_{i-1/2,j-1/2} = \Phi(\xi_{i-1/2}, \zeta_{j-1/2})$ ($\xi_\upsilon = \xi_0 + \upsilon æ$) are placed in the centers of these cells, nodes $U_{i-1,j-1/2}$, $U_{i-1,j-1/2}$ lie in the middle points of the left and right sides of cells and nodes $V_{i-1/2,j-1}$, $U_{i-1/2,j}$ are disposed in the middle points of the upper and lower sides cells ($i=1, 2, \cdots, M$; $j=1,2, \cdots, N$). In accordance with the boundary conditions, nodes $U_{0,j-1/2}$, $U_{M,j-1/2}$ are placed on side boundaries $\xi=\xi_0$ and $\xi=\xi_K$, nodes $V_{i-1/2,0}$- on bottom $\zeta=\zeta_0$, nodes $\Phi_{i-1/2,N-1/2}$- on free surface $\zeta=\zeta_K$ (this occurs to be possible for arbitrary æ, M, N due to the proper choice of the rectangle size: $\xi_K - \xi_0 = Mæ$, $\zeta_K - \zeta_0 = (N-1/2)æ$). In doing so, nodes $V_{i-1/2,N}$ turn out to be located outside the flow domain and, as usually in such cases, to find the fictive values of V, it is necessary to implement boundary condition (3.24) as well as equation(3.21).

The chosen disposition of grid nodes makes it possible to approximate derivatives by the symmetric differences only, values of functions between their nodes are defined, if it is necessary, by the half-sums, for example

$$(U_\xi)_{i-1/2,\ j-1/2} = (U_{ij-1/2} - U_{i-1j-1/2}) / æ$$

$$U_{i-1/2,\ j-1/2} = (U_{ij-1/2} + U_{i-1j-1/2}) / 2$$

Scheme realization is reduced to solving the set of linear-equation systems with tri-diagonal matrices. These systems are obtained by discretizing equation (3.17) (at the first half-step) and substituting, instead of U, discrete analogies of expression (3.18) or boundary values (3.19), (3.20). At the second half-step the procedure is similar.

The scheme is developed for an arbitrary variable exchange, in particular, an implicit dependence (by means of differential equations) of new variables on old ones may be used, which allows the computational grid to be fitted to a boundary (a free surface or a bottom), dependent non-uniquely on the horizontal coordinate(see Haussling[7], Thompson et al.[11], Yeung [14], Yeung& Vaidhyanathan[15]). Here, however, non-unique boundaries will not occur in the problems considered below, so that their solution may be obtained with use of simple algebraic dependencies $\xi=\xi(xy)$, $\zeta=\zeta\ (xy)$, making the surface and bottom straight (shift and stretching along axis are assumed to be absent, i. e. $\xi_0 = -l_1$, $\xi_K = l_2$)

$$\xi = x,\ \ \zeta = (\zeta_0(\eta - y) + \xi_K(h = y)) / d \quad (3.25)$$

Relations (3.25) lead to the following expressions for the coefficients, involved in equations (3.18), (3.22) and boundary conditions (3.19), 93.20), (3.23)

$$\begin{aligned} &x_\xi = 1,\ \ x_\zeta = 0, \\ &y_\xi = ((\zeta - \zeta_0)\eta_\xi + (\zeta - \zeta_K)h_\xi) / (\zeta_K - \zeta_0), \\ &y = d((\zeta_K - \zeta_0), \\ &\alpha = y_\zeta,\ \ \beta = -y_\xi,\ \ \gamma = (1+\beta^2)/\alpha \qquad (3.26) \end{aligned}$$

In this case the Jacobian of variable transformation is equal to y_ζ and the condition of its being different from zero is equivalent to the natural condition of free-surface being not intersected with bottom: $d \neq 0$.

4. Numerical algorithm for calculating the evolutionary problem

Evolutionary equations (2.13), (2.14) are approximated with the Crank-Nicolson Scheme

$$d^{n+1,\ k+1} = d^n - \frac{\tau}{2}(q_x^{n+1,\ k} + q_x^n) \quad (4.1)$$

$$u^{n+1,\ k+1} = u^n - \frac{\tau}{2}(\delta_x^{n+1,\ k} + \delta_x^n) \quad (4.2)$$

Where τ is a time step, the first superscript denotes a time step number, the second one - an iteration number. An iterative procedure is necessary at every time step because of the scheme implicity and the nonlinearity of dependencies of functions q, δ, on quantities d, u to be found(relations (2.15), (2.16)).

The Crank-Nicolson scheme is used extensively for calculations of fluid motion with a free surface (see Yeung [14], Asaithambi[1], Haussling[7]. Scheme popularity may be explained by the following its advantages: the second-order accuracy for smooth solutions, absolute stability and absence of dissipation in linear problems, simplicity accuracy for smooth solutions, absolute stability and absence of treatment.

Space discretization is based, like in the elliptic problem, on the idea of disposition of grid nodes for different functions: nodes $d_{i-1/2}$, $\delta_{i-1/2}$, $\eta_{i-1/2}$, $v_{i-1/2}$, $h_{i-1/2}$, $P_{i-1/2}$ are placed at points $x_{i-1/2}$ (i= 1, 2, ⋯, M), nodes q_i, u_i lie at points x_i (i= 0, 1, ⋯, M). Here x_v is defined as $-l_1+v_æ$. Values of derivatives q_x, δ_x between nodes of functions q,δ are approximated by the symmetric differences.

Nonlinearity iteration in the evolutionary problem is combined with iteration for solving the elliptic problem, i. e. when dependence $q^k = Q(\eta^k)u^k$ is being realized(the first superscript is omitted) only one step of scheme (3.17)-(3.24) is fulfilled. In other words, in the elliptic problem the boundary value of potential as well as the coefficients become dependent upon an iteration number: in equations(3.17-(3.24), instead of ϕ, α, β, γ there should be written ϕ^{k-1}, α^{k-1}, β^{k+1}, γ^{k-1} with the latter being defined by (2.22), (3.26) with use of u^{k-1}, η^{k-1}.

5. Test calculations of the liquid ellipse problem

As a test, here has been considered the liquid ellipse problem, a seldom example of a free-surface problem having an exact solution in nonlinear formulation (more precisely, solving the problem is reduced to solving a single ordinary, differential equation: Ovsyannikov[10]). This solution describes such kind of fluid motion in the absence of gravitation, when the flow domain boundary at every moment of time coincides with ellipse $(x/\alpha)^2+(\alpha y)^2=1$, where $\alpha=\alpha(t)$is its horizontal half-axis, and the velocity field possesses potential $\tilde{\Phi}(x, y, t)=(\dot{\alpha}/2\alpha)(x^2-y^2)$, which is harmonic and also satisfies the kinematic and dynamic conditions(nonlinear) on the boundary if only evolution of the ellipse half-axis is governed by equation(point above denotes differentiation with respect to t)

$$\dot{\alpha}=c\frac{\alpha^2}{\sqrt{1+\alpha^4}}$$

with some initial value $\alpha_0=\alpha(0)$ and constant, defining the direction - along which axis, x(c>0) or y(c<0), - and the quickness of ellipse stretching in time. The formulation of given problem can be written in form of (2.13)-(2.20), (2.10), (2.2), (2.4)-(2.6), (2.21), (2.22), if to take, as fluid domain Ω(t), the piece of ellipse bounded at the left and below by symmetry axes $x=0$ and $y=0$, at the right - by vertical penetrable boundary $x=l$ ($\iota<\alpha_0$), above - by a free surface, and in the boundary and initial conditions to state

$$\begin{gathered} h(x,t)\equiv 0,\ p(x,\ t)\equiv -\eta(x,t), \\ l_1=0,\ l_2=l,\ U_1(y,t)\equiv 0,\ U_2(y,t)=\dot{\alpha}l/\alpha, \\ \eta_o(x)=\frac{1}{\alpha_o}\sqrt{1-(x/\alpha_o)^2},\ u_0(x)=\alpha_1(1+1/\alpha_0^4(x/\alpha_0) \end{gathered} \quad (5.1)$$

where $\alpha_1=\dot{\alpha}(0)=c\alpha_0^2/\sqrt{1+\alpha_0^4}$.

According to the problem statement, gravitation is absent that means member η in the right-hand side of (2.16) should be omitted (multiplied by the zero coefficient). However, the absence of gravitation is possible to be simulated without rewriting equation(2.16) but with posing fictive surface pressure to neutralize gravitation ($\eta+p=0$) what is expressed with the second relations (5.1).

In calculations, there have been used values $\alpha_0=1$, $\alpha_1=-1$($\because c=-2$), defining the initial condition of fluid and the direction of ellipse stretching - vertical, which is more interesting in comparison with horizontal, because essential stretching of the flow domain along y-axis in combination with sufficient surface deformation provides extreme conditions for testing the algorithm. Value $l=1/4$ of domain length used here makes it possible to calculate the problem up to moment $t=\sqrt{2}$(computation time is limited by the chosen value of l:$0\le t<t_*$,$\alpha(t_*)=l$).

Comparison of calculated values with exact ones(marked above by a wave) has been carried out for three functions - d, u, q, i. e. the following relative errors have been checked

$$\gamma_1(t)=\|d-\tilde{d}\|/\|\tilde{d}\|$$

$$\gamma_2(t)=\|u-\tilde{u}\|/\|\tilde{u}\|$$

$$\gamma_3(t)=\|q-\tilde{q}\|/\|\tilde{q}\|$$

where $\|f(x,t)\|$should be understood as

$$\|f(x,t)\|_{C_{\mathfrak{A}}}=\max_i|f_i(t)|=\max_i|f(x_i,t)|$$

and functions $\tilde{d}(x,t)$, $\tilde{u}(x,t)$, $\tilde{q}(x,t)$, in accordance with the written above exact expressions for the domain boundary and potential $\tilde{\Phi}(x,y,t)$, have forms

$$\tilde{d}(x,t)=\sqrt{1-(x/\alpha)^2}/\alpha=\tilde{\eta}(x,t)$$

$$\tilde{u}(x,t)=\dot{\alpha}(1+1/\alpha^4)(x/\alpha)$$

$$\tilde{q}(x,t) = \dot{\alpha}(x / \alpha)\sqrt{1-(x / \alpha)^2} / \alpha = \dot{\alpha}(x / \alpha)\tilde{\eta}$$

The results of calculation of the fluid surface evolution are presented on Fig. 2, where the surface shape is shown for time moments t=0, 1, 2(in order of numeration). The solid lines correspond to the exact solution, the points - to the

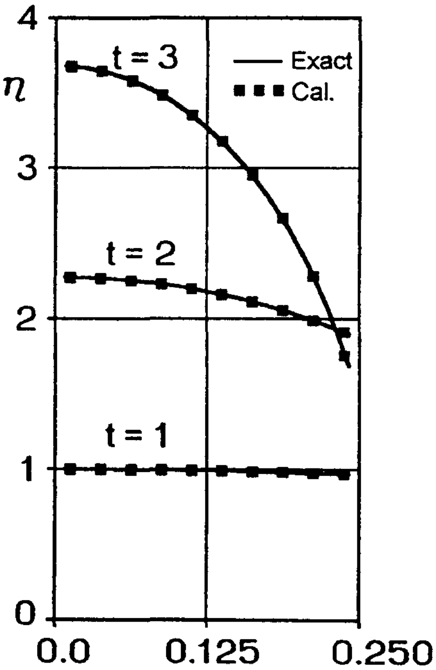


Fig.2 Free surface evolution in the liquid ellipse problem

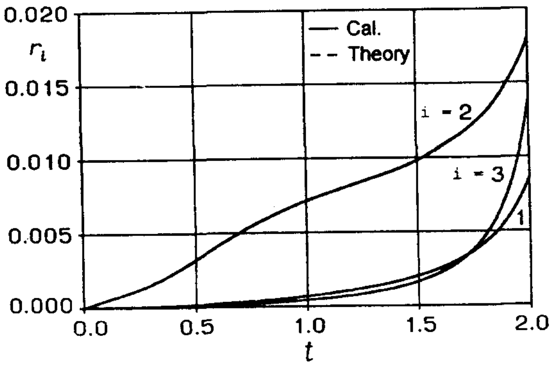


Fig. 3 Relative erroes $\gamma_i(t)$, i= 1,2,3

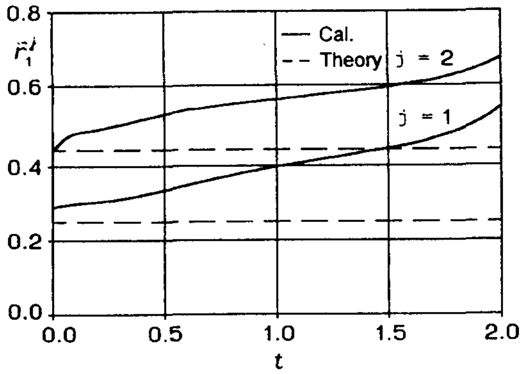


Fig. 4 Convergence rate $\gamma^{j}{}_{1}(t)$= 1,2

calculation with grid parameters M=10, N= 20, $\tau/_{\ae}$- 1/2. To a scale of the Figure, the numerical results actually coincide with the exact solution. But difference yet exists and can be observed on Fig. 3, where relative errors $\gamma_i(t)$ (i=1,2,3) are presented. The significant increase of functions $\gamma_i(t)$ may be explained by the essential deformation of flow domain: as it can be seen from Fig. 2, the liquid ellipse becomes almost 4 times greater in vertical and, besides, the slope of tangent to the surface at the right-side boundary grows quickly(in an absolute value), achieving the great enough value by time moment t=2: $\eta_x(l,2)\approx$-35.

In order to analyze the convergence of the solution of finite-difference problem to the solution of differential one, there have been performed the calculations with use of different grids -with steps $\ae$= $2l/(jM)$, where values of l, M are mentioned above, j= 1, 2, 3. When the grid step was varied, ratios M/N, $\tau/_{\ae}$ and $\omega/_{\ae}^{2}$ were kept constant. The computation results are shown on Fig. 4, Where each curve with number j(j= 1, 2) represents function $\gamma^{j}{}_{1}(t)$ $=\gamma^{j+1}{}_{1}(t)$ /$\gamma^{j}{}_{1}(t)$, with $\gamma^{j}{}_{1}(t)$ being error $\gamma_1(t)$ calculated with grid step $\ae_j$. All over the time interval considered, the convergence of the numerical solution to the exact one, i. e. the error reduction with the grid step decrease, may be observed: $\gamma^{j}{}_{1}$<1. But the actual rate of convergence is lower than the theoretical one(curves 1, 2 lie, correspondingly, above levels 1/4 and 4/9 drawn by the dashed lines) and, besides, decreases in time (the curves rise). The mentioned features of the convergence rate behavior are most likely to be also resulted from the strong deformation of computational domain.

Thus, taking into account the conditions of scheme testing, being extreme enough because of the essential domain deformation, and the use of relatively rough grids, the results of test calculations should be estimated as quite satisfactory.

6. Wave generation by lifting a bottom portion(test calculation)

The problem on wave generation by lifting a bottom portion dosen't possess an exact analytical solution but has been investigated in detail by means of experiment (Hammach [6]) and numerically (Nakayama [8], [9]) as well. Good agreement of the experimental and numerical results mentioned indicates their reliability and makes it possible to explore this problem as a test. In addition, the problem allows the constructed numerical algorithm to be compared with another one(in the present case, with the BIE method (Nakayama [8], [9]) in efficiency.

The problem considered can be deduced from general formulation (2.13)-(2.20), (2.10), (2.2), (2.4)-(2.6), (2.21), (2.22) by posing

$$h(x,t)=\begin{cases}1-h_0(1-\exp\{-\alpha t)), & 0\le x\le b,\\ 1, & b<x\le l,\end{cases}$$

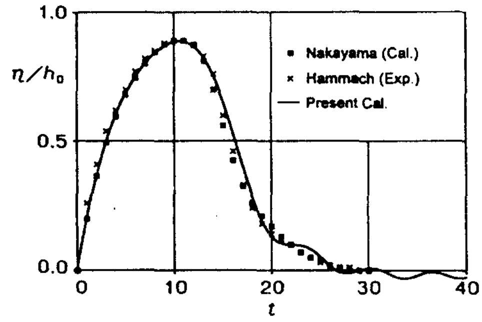
$$\iota_1=0,\ \iota_2=l,\ U_1(y,t)\equiv U_2(y,t)\equiv 0$$

$$p(x,t)\equiv 0,\quad \eta_0(x)\equiv u_0(x)\equiv 0$$

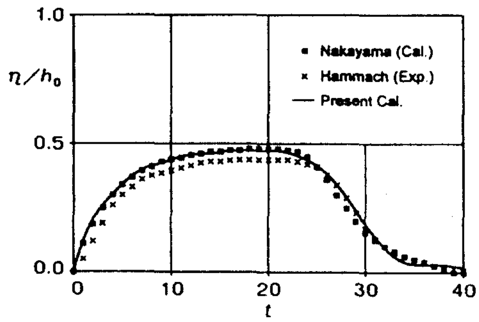
Similar to Nakayama [8], [9], the following values of constants have been used: j = 36 ; b= 12.2 ; h_o = 0.1 ; a = 0.231.The problem has calculated until time moment t = 40 with step τ= 0.1 on mesh 32×4 what is in close correspondence to the disposition of boundary elements in Nakayama [8].

The results of present computation, demonstrated by the curves on Fig. 5(where, in dependence on time, elevation of water surface at fixed points, $\eta(0,t)$ (a) and $\eta(b,t)$ (b), is shown), agree quite well with the results of similar calculation from Nakayama [8] (points 1) and experimental data of Hammach [6] (points 2). As far as the computer resource expanse is concerned, here it amounts to 5.4 sec per one time step for IBM PC 286 AT and in Nakayama [8] - 2.7 sec for HITAC 8800/8700, with the latter being approximately two orders faster than IBM PC. The given values count in favor of high economic feasibility of the numerical method constructed here.

7. Conclusion



(a) $\eta(0,t)$



(b) $\eta(b,t)$

Fig. 5 Time history of wave elavations ar x=0 and b

The numerical algorithm for calculating the problem on unsteady motion of ideal fluid with a free surface is developed. This method possesses the following primary features :

1. The finite-difference scheme is implicit that results in a high reserve of stability.
2. Application of symmetrical approximations only (the symmetrical differences and the half-sums) provides the second-order scheme accuracy.
3. Due to fitting the computational grid to the boundary of time-dependent domain $\Omega(t)$, introducing the additional functions U and V to be found and disposing the grid nodes in the most optimal way, the boundary conditions in the elliptic problem are implemented exactly.
4. The scheme is conservative: there are fulfilled exactly mass and momentum conservation laws.
5. The side effect caused by application of the implicit scheme, namely the necessity of iterative procedure at every time step, is weakened by double use of this procedure : simultaneously with iterating the nonlinearity, the solution of the elliptic problem is being constructed by means of establishing the solution of the corresponding parabolic problem.
6. The parabolic problem mentioned is calculated by the economic method of fractional steps.
7. The algorithm is universal in sense of possibility to use variable exchange general enough, in order to fit the computational grid to the boundary of time-dependent domain.
8. The algorithm efficiency may be improved by applying some technical details as constructing the initial approach for the iterative procedure by means of linear extrapolation of solution from two previous time steps, managing the iterative parameter in dependence on a divergence value, and so on.

Due to the numerical method features mentioned above, its high efficiency (accuracy × economic feasibility) has been achieved what has been proved by the results of test calculations.

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REFERENCES

[1] Asaithambi N. S. 1987 Computation of free-surface flows.- *J. Comp. Phys.* 73, 380-394.

[2] Daigyji H., Shin B. R. 1992 Some numerical schemes using curvilinear coordinate grid for incompressible and compressible Navier-Stokes equations. - Proc. 5th Asian congr. *fluid Mech.* (*August* 10-14, 1992, *Taejon, Korea*).

[3] Douglas J., Rachford H. H. 1956 On the numerical solution of heat conduction problems in two and three space variables. -*Trans. Amer. Math. Soc.* 82, No 2, 421-439

[4] Fletcher C. A. J. 1988 *Computational Techniques for fluid Dynamics. Fundamental and General Techniques..* - Berlin, New-york: Springer- verlag, 409 p.

[5] Fletcher C. A. J. 1988 *Computational Techniques for Fluid Dynamics. V.II. Specific Techniques for Different Flow Categories.* - Berlin, New-York:

Springer-verlag, 484 P.

[6] Hammach J. L. 1973 A note on tsunamis: their generation and propagation in an ocean of uniform depth. - *J. Fluid Mech.* 60, No 4, 769-799.

[7] Huassling H. J. 1982 Solutions of nonlinear water wave problems using boundary-fitted coordinate systems. - *Appl. Math. and Comput.*, No 10-11, 385-407.

[8] Nakayama T. 1983 Boundary element analysis of nonlinear water wave problems. - *Intl J. Num. Meth. Engng* 19, 953-970.

[9] Nakayama T. 1990 A computational method for simulating transient motions of an incompressible inviscid fluid with a free surface. - *Intl J. Num. Meth. Fluids* 10, 683-695.

[10] Ovsyannikov L. V. 1967 General equations and examples. - In : *The problem on Unsteady Motion of Fluid with a Free Boundary. Novosibirsk*: Nauka, pp. 67-68(in Russian).

[11] Thompson J. F., Warsi Z. U. A., Mastin C. W. 1982 Boundary-fitted coordinate systems for numerical solution of partial differential equations. A review. - *J. Comp. Phys.* 47, No 1, 1-108.

[12] Whitham G. B. 1974 *Linear and Nonlinear Waves.* - New York: Wiley - Interscience, 636 P.

[13] Yanenko N. N. 1967 *Method of Fractional Steps for solving Poly-Dimensional Problems of Mathematical Physics.* - Novosibirsk: Nauka, 196 p.(in Russian).

[14] Yeung R. W. 1982 Numerical methods in free-surface flows. - *Ann. Rev. Fluid Mech.* 14, 395-442.

[15] Yeung R. W., Vaidhyanathan M. 1992 Non-linear inter-action of water waves with submerged obstacles. - *Intl J. Num. Meth. Fluids* 14, 1111-1130.