Journal of the Korean
Statistical Society
Vol. 24, No. 2, 1995

A Random Shock Model for a Linearly Deteriorating System†

Jiyeon Lee[1] and Eui Yong Lee[2]

ABSTRACT

A random shock model for a linearly deteriorating system is introduced. The system deteriorating linearly with time is subject to random shocks which arrive according to a Poisson process and decrease the state of the system by a random amount. The system is repaired by a repairman arriving according to another Poisson process if the state when he arrives is below a threshold. Explicit expressions are deduced for the characteristic function of the distribution function of $X(t)$, the state of the system at time t, and for the distribution function of $X(t)$ if $X(t)$ is over the threshold. The stationary case is briefly discussed.

KEYWORDS: Random shock model, Poisson process, Integro differential equation, Characteristic function, Renewal process, Stationary distribution.

---

†This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1993 and by the Basic Science Research Institute Program, Ministry of Education, 1994, Project No. BSRI-94-1403.

[1]Department of Statistics, Yeungnam University, Kyongsan, 712-749, Korea.

[2]Department of Mathematics, Pohang University of Science and Technology, Pohang, 790-784, Korea.

1. INTRODUCTION

In this paper, we introduce a random shock model for a system which deteriorates linearly with time. It is assumed that the state of the system is initially $\beta > 0$ and thereafter decreases linearly at a constant rate $\mu > 0$ until a shock arrives at the system. The shocks come to the system according to a Poisson process of rate $\nu > 0$. Each shock instantaneously decreases the state of the system by a random amount Z, where Z is a nonnegative random variable with distribution function H. It is further assumed that the system is repaired by a repairman who arrives at the system according to another Poisson process of rate $\lambda > 0$; if the state of the system when he arrives exceeds a threshold α, he does nothing, otherwise he instantaneously increases the state of the system up to β(or if necessary replaces the system with a new one). This model is illustrated in FIG 1.

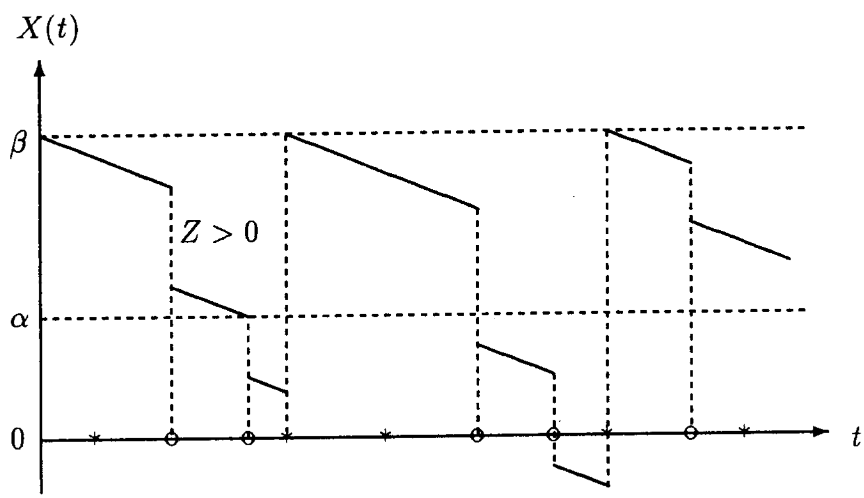

FIG 1 (o:shock, $*$:repairman)

Lee and Lee (1993) introduced a similar random shock model for a system. In the model, however, the state of the system was assumed to deteriorate purely jumpwise due to the shocks. In this paper, the earlier model is generalized by assuming that the state of the system deteriorates linearly between shocks, which may be a more realistic assumption. The probabilistic structure of the model is quite different from that of the earlier model. Firstly, the discrete probability at state β no more exists in the present model. Secondly, any state less than β can be passed through either jumpwise due to the shock or continuously due to the linear deterioration. Notice that in the earlier model, any state less than β can be passed through only jumpwise with probability one. Thirdly, the first passage time from state β to state $x, \alpha \leq x < \beta$, is now bounded by $(\beta - x)/\mu$ with probability one, since the state of the system decreases constantly at rate μ, even if no shock arrives. These differences lead us to adopt new approaches in the key steps on the way of analyzing the model.

Our model can also be applied to an inventory with Poisson restocking (see Baxter and Lee (1987)) by assuming that the level of the inventory decreases linearly due to regular orders and jumpwise when irregular orders come. It is assumed that the irregular orders come according to a Poisson process and the amount of each is a random variable.

Let $X(t)$ denote the state of the system at time t and let $F(x,t) = P\{X(t) \leq x\}$ denote the distribution function of $X(t)$, we derive an integro-differential equation for $F(x,t)$ in section 2, and solve the equation in section 3 to obtain an expression for the characteristic function of $F(x,t)$, where some terms are still needed to be evaluated. In section 4, we obtain an explicit formula for $F(x,t)$, for $\alpha \leq x < \beta$, by making use of a renewal argument, and use this result to complete the expression for the characteristic function. The stationary case is discussed in section 5.

For convenience to analyze the model, we allow that $X(t)$ takes on the negative values, which might be regarded as undesirable in some applications such as the inventory. Notice, however, that in the case where the state 0 is considered as an absorbing barrier to prevent the state from being negative, the probability of the system in state 0 at time t is exactly same as $F(0,t)$ in our original model.

2. INTEGRO-DIFFERENTIAL EQUATION FOR $F(x,t)$

During the interval $(t, t+\delta t)$, one of the following four mutually exclusive events will occur:

(a) Neither the repairman nor the shock comes, then

$$X(t+\delta t) = X(t) - \mu\delta t \quad \text{almost surely.}$$

(b) The repairman does not come but the shock comes, then

$$X(t+\delta t) = X(t) - Z - \mu\delta t \quad \text{almost surely.}$$

(c) The repairman comes but does nothing since $X(t) > \alpha$, and the shock does not come, then

$$X(t+\delta t) = X(t) - \mu\delta t \quad \text{and} \quad X(t) > \alpha \quad \text{almost surely.}$$

(d) The repairman comes and repairs the system since $X(t) \leq \alpha$, and the shock does not come, then

$$X(t+\delta t) = \beta - \mu\delta t' \quad \text{for some} \quad \delta t' \leq \delta t, \quad \text{and} \quad X(t) \leq \alpha \quad \text{almost surely.}$$

Notice that the probability of the event that both the repairman and the shock come during the interval $(t, t+\delta t)$ is $o(\delta t)$. Thus, for $x \leq \beta$,

$$\begin{aligned} F(x, t+\delta t) &= P\{X(t+\delta t) \leq x\} \\ &= (1-\lambda\delta t)(1-\nu\delta t)P\{X(t) - \mu\delta t \leq x\} \\ &\quad +(1-\lambda\delta t)(\nu\delta t)P\{X(t) - Z - \mu\delta t \leq x\} \\ &\quad +\lambda\delta t(1-\nu\delta t)P\{X(t) - \mu\delta t \leq x, X(t) > \alpha\} \\ &\quad +\lambda\delta t(1-\nu\delta t)P\{\beta - \mu\delta t' \leq x, X(t) \leq \alpha\} \\ &\quad +o(\delta t), \end{aligned}$$

where $P\{X(t)-\mu\delta t\le x\}=F(x+\mu\delta t,t)=F(x,t)+\mu\delta t\frac{\partial}{\partial x}F(x,t)+o(\delta t)$ on performing a Taylor series expansion by assuming that $\partial F(x,t)/\partial x$ exists. Subtracting $F(x,t)$ from each side of the above equation, dividing by δt, and letting $\delta t\to 0$ gives us the following integro-differential equation for $F(x,t)$:

$$\frac{\partial}{\partial t}F(x,t)=\mu\frac{\partial}{\partial x}F(x,t)-\nu F(x,t)-\lambda F(x\wedge\alpha,t)+\nu\int_0^\infty F(x+z,t)dH(z), \quad (2.1)$$

for $x<\beta$, and $F(\beta,t)=1$ for all $t\ge 0$.

3. THE CHARACTERISTIC FUNCTION OF $F(x,t)$

Taking the ordinary Fourier transform of equation (2.1) with respect to x yields

$$\begin{aligned}\frac{\partial f^\circ(s,t)}{\partial t}&\\ =\mu\{\exp(is\beta)-isf^\circ(s,t)\}-\nu f^\circ(s,t)&\\ -\lambda\Big\{f^\circ(s,t)-\int_\alpha^\beta\exp(isx)F(x,t)dx+F(\alpha,t)\frac{\exp(is\beta)-\exp(is\alpha)}{is}\Big\}&\\ +\nu\Big\{h^*(-s)f^\circ(s,t)+\frac{\exp(is\beta)}{is}-\frac{\exp(is\beta)h^*(-s)}{is}\Big\},& \quad (3.1)\end{aligned}$$

where $f^\circ(s,t)=\int_{-\infty}^\beta\exp(isx)F(x,t)dx$ and $h^*(-s)=\int_0^\infty\exp(-isz)dH(z)$. Rearranging the above equation and putting $g(s,t)=\int_\alpha^\beta\exp(isx)F(x,t)dx$, we obtain the following ordinary differential equation for $f^\circ(s,t)$:

$$\begin{aligned}\frac{df^\circ(s,t)}{dt}+\{i\mu s+\lambda+\nu(1-h^*(-s))\}f^\circ(s,t)&\\ =\mu\exp(is\beta)+\lambda g(s,t)&\\ -\frac{\lambda F(\alpha,t)\{\exp(is\beta)-\exp(is\alpha)\}}{is}+\frac{\nu\exp(is\beta)(1-h^*(-s))}{is}.& \quad (3.2)\end{aligned}$$

On solving the above equation with boundary condition $f^\circ(s,0) = 0$, we see that

$$\begin{aligned} f^\circ(s,t) &= \exp(-\{i\mu s + \lambda + \nu(1-h^*(-s))\}t) \\ &\times \int_0^t \exp(\{i\mu s + \lambda + \nu(1-h^*(-s))\}u)\big\{\mu \exp(is\beta) + \lambda g(s,u) \\ &- \frac{\lambda F(\alpha,u)[\exp(is\beta)-\exp(is\alpha)]}{is} + \frac{\nu \exp(is\beta)(1-h^*(-s))}{is}\big\}du. \end{aligned} \tag{3.3}$$

Since the characteristic function of $F(x,t)$ with respect to x, $f^*(s,t)$ say, satisfies $f^*(s,t) = \exp(is\beta) - isf^\circ(s,t)$, it can be shown that

$$\begin{aligned} f^*(s,t) &= \int_{-\infty}^{\beta} \exp(isx)dF(x,t) \\ &= \frac{\exp(is\beta)}{i\mu s + \lambda + \nu(1-h^*(-s))}\{\lambda + \exp(-\{i\mu s + \lambda + \nu(1-h^*(-s))\}t) \\ &\times [i\mu s + \nu(1-h^*(-s))]\} - \exp(-\{i\mu s + \lambda + \nu(1-h^*(-s))\}t) \\ &\times \int_0^t \exp(\{i\mu s + \lambda + \nu(1-h^*(-s))\}u)\{is\lambda g(s,u) \\ &- \lambda F(\alpha,u)[\exp(is\beta)-\exp(is\alpha)]\}du. \end{aligned} \tag{3.4}$$

The terms $F(\alpha,u)$ and $g(s,u)$ will be evaluated in the next section.

4. A FORMULA FOR $F(x,t)$, $\alpha \le x \le \beta$

Consider the points where the actual repair occurs. The sequence of these points forms an embedded renewal process. Let T^* be the generic random variable denoting the time between successive renewals. Then

$$T^* \stackrel{D}{=} T_\alpha + E^\lambda, \tag{4.1}$$

where $\stackrel{D}{=}$ denotes equality in distribution, T_α is the first passage time from state β to the state α, and E^λ is an exponential random variable with parameter λ.

We, first, derive the distribution function of T_α, then we can use the renewal argument of Lee and Lee (1993) to deduce an expression for $F(x,t), \alpha \le x \le \beta$. To do so, we project the process $\{X(t), t \ge 0\}$ between two successive renewals onto the vertical axis. Then we obtain an alternating renewal process $\{E_n^{\nu/\mu}, Z_n\}$, $n \ge 1$, starting from β and moving to the negative direction, where $E_n^{\nu/\mu}$'s denoting 'up' times are independent exponential random variables with parameter ν/μ and Z_n's denoting 'down' times are independent random variables with distribution function H. Let $A(\beta-\alpha)$ be the total time spent in 'up' state during the interval (α,β). Observe that

$$T_\alpha \stackrel{D}{=} A(\beta-\alpha)/\mu. \tag{4.2}$$

Takács (1957) showed that the distribution function of $A(\beta-\alpha)$ is given by

$$P\{A(\beta-\alpha) \le t\} = 1 - \sum_{n=0}^{\infty} \exp(-\nu t/\mu) \frac{(\nu t/\mu)^n}{n!} H^{(n)}(\beta-\alpha-t), \tag{4.3}$$

where $H^{(n)}$ is the n-fold recursive Stieltjes convolution of H, $H^{(0)}$ being the Heaviside function. It follows from equation (4.2) that the distribution function of T_α, U_α say, is given by

$$\begin{aligned} U_\alpha(t) &= P\{T_\alpha \le t\} \\ &= \sum_{n=0}^{\infty} [1 - H^{(n)}(\beta-\alpha-\mu t)] \exp(-\nu t) \frac{(\nu t)^n}{n!}, \quad 0 \le t < \frac{\beta-\alpha}{\mu}. \end{aligned} \tag{4.4}$$

Further, notice that, by the similar argument to the above, the distribution function of T_x, the first passage time from state β to the state x, $\alpha \le x < \beta$, U_x say, is given by

$$U_x(t) = \sum_{n=0}^{\infty} [1 - H^{(n)}(\beta-x-\mu t)] \exp(-\nu t) \frac{(\nu t)^n}{n!}, \quad 0 \le t < \frac{\beta-x}{\mu}. \tag{4.5}$$

Then, by the same argument as that of Lee and Lee (1993), it can be shown that

$$F(x,t) = U_x(t) - \int_0^t \{1 - U_x(t-u)\} dW(u), \tag{4.6}$$

where $W(t) = \sum_{n=1}^{\infty} V^{(n)}(t)$ is the renewal function of the embedded renewal process, $V(t)$ being the distribution function of T^* which is, from equation (4.1), given by

$$V(t) = P\{T^* \leq t\} = \int_0^t U_\alpha(t-u)\lambda \exp(-\lambda u)du.$$

Equation (4.6) enables us to determine the terms $F(\alpha, t)$ and $g(s,t)$ in the expression for the characteristic function of $F(x,t)$ obtained in section 3.

5. STATIONARY CASE

In this section, we briefly consider the case where the distribution function of $X(t)$ does not depend on time t, that is, $\partial F(x,t)/\partial t = 0$; we denote the corresponding distribution function by $F(x)$. From equation (3.1), it follows that

$$\begin{aligned} f^\circ(s) &= \int_{-\infty}^{\beta} \exp(isx)F(x)dx \\ &= \frac{1}{i\mu s + \lambda + \nu(1 - h^*(-s))}\Big\{\mu \exp(is\beta) + \lambda g(s) \\ &- \frac{\lambda F(\alpha)[\exp(is\beta) - \exp(is\alpha)]}{is} + \frac{\nu \exp(is\beta)(1 - h^*(-s))}{is}\Big\}, \end{aligned} \tag{5.1}$$

where $g(s) = \int_\alpha^\beta \exp(isx)F(x)dx$.

Applying the key renewal theorem to equation (4.6), we obtain, for $\alpha \leq x \leq \beta$,

$$F(x) = \frac{E(T^*) - E(T_x)}{E(T^*)} = \frac{\lambda[E(T_\alpha) - E(T_x)] + 1}{\lambda E(T_\alpha) + 1}, \tag{5.2}$$

where

$$E(T_x) = \int_0^{\frac{\beta - x}{\mu}} \sum_{n=0}^{\infty} \frac{\exp(-\nu t)(\nu t)^n}{n!} H^{(n)}(\beta - x - \mu t)dt. \tag{5.3}$$

In summary, the characteristic function of the stationary distribution function $F(x)$ is given by

$$\begin{aligned} f^*(s) &= \int_{-\infty}^{\beta} \exp(isx)dF(x) \\ &= \frac{\lambda\{F(\alpha)[\exp(is\beta) - \exp(is\alpha)] + \exp(is\beta) - isg(s)\}}{i\mu s + \lambda + \nu(1 - h^*(-s))}, \end{aligned} \tag{5.4}$$

where $F(\alpha)$ and $g(s)$ can be determined from equation (5.2) and (5.3).

REFERENCES

(1) Baxter, L.A. and Lee, E.Y. (1987). An Inventory with Constant Demand and Poisson Restocking. *Probability in the Engineering and Informational Sciences*, **1**, 203–210.

(2) Lee, E.Y. and Lee, J. (1993). A Model for a System Subject to Random Shocks. *Journal of the Applied Probability*, **30**, 979–984.

(3) Takács, L. (1957). On Certain Sojourn Time Problems in the Theory of Stochastic Processes. *Acta Mathematica Academiae Scientiarum Hungaricae*, **8**, 169–191.