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Flows and Some Extreme Values in Multiple Server Open Jackson Network †

YouSung Park [1], Haeyong Lee [2], and Keeyoung Kim[1]

ABSTRACT

Output processes emanating from exit arcs in a multiple server open Jackson network with node i having s_i servers are determined. Beutler and Melamed(1978) showed, for traffics on all exit arcs of single server open Jackson network in equilibrium, that the customer streams leaving any exit set are Poisson and that the collections over all nodes which yield the Poisson departure processes are mutually independent. In this paper we generalize the above results to multiple servers open Jackson network in equilibrium. While no weak limit result is possible under the equilibrium condition, nonetheless approximations to the distributions of maximum queue lengths for no feedback nodes in multiple servers open Jackson network are established.

KEYWORDS: Output processes, Exit arcs, Open Jackson network, Equilibrium distribution, Maximum queue length.

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[1]Department of Statistics, Korea University, Seoul, 136-701, Korea.
[2]Department of Statistics, Sungshin Women's University, Seoul, 136-742, Korea.

1. INTRODUCTION

We consider a network of N server nodes with node i having s_i servers. Customers can arrive from outside to node i independently according to respective Poisson processes with intensity λ_i. The servers at node i work independently according to an exponential distribution with parameter μ_i. When a customer completes service at node i, he goes next to node j with probability P_{ij} (independent of the system rates, λ_i, μ_i for $i, j = 1, 2, \ldots, N$). There is a probability P_{i0} that a customer will leave the network at node i upon completion of service such that $P_{i0} = 1 - \sum_{j=1}^{N} P_{ij}$. There is no limit to queue capacity at any node.

Throughout this paper, we assume that

$$\frac{\rho_i}{s_i} < 1, \quad \text{where } \delta_i = \lambda_i + \sum_{j=1}^{N} \delta_j P_{ji} \qquad (1.1)$$
$$\text{and } \rho_i = \frac{\delta_i}{\mu_i}, i = 1, 2, \ldots, N.$$

so that, since we consider output processes in equlibrium, $P\{Q_1(t) = n_1, \ldots, Q_N(t) = n_N\}$ is time invariant to with Q_i being the queue length of node i at time t and δ_i given in (1.1) is uniquely determined by ergodicity of $Q_i(t)$.

It is well known that each node (say the i-th) having s_i servers in equilibrium for the open Jackson network defined as above behaves as if it were an independent $M/M/s_i$ system with a Poisson input rate δ_i (cf. Jackson(1963)) even though in general total input to the node i is not a Poisson process. However, as Beutler and Melamed(1978) indicated, there is a class of arcs along which the traffic consists of Poisson processes. These are so-called exit arcs. Node j is said to be accessible from node i(notationally, $i \to j$)if, for some $n \geq 0, P_{ij}^n > 0$. Both nodes i and j accessible to each other are denoted by $i \leftrightarrow j$.

Definition 1.1. The canonical decomposition of an open Jackson network with N nodes is the partition

$$\wp = \{C_q : q = 1, 2, \ldots, \ell \leq N\} \qquad (1.2)$$

induced by the communication relation $\leftrightarrow$ with $C_\ell = \{0\}$. An arc $(i,j), i \in C_p, j \in C_q$ is an exit arc if $P_{ij} > 0$ but $P_{ji} = 0$ for $i \neq j$.

We assume without loss of generality that $q > p$ if there exist some nodes $i \in C_p, j \in C_q$ such that $i \to j$ but $j \not\to i$. Following the notation of Beutler and Melamed(1978), define $K_{ij}(t)$ to be the counting process on the arc (i,j) over the time interval $(0,t]$. That is, $K_{ij}(t)$ is the number of customers leaving node i and arriving instantaneously at node j in the period $(0,t]$.

Section 2 establishes that the queueing processes $Q_i(t)$ and counting processes $K_{ij}(t), i = 1, \ldots, N, j = 0, 1, \ldots, N, i \neq j$ on exit arcs (i,j) are mutually independent and moreover, $K_{ij}(t)$ are mutually independent Poisson processes in equilibrium. These generalize the results of Beutler and Melamed(1978) in the sense that we consider the output processes in a multiple servers open Jackson network and, in addition, in any subset of the canonical decomposition $\wp$ in (1.2). Using the results in section 2, we investigate in section 3 under what conditions can we assert that the maximum queue length of node i linearly normalized will have a non-degenerate limit G with appropriate norming constants a_{ti}, b_{ti} such that

$$\lim_{t\to\infty} P\{b_{ti}(M_{ti} - a_{ti}) \leq x\} = G(x),$$

where M_{ti} is the maximum queue length of node i in the time interval $(0,t]$.

2. OUTPUT PROCESSES FROM EXIT SETS

One is often interested, for networks, in output processes from individual nodes since they influence input processes to other nodes. For example, the output process from a previous node is the basic requirement in order to determine the input process to the next node in series queueing systems like tandem queues. Melamed(1979) showed that output processes for no feedback nodes(i.e, there is no path a departing customer can follow that will eventually feedback, prior to exiting a network) in a single server open Jackson network are mutually independent Poisson processes. Disney, et al.(1980) showed that, for a feedback node, the output process is never a Poisson process nor is it a renewal process. In this section , we only consider the output processes for no feedback nodes in a multiple server open Jackson network. We now define a certain set, so called, a set of exit arcs as follows.

Definition 2.1. A set V_1 of nodes in an open Jackson network is said to be an exit set with respect to a set V_2 of nodes if $P_{ij} = 0$ whenever $j \in V_1, i \in V_2$.

Let $V_p = \{C_1, \dots, C_p\}$ be a set of nodes numbered $1, 2, \dots, r_1, r_1+1, \dots, r_2, \dots, r_{p-1}, r_{p-1}+1, \dots, r_p$ and $V_q = \{C_{p+1}, \dots, C_q\} \cup \{0\}$, $q \leq N$ be a set of nodes with numbers $r_p + 1, \dots, r_q, 0$ without loss of generality. Moreover, we assume throughout this paper that $(V_p \cup V_q)^c$ is the complement of $V_p \cup V_q$ and $i \to k$ for all $i \in V_p$ and $k \in (V_p \cup V_q)^c$. Then we may consider V_p as an exit set with respect to V_q by Definition 2.1.

Remark 2.2. Let $\underline{Q}_p(t) = (Q_1(t), \dots, Q_p(t))$ and $\underline{K}_{pq}(t) = (K_{ij}(t), i = 1, \dots, p, j = 1, \dots, q), p, q \leq N$ in multiple servers Open Jackson network as defined before. Then $(\underline{Q}_p(t), \underline{K}_{pq}(t)), p \neq q$ is a Markov jump process with denumerable state space and $a.s$ a finite number of jumps in each finite time interval. This assertion can be easily shown since $\underline{Q}_p$ is a Markov process by the nature of Poisson inputs from outside and of exponential servive times(cf. Beutler and Melamed(1978)).

Let $(\underline{n}_p, \underline{k}_{pq})_t$ be the event that is $Q_i(t) = n_i$ and $K_{ij}(t) = k_{ij}$ for $i \in V_p, j \in V_q$ at time t. Then the marginal probabilities of $P\{(\underline{n}_p, \underline{k}_{pq})_{t+h} \mid Q_i(t), K_{ij}(t), i \in V_p, j \in V_q\}$ can be shown as follows:

If $Q_i(t) = n_i$ and $K_{ij}(t) = k_{ij}$ for each $i \in V_p, j \in V_q$, with and

$$\zeta(n_i) = \begin{cases} n_i & \text{if } n_i \leq s_i \\ s_i & \text{if } n_i > s_i, \end{cases}$$

then

$$1 - h \sum_{i=1}^{r_p} \lambda_i - h \sum_{j=0}^{N} \sum_{i=1}^{r_p} \zeta(n_i) \mu_i P_{ij} + o(h). \tag{2.1}$$

Or, similar approach as above, if $Q_\ell(t) = n_\ell, k_{\ell m}(t) = k_{\ell m}$, each $\ell \in V_p, m \in V_q$ except $Q_i(t) = n_i - 1$ for $i \in V_p$,

$$\sum_{i=1}^{r_p} \lambda_i h + o(h). \tag{2.2}$$

Or, if $Q_\ell(t) = n_\ell, k_{\ell m}(t) = k_{\ell m}$, each $\ell \in V_p, m \in V_q$ except $Q_i(t) = n_i + 1$ for $i \in V_p$,

$$\sum_{j \in (V_p \cup V_q)^c} \sum_{i=1}^{r_p} \zeta(n_i + 1)\mu_i P_{ij} h + o(h). \tag{2.3}$$

Or, if $Q_\ell(t) = n_\ell, k_{\ell m}(t) = k_{\ell m}$, each $\ell \in V_p, m \in V_q$ except $Q_i(t) = n_i + 1$ and $k_{i0}(t) = k_{i0} - 1$ for $i \in V_p$,

$$\sum_{i=1}^{r_p} \zeta(n_i + 1)\mu_i P_{i0} h + o(h). \tag{2.4}$$

Or, if $Q_\ell(t) = n_\ell, k_{\ell m}(t) = k_{\ell m}$, each $\ell \in V_p, m \in V_q$ except $Q_i(t) = n_i + 1$ and $Q_j(t) = n_j - 1$ for $i, j \in V_p$,

$$\sum_{j=1}^{r_p} \sum_{i=1}^{r_p} \zeta(n_i + 1)\mu_i P_{ij} h + o(h). \tag{2.5}$$

Or, if $Q_\ell(t) = n_\ell, k_{\ell m}(t) = k_{\ell m}$, each $\ell \in V_p, m \in V_q$ except $Q_i(t) = n_i + 1$ and $k_{ij}(t) = k_{ij} - 1$ for $i \in V_p, j \in V_q$,

$$\sum_{j=r_p+1}^{r_q} \sum_{i=1}^{r_p} \zeta(n_i + 1)\mu_i P_{ij} h + o(h). \tag{2.6}$$

Now, we assume for a moment(in fact, this is the solution of Theorem 2.6) that

$$\begin{aligned} P_t(\underline{n}_p, \underline{k}_{pq}) &= P\{Q_i(t) = n_i, K_{ij}(t) = k_{ij}, i \in V_p, j \in V_q\} \\ &= \prod_{i=1}^{r_p} \frac{\rho_i^{n_i}}{a_i(n_i)} P_{0i} \prod_{j \in V_q} \frac{\exp(-\delta_i P_{ij} t)(\delta_i P_{ij} t)^{k_{ij}}}{k_{ij}!}, \end{aligned} \tag{2.7}$$

where

$$a_i(n_i) = \begin{cases} n_i! & \text{if } n_i \le s_i \\ s_i^{n_i - s_i} s_i! & \text{if } n_i > s_i, \end{cases}$$

ρ_i and δ_i are given in (1.1) and P_{0i} such that $P_{0i} \cdot \sum_{n_i=0}^{\infty} \rho_i^{n_i}/a_i(n_i) = 1$. Then we have the following results.

Lemma 2.3. Suppose that $P_t(\underline{n}_p, \underline{k}_{pq})$ is given in (2.7). Then

$$\begin{aligned} &P_t'(\underline{n}_p, \underline{k}_{pq}) + \sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\lambda_i \\ &= \sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i + 1, \ldots, n_{r_p}, k_{10}, k_{1r_p+1}, \ldots, k_{1r_q}, \ldots, k_{(i-1)r_q}, \\ &\quad k_{i0} - 1, k_{ir_p+1}, \ldots, k_{r_pr_q})\zeta(n_i+1)\mu_i P_{i0} \\ &+ \sum_{j=r_p+1}^{r_q} \sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i+1, \ldots, n_{r_p}, k_{10}, \ldots, k_{ij} - 1, \ldots, k_{r_pr_q}) \\ &\quad \cdot\zeta(n_i+1)\mu_i P_{ij} \\ &+ \sum_{j\in(V_p\cup V_q)^c} \sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i+1, \ldots, n_{r_p}, \underline{k}_{pq})\zeta(n_i+1)\mu_i P_{ij}. \end{aligned}$$

Proof. Take differentiation with respect to t in (2.7). Then we have

$$\begin{aligned} P_t'(\underline{n}_p, \underline{k}_{pq}) &= P_t(\underline{n}_p, \underline{k}_{pq})\Big[\frac{1}{t}\sum_{i=1}^{r_p}\sum_{j\in V_q} k_{ij} - \sum_{i=1}^{r_p}\sum_{j\in V_q}\delta_i P_{ij}\Big] \qquad (2.8) \\ &= P_t(\underline{n}_p, \underline{k}_{pq})\Big[\frac{1}{t}\sum_{i=1}^{r_p}\sum_{j\in V_q} k_{ij} - \sum_{i=1}^{r_p}\delta_i\Big(1 - \sum_{j=1}^{r_p} P_{ij} - \sum_{j\in(V_p\cup V_q)^c} P_{ij}\Big)\Big] \end{aligned}$$

Since $\delta_i = \lambda_i + \sum_{j=1}^{r_p}\delta_j P_{ji}$ for $i = 1, \ldots, r_p$ by the assumption of the set of $(V_p \cup V_q)^c$, we have that

$$\sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\lambda_i = \sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\Big(\delta_i - \sum_{j=1}^{r_p}\delta_j P_{ji}\Big). \qquad (2.9)$$

Next, one may easily show that

$$\frac{a_i(n_i)\zeta(n_i+1)}{a_i(n_i+1)} = 1 \text{ with } a_i(-1) \equiv 1 \quad \text{for all } n_i \geq 0. \tag{2.10}$$

Using (2.10),

$$\sum_{i=1}^{r_p} P_t(n_1,\ldots,n_i+1,\ldots,n_{r_p},k_{10},k_{1r_p+1},\ldots,k_{1r_q},\ldots,k_{(i-1)r_q},$$

$$k_{i0}-1,k_{ir_p+1},\ldots,k_{r_pr_q})\zeta(n_i+1)\mu_iP_{i0}$$

$$=\sum_{i=1}^{r_p} P_t(\underline{n}_p,\underline{k}_{pq})\rho_i\frac{a_i(n_i)}{a_i(n_i+1)}\frac{k_{i0}}{\delta_iP_{i0}t}\zeta(n_i+1)\mu_iP_{i0}$$

$$=\frac{1}{t}\sum_{i=1}^{r_p} P_t(\underline{n}_p,\underline{k}_{pq})k_{i0}. \tag{2.11}$$

And

$$\sum_{j=r_p+1}^{r_q}\sum_{i=1}^{r_p} P_t\ (n_1,\ldots,n_i+1,\ldots,n_{r_p},k_{10},\ldots,k_{ij}-1,\ldots,k_{r_pr_q})\zeta(n_i+1)\mu_iP_{ij}$$

$$=\sum_{j=r_p+1}^{r_q}\sum_{i=1}^{r_p} P_t(\underline{n}_p,\underline{k}_{pq})\rho_i\frac{a_i(n_i)}{a_i(n_i+1)}\frac{k_{ij}}{\delta_iP_{ij}t}\zeta(n_i+1)\mu_iP_{ij}$$

$$=\frac{1}{t}\sum_{j=r_p+1}^{r_q}\sum_{i=1}^{r_p} P_t(\underline{n}_p,\underline{k}_{pq})k_{ij}. \tag{2.12}$$

Finally,

$$\sum_{j\in(V_p\cup V_q)^c}\sum_{i=1}^{r_p} P_t(n_1,\ldots,n_i+1,\ldots,n_{r_p},\underline{k}_{pq})\zeta(n_i+1)\mu_iP_{ij}$$

$$=\sum_{j\in(V_p\cup V_q)^c}\sum_{i=1}^{r_p} P_t(\underline{n}_p,\underline{k}_{pq})\rho_i\frac{a_i(n_i)}{a_i(n_i+1)}\zeta(n_i+1)\mu_iP_{ij}$$

$$\sum_{j\in(V_p\cup V_q)^c}\sum_{i=1}^{r_p} P_t(\underline{n}_p,\underline{k}_{pq})\delta_iP_{ij}. \tag{2.13}$$

Thus, by (2.8),(2.9),(2.11),(2.12) and (2.13), the claim follows.

Lemma 2.4. With $P_t(\underline{n}_p, \underline{k}_{pq})$ given in (2.7), we have

$$\sum_{j=0}^{N}\sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\zeta(n_i)\mu_i P_{ij}$$

$$= \sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i - 1, \ldots, n_{r_p}, \underline{k}_{pq})\lambda_i$$

$$+ \sum_{j=1}^{r_p}\sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i + 1, \ldots, n_j - 1, \ldots, n_{r_p}, \underline{k}_{pq})\zeta(n_i+1)\mu_i P_{ij}.$$

Proof. Since $\sum_{j=0}^{N} P_{ij} = 1$,

$$\sum_{j=0}^{N}\sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\zeta(n_i)\mu_i P_{ij} = \sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\zeta(n_i)\mu_i. \tag{2.14}$$

And, by (2.7),

$$\sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i - 1, \ldots, n_{r_p}, \underline{k}_{pq})\lambda_i = \sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\zeta(n_i)\frac{\lambda_i}{\rho_i}. \tag{2.15}$$

Note that $a_i(n_i)/a_i(n_i - 1) = \zeta(n_{\oe i})$ for all $n_i \geq 0$. Thus, with (2.10) and $\delta_i = \lambda_i + \sum_{j=1}^{r_p} \delta_j P_{ji}$ for $i \in V_p$,

$$\sum_{j=1}^{r_p}\sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i + 1, \ldots, n_j - 1, \ldots, n_{r_p}, \underline{k}_{pq})\zeta(n_i+1)\mu_i P_{ij}$$

$$= \sum_{j=1}^{r_p}\sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\frac{a_j(n_j)}{\rho_j a_j(n_j - 1)}\rho_i\frac{a_i(n_i)}{a_i(n_i+1)}\zeta(n_i+1)\mu_i P_{ij}$$

$$= \sum_{j=1}^{r_p}\sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\frac{\mu_j}{\delta_j}\zeta(n_j)\delta_i P_{ij}$$

$$= \sum_{j=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq})\frac{\mu_j}{\delta_j}\zeta(n_j)\sum_{i=1}^{r_p}\delta_i P_{ij}$$

$$= \sum_{j=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq}) \frac{\mu_j}{\delta_j} \zeta(n_j)(\delta_j - \lambda_j)$$

$$= \sum_{j=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq}) \mu_j \zeta(n_j) - \sum_{j=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq}) \zeta(n_j) \frac{\lambda_j}{\rho_j}. \tag{2.16}$$

Hence, by (2.14)–(2.16), the proof follows.

Theorem 2.5. $Q_i(t), K_{ij}(t), i \in V_p, j \in V_q$ in equilibrium are mutually independent and each $K_{ij}(t), i \in V_p, j \in V_q$ is independently Poisson distributed with intensity $\delta_i P_{ij}$.

Proof. From (2.1)–(2.6), we have the marginal differential equation:

$$\begin{aligned}
&P_t'(\underline{n}_p, \underline{k}_{pq}) \\
&= \sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq}) \lambda_i - \sum_{j=0}^{r_q} \sum_{i=1}^{r_p} P_t(\underline{n}_p, \underline{k}_{pq}) \zeta(n_i) \mu_i P_{ij} \\
&\quad + \sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i - 1, \ldots, n_{r_p}, \underline{k}_{pq}) \lambda_i \\
&\quad + \sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i + 1, \ldots, n_{r_p}, k_{10}, k_{1 r_p+1}, \ldots, k_{1 r_q}, \ldots, k_{(i-1) r_q}, \\
&\qquad k_{i0} - 1, k_{i r_p+1}, \ldots, k_{r_p r_q}) \zeta(n_i + 1) \mu_i P_{i0} \\
&\quad + \sum_{j=1}^{r_p} \sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i + 1, \ldots, n_j - 1, \ldots, n_{r_p}, \underline{k}_{pq}) \zeta(n_i + 1) \mu_i P_{ij} \\
&\quad + \sum_{j=r_p+1}^{r_q} \sum_{i=1}^{r_p} P_t(n_1, \ldots, n_i + 1, \ldots, n_{r_p}, k_{10}, \ldots, k_{ij} - 1, \ldots, k_{r_p r_q}) \\
&\qquad \cdot \zeta(n_i + 1) \mu_i P_{ij}.
\end{aligned} \tag{2.17}$$

Since $K_{ij}(t), i \in V_p, j \in V_q$ are counting processes, they must equal to 0 for all i, j at time $t = 0$ almost surely. And since the queueing network is supposed to be in equilibrium at $t = 0$, (2.16) with a proper form of $P_t(\underline{n}_p, \underline{k}_{pq})$ is subject to the initial condition

$$P_0(\underline{n}_p, \underline{k}_{pq}) = \begin{cases} \prod_{i=1}^{r_p} \frac{\rho_i^{n_i}}{a_i(n_i)} P_{0i} & \text{if all } k_{ij} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

To verify the assertion of the Theorem, it suffices to show that (2.7) satisfies (2.17) since (2.7) is trivially true for the condition (2.16). But this claim hold immediatly from Lemma 2.3 and 2.4.

Theorem 2.6. The output processes $K_{ij}, i \in V_p, j \in V_q$ in equilibrium are mutually independent Poisson processes with respective intensities $\delta_i P_{ij}$.

Proof. The vector $(Q_i(t), K_{ij}(t), i \in V_p, j \in V_q)$ is a Markov process as stated in Remark 2.2. Thus, by Corollary 1 and Theorem 4 of Beutler and Melamed and Theorem 2.5, we have the claim.

For specific case,let $\ell = 3$ in the canonical decomposition $\wp$ given in (1.2) and $V_p = C_1, V_q = C_2 \cup \{0\}$ for an open Jackson network with all nodes having single server. Then, we have the following result which is the work made by Beutler and Melamed(1978).

Corollary 2.7. When we define $V_p = \{1, 2, \ldots, r\}$ and $V_q = \{0, r+1, \ldots, N\}$ in a single server open Jackson network, the same claims of Theorem 2.5 and 2.6 hold.

Owing to Theorem 2.6 and Corollary 2.7, one may say that the output process on exit arc (i,j) depends only on the total input parameter δ_i and the transition probability P_{ij} regardless of the number of servers working on nodes. Moreover, we may also consider an exit set as a sub-open Jackson network of the entire open Jackson network when the entire open Jackson network is in equilibrium and departure processes of the sub-open Jackson network is again Poisson processes as shown in the entire open Jackson network.

3. EXTREME VALUES FOR NO FEEDBACK NODES IN A MULTIPLE OPEN JACKSON NETWORK

Jackson networks have been proved useful models for queueing and computer systems with applications to military command and control modelling, public utility models for police, fire, and medical emergency system design etc.. A typical question is how to determine the size of some subset of interested nodes in order to successfully manage service demand to a system for a long future period. Namely, this section gives emphasis on derivation of an asymptotic behavior of maximum queue length for no feedback nodes in multiple servers open Jackson network. Anderson(1970) showed that when $X_1, X_2, \ldots, X_n$ are integer valued i.i.d. r.v's with distribution function F such that $(1-F(n))/(1-F(n+1)) \to e^{\zeta}, \zeta > 0$ as $n \to \infty$, bounds of the limiting distribution of $M_n = \max\{X_1, X_2, \ldots, X_n\}$ are given by

$$\limsup_{n\to\infty} P\{M_n - \beta_n \le x\} \ge \exp\left[-e^{-\zeta(x-1)}\right]$$

and

$$\liminf_{n\to\infty} P\{M_n - \beta_n \le x\} \le \exp\left[-e^{-\zeta x}\right] \quad \text{for all } x,$$

where β_n is determined by

$$1 - F_c(\beta_n) = \frac{1}{n} \quad \text{for sufficiently large } n \tag{3.1}$$

in which $F_c(x) = 1 - \exp(-h_c(x)), h_c(x) = h([x]) + (x - [x])h(([x+1]) - h([x])), h(n) = -\ln(1-F(n))$ and $[.]$ is integer operator.

Many authors have dealt with extreme value theory for a single queueing system other than networks for the case of null recurrent queueing systems which is, in fact, the basic requirement of existence for maximum queue length distributions. Let $X(t)$ be the queue length at time t and T_n be the n-th visit time to state 0 of process $X(t)$ in which $[T_i, T_{i+1})$ is called i-th busy cycle. Serfozo(1988a,b) provided asymptotic distributions of $M_n = \max\{X(t), 0 \le t \le T_n\}$ in $M/M/s, M/G/1$ and $GI/M/1$ queue systems even for positive recurrent processes with appropriate assumptions. However, Serfozo's works have a defect in practical applications since the time period he considered ,i.e., $[0, T_n]$, is a random time period and thus not observable in advance. McCormick and Park(1992) filled this gap for $M/M/s$ queue system by considering a general time interval $[0, t]$ instead of $[0, T_n]$.

We will use Anderson and McCormick and Park's method for the distribution of extreme value for no feedback nodes in multiple servers open Jackson network with the results of section 2. Let $Y_i = \max\{X(t), T_{i-1} \leq t \leq T_i\}, i \geq 1$. Then we know that if $X(0) = 0, Y_i$ are i.i.d. r.v's by the strong Markovian property. Furthermore, when $X(t)$ is a queueing process of $M/M/s$ queues with arrival and service rate λ, μ respectively, then

$$P(Y_i \leq x) = 1 - \Big[\sum_{k=0}^{s-1} k!\Big(\frac{\mu}{\lambda}\Big)^k + \sum_{k=s}^{[x]}\Big(\frac{\lambda}{\mu s}\Big)^{-k}\Big]^{-1} \quad \text{for } x \geq s \tag{3.2}$$

by Chung(1967) and

$$E(T_i - T_{i-1}) = \frac{1}{\lambda(1 - \lambda/\mu s)} \sum_{r=0}^{s-1} \frac{(1 - r/s)(\frac{\lambda}{\mu})^r}{r!} \tag{3.3}$$

by McCormick and Park(1992).

As indicated in Section 2, the total input processes to each node in a network is generally not Poisson process but there are a set of nodes whose total input processes are Poisson. Those nodes are as follows. Let U_i be the set of all nodes j such that $P_{ji} > 0$.

Definition 3.1. A node i is said to be a *bridge* if U_i is nonempty and is not accessible from node i.

From the definition 3.1, U_i is the exit set with respect to node i. This implies that if a node i is a bridge then the input process into the node i is the Poisson process with intensity $\delta_i = \lambda_i + \sum_{j \in U_i} \delta_j P_{ji}$. Let $M_{ti} = \max_{0 \leq u \leq t} Q_i(u)$ and $P_0(.)$ and $P_\pi(.)$ be probability distributions with initial probabilities $P_0(Q_i(0) = 0) = 1$ and $P_\pi(Q_i(0) = n_i) = \pi_i = \rho_i^{n_i}(a_i(n_i))^{-1} P_{0i}$ as given in (2.18), respectively.

Theorem 3.2. Suppose that the processes $\{Q_1(t), Q_2(t), \ldots, Q_N(t)\}$ in a multiple server open Jackson network are in equilibrium. If a node i is a bridge, we then have

$$\limsup_{t\to\infty} P_\pi\{M_{ti} \le \beta_{ti} + x\} \le \exp\Big[-\delta_i\Big(1 - \frac{\rho_i}{s_i}\Big)\frac{(\rho_i/s_i)^x}{\sum_{r=0}^{s_i-1}(r!)^{-1}(1-r/s_i)\rho_i^r}\Big]$$

and

$$\liminf_{t\to\infty} P_\pi\{M_{ti} \le \beta_{ti} + x\} \ge \exp\Big[-\delta_i\Big(1 - \frac{\rho_i}{s_i}\Big)\frac{(\rho_i/s_i)^{x-1}}{\sum_{r=0}^{s_i-1}(r!)^{-1}(1-r/s_i)\rho_i^r}\Big],$$

where $\beta_{ti} = (-\ln(\rho_i/si))^{-1}\{\ln([t]) + \ln(1-\rho_i/si)\}$ and δ_i and ρ_i are given as (1.2).

Proof. Let N_{ti} be the number of busy cycles of node i up to time t. Since node i is a bridge, the queueing process $Q_i(t)$ of node i is a $M/M/s_i$ queueing process with arrival rate δ_i and service rate μ_i in equilibrium. And thus the expected length of busy cycle of node i, η_i, is by (3.3)

$$\eta_i = \frac{1}{\delta_i(1-\rho_i/s_i)}\sum_{r=0}^{s_i-1}\frac{(1-r/s_i)\rho_i^r}{r!}, \tag{3.4}$$

where $\rho_i = \delta_i/\mu_i$. Then it is obvious by renewal theory that

$$\frac{N_{ti}}{t} \to \frac{1}{\eta_i} \text{ in probability as } t \to \infty. \tag{3.5}$$

One can show by (3.5)(cf. Berman(1986)) that for any $\epsilon > 0$ and for sufficiently large t,

$$P_0\{\max_{1\le k\le[t\eta_i^{-1}(1+\epsilon)]} Y_{ki} \le x_{ti}\} + o(1) \tag{3.6}$$

$$\le P_0\{M_{ti} \le x_{ti}\}$$

$$\le P_0\{\max_{1\le k\le[t\eta_i^{-1}(1-\epsilon)]} Y_{ki} \le x_{ti}\} + o(1), \tag{3.7}$$

where $Y_{ki} = \max_{T_{k-1} \le u \le T_k} Q_i(u)$ with $T_0 = 0$ a.s.. Then by Markovian property, Y_{ki} are i.i.d. sequences for each i with its distribution function $F_i(x)$ given as (3.2) indexed by i. It is then elementary to check that

$$\lim_{n\to\infty} \frac{1 - F_i(n)}{1 - F_i(n+1)} = \frac{\rho_i}{s_i}$$

and $\beta_{ni} = (-\ln(\rho_i/s_i))^{-1}\{\ln(n) + \ln(1 - \rho_i/s_i)\}$ satisfying (3.1).
Thus, applying Anderson's approach (or Theorem 2.1 in McCormick and Park(1992)) into (3.6) and (3.7), we get

$$\limsup_{t\to\infty} P_0\{M_{ti} \le \beta_{ti} + x\} \le \exp\Big[-\eta_i^{-1}(1+\epsilon)\Big(\frac{\rho_i}{s_i}\Big)^x\Big] \tag{3.8}$$

and

$$\liminf_{t\to\infty} P_0\{M_{ti} \le \beta_{ti} + x\} \ge \exp\Big[-\eta_i^{-1}(1-\epsilon)\Big(\frac{\rho_i}{s_i}\Big)^{x-1}\Big]. \tag{3.9}$$

Let $\tau_i = \inf\{u > 0; Q_i(u^-) \ne 0 = Q_i(u)\}$ and $\widetilde{M}_{ti} = \max_{\tau_i \le u \le t} Q_i(u)$. And note that, for a fixed positive integer j,

$$\begin{aligned} P_0\ \{M_{ti} \le x_{ti}\} &\le P_\pi\{\widetilde{M}_{ti} \le x_{ti}\} \\ &\le P_0\{\max_{0\le u\le t-j} Q_i(u) \le x_{ti}\} + P_\pi\{\tau_i > j\}. \end{aligned} \tag{3.10}$$

Assume for a moment that

$$\lim_{t\to\infty} P_\pi\{\widetilde{M}_{ti} = M_{ti}\} = 1. \tag{3.11}$$

Then for fixed j, by (3.8)–(3.11) with $x_{ti} = \beta_{ti} + x$

$$\begin{aligned} \exp\Big[-\eta_i^{-1}(1-\epsilon)\Big(\frac{\rho_i}{s_i}\Big)^{x-1}\Big] &\le \lim P_\pi\{M_{ti} \le \beta_{ti} + x\} \\ &\le \exp\Big[-\eta_i^{-1}(1+\epsilon)\Big(\frac{\rho_i}{s_i}\Big)^{x}\Big] + P_\pi\{\tau > j\}. \end{aligned}$$

And then letting $j \to \infty$, we have the proof since ϵ is arbitrarily small.
To end the proof, note that for a positive integer k,

$$P_\pi\{\widetilde{M}_{ti} < M_{ti}\} = P_\pi\{\max_{0\le u\le \tau_i} Q_i(u) > \max_{\tau_i\le u\le t} Q_i(u)\}$$
$$\le P_\pi\{\max_{0\le u\le \tau_i} Q_i(u) > k\} + P_\pi\{\max_{\tau_i\le u\le t} Q_i(u) \le k\}. \quad (3.12)$$

Since $\max\{Q_i(u), 0 \le u \le \tau_i\}$ is nondefective by the assumption (1.1), the first term of (3.12) will be vanished as $k \to \infty$. It is clear by virtue of (3.9) that

$$\max_{\tau_i\le u\le t} Q_i(u) \to \infty \text{ in probability}$$

to make the second term of (3.12) to be disappear as $t \to \infty$.
Hence we have

$$\lim_{t\to\infty} P_\pi\{\widetilde{M}_{ti} < M_{ti}\} = 0$$

for (3.11) to be true. This completes the proof.

We now modify the process $\{Q_1(t), Q_2(t), \ldots, Q_N(t)\}$ defined from multiple servers open Jackson network into a sequence of the process $\{Q_1^{[t]}(t), Q_2^{[t]}(t), \ldots, Q_N^{[t]}(t)\}$ such that for each node i, the $[t]$-th sequence of queueing processes $\{Q_i^{[t]}(t), t \ge 0\}$ has total arrival rate $\delta_i = \delta_{[t]i}$ and service rate $\mu_i = \mu_{[t]i}$ depending on t. As indicated by Serfozo(1988a) or McCormick and Park(1992), an advantage from the above relaxation is that we can avoid to the nonconvergence of M_{ti} exhibited in Theorem 3.2. Since we only consider bridge nodes for asymptotic distribution of maximum queue length of the $[t]$-th sequence in equilibrium up to time t, the problem is equivalent to investigation of distributional behavior for maximum queue length from a sequence of $M/M/s$ queues via results in section 2. McCormick and Park(1992) provided an asymptotic results for extreme values generated from a single $M/M/s$ queueing processes with the dependency of parameters on time t. Thus we can extend the result of McCormick and Park to multiple server Open Jackson network as follows.

Proposition 3.3. Suppose that a sequence of the process $\{Q_1^{[t]}(t), \ldots, Q_N^{[t]}(t)\}$ in multiple servers open Jackson network with arrival rates $\lambda_{[t]1}, \ldots, \lambda_{[t]N}$, service rates $\mu_{[t]1}, \ldots, \mu_{[t]N}$ and transition probabilities $P_{ij}^{[t]}, i, j = 1, \ldots, N$. If node i is a bridge and for $0 < c_i, d_i < \infty$, with η_i given as (3.4),

$$\lim_{t\to\infty} t\Big(1 - \frac{\rho_{[t]i}}{s_i}\Big)^2 = c_i \quad \text{and} \lim_{t\to\infty} \delta_{[t]i}\Big(1 - \frac{\rho_{[t]i}}{s_i}\Big) = d_i,$$

then

$$\lim_{t\to\infty} P\Big[\Big(M_{ti}^{[t]} - \frac{\ln(1 - s_i^{-1}\rho_{[t]i})}{\ln s_i^{-1}\rho_{[t]i}} - (s_i - 1)\Big) \ln \frac{s_i}{\rho_{[t]i}} \le x\Big]$$

$$= \exp\Big[-\frac{c_i}{\eta_i e^x}\Big] \quad \text{in equilibrium,}$$

where $M_{ti}^{[t]} = \max_{0\le u\le t} Q_i^{[t]}(u)$, $\delta_{[t]i} = \lambda_{[t]i} + \sum_{j=1}^N \delta_j^{[t]} P_{ji}^{[t]}$.

Proof. Since the node i is a bridge, the queueing process of node i in equilibrium is exactly same as $M/M/s_i$ queues with arrival rate $\delta_{[t]i}$ and service rate $\mu_{[t]i}$ by Theorem 2.6. Thus by Theorem 2.4 of McCormick and Park the claim holds.

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