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On a Stopping Rule for the Random Walks with Time Stationary Random Distribution Function

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ABSTRACT

Sums of independent random variables $S_n = X_1 + \cdots + X_n$ are considered, where the X_n are chosen according to a stationary process of distributions. For $c > 0$, let t_c be the smallest positive integer n such that $|S_n| > cn^{\frac{1}{2}}$. In this set up we are concerned with finiteness of expectation of t_c and we have some results of sign-invariant process as applications.

KEYWORDS: Stopping rule, Random walks, Random distribution function, Stationarity, Ergodicity.

1. INTRODUCTION

Let $\mathcal{F}$ be a set of distributions on $\mathcal{R}^1$ with the topology of weak convergence, and let $\mathcal{A}$ be the σ-field generated by the open sets. We denote by $\mathcal{F}_1^\infty$ the space consisting of all infinite sequence $(F_1, F_2, \ldots)$, $F_n \in \mathcal{F}$ and $\mathcal{R}_1^\infty$ the space consisting of all infinite sequences $(x_1, x_2, \ldots)$ of real numbers. Take the σ-field

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$\mathcal{A}_1^\infty$ to be the smallest σ-field of subsets of $\mathcal{F}_1^\infty$ containing all finite dimensional rectangles and take $\mathcal{B}_1^\infty$ to be the Borel σ-field of $\mathcal{R}_1^\infty$. Let $\omega = (F_1^\omega, F_2^\omega, \ldots)$ be the coordinate process in $\mathcal{F}_1^\infty$ and ν its distribution on $\mathcal{A}_1^\infty$. Let θ be the coordinate shift: $\theta^k(\omega) = \omega'$ with $F_n^{\omega'} = F_{n+k}^\omega$, $k = 1, 2, \ldots$. On $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ we also define the shift transformation $\sigma : \mathcal{R}_1^\infty \to \mathcal{R}_1^\infty$ by $\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. ν is called stationary if for every $A \in \mathcal{A}_1^\infty, \nu(\theta^{-1}(A)) = \nu(A)$ and we let π be its marginal distribution.

Let $\mathcal{J}$ be the σ-field of invariant sets in $\mathcal{A}_1^\infty$, that is, $\mathcal{J} = \{A|\theta^{-1}(A) = A, A \in \mathcal{A}_1^\infty\}$ and let $\mathcal{I}$ be the σ-field of invariant sets in $\mathcal{B}_1^\infty$, that is, $\mathcal{I} = \{B|\sigma^{-1}(B) = B, B \in \mathcal{B}_1^\infty\}$. For each ω define a probability measure P_ω on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ so that $P_\omega = \prod_{i=1}^\infty F_i^\omega$. A monotone class argument shows that $P_\omega(B), B \in \mathcal{B}_1^\infty$, is $\mathcal{A}_1^\infty$-measurable as a function of ω. So we can define a new probability measure such that $P(B) = \int P_\omega(B)\nu(d\omega)$. Define the process $\{X_n\}$ on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ such that $X_n(x_1, x_2, \ldots) = x_n$ and set $S_n = X_1 + X_2 + \cdots + X_n$. By the definition of $P_\omega, \{X_n\}$ are independent with respect to P_ω and we also note that $\{X_n\}$ is a sequence of independent and identically distributed random variables when $\mathcal{F}$ has just one element. For $c > 0$, let t_c be the smallest positive integer n such that $|S_n| > cn^{\frac{1}{2}} (= \infty$ if no such that n exist). Our principal aim is to prove the following :

Theorem 1. Let $\mathcal{F} = \left\{F \middle| \int x dF(x) = 0\right\}$ and let ν be stationary and ergodic with $\iint x^2 dF(x)\pi(dF) = 1$. Then

$$E_\omega(t_c) = \int t_c dP_\omega < \infty \quad \nu\text{-a.e. } \omega$$

if $0 \le c < 1$.

Special case of the theorem has appeared previously. Blackwell and Freedman (1964) have treated the "coin-tossing" case in which $X_k, k \geq 1$, are symmetric and assume the values ± 1. Many other extensions of Blackwell and Freedman's results have appeared in Chow, Robbins and Teicher(1965) and Gundy and Siegmund(1968). We also treat the case of infinite variance . Results can be applied to sign-invariant random variables.

2. PROOF AND RELATED RESULTS

We fix c and may dispence with it as a subscript in the sequal. In what follows, we deal with the sequence of the stopping rules $\tau = \tau(n) = \min(t, n), n \geq 1$.

We need the following two lemmas.

Lemma 1. Under the conditions of Theorem 1, we have for ν-a.e. ω

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \int_{\{X_k^2 > \epsilon n\}} X_k^2 dP_\omega = 0$$

for every $\epsilon > 0$.

Proof. By the ergodic theorem, we have for ν-a.e. ω

$$\begin{aligned} & n^{-1} \sum_{k=1}^{n} \int_{\{X_k^2 > M\}} X_k^2 dP_\omega \\ &= n^{-1} \sum_{k=1}^{n} \int_{\{x^2 > M\}} x^2 dF_k^\omega(x) \\ &\to \int\int_{\{x^2 > M\}} x^2 dF(x)\pi(dF). \end{aligned}$$

Let $X_M(\omega) = \int_{\{x^2 > M\}} x^2 dF_1^\omega(x)$, then $X_M(\omega) \to 0$ ν-a.e. ω as $M \to \infty$ by the dominated convergence theorem. Since for any $M > 0$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} \int_{\{X_k^2 > \epsilon n\}} X_k^2 dP_\omega \leq \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} \int_{\{X_k^2 > M\}} X_k^2 dP_\omega = EX_M,$$

by letting M go to ∞, right hand side of above goes to 0 by the dominated convergence theorem which proves the lemma.

The following lemma is due to Chow, Robbins and Teicher(1965).

Lemma 2. Let $Y_1, Y_2, \ldots$ be independent with $EY_n = 0, EY_n^2 = \sigma_n^2 < \infty (n \geq 1)$ and set $S_n = \sum_{k=1}^n Y_k$. Then if t is a stopping rule, $E\sum_{k=1}^t \sigma_k^2 < \infty$ implies that $ES_t^2 = E\sum_{k=1}^t \sigma_k^2$.

Lemma 3. Suppose that there exists a ω such that

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^n \int X_k^2 dP_\omega = 1 \quad \text{and} \quad \int t dP_\omega = \infty.$$

Then there exist positive real numbers α, β and positive integer N such that

$$0 < \alpha E_\omega \tau \leq E_\omega X_\tau^2 \leq \beta E_\omega \tau, \quad \text{for all} \quad n \geq N.$$

Proof. Since $\lim_{n\to\infty} n^{-1} \sum_{k=1}^n \int X_k^2 dP_\omega = 1$, ν-a.e. ω by ergodic theorem, for given $\epsilon > 0$ there exist M such that

$$1 + \epsilon \geq n^{-1} \sum_{k=1}^n \int X_k^2 dP_\omega \geq 1 - \epsilon, \quad \text{for all} \quad n \geq M.$$

Then for $m > M$,

$$\begin{aligned} E_\omega \tau(m) &= E_\omega \sum_{k=1}^{\tau(m)} 1 \\ &= E_\omega \Big(\sum_{k=1}^{\tau(m)} 1\Big)\Big(I\{t \leq M\}\Big) + E_\omega \Big(\sum_{k=1}^{\tau(m)} 1\Big)\Big(I\{t > M\}\Big) \\ &\leq M + \frac{1}{1+\epsilon} E_\omega \sum_{k=1}^{\tau(m)} \int X_k^2 dP_\omega \\ &= M + \frac{1}{1+\epsilon} E_\omega S_\tau^2, \end{aligned}$$

where the last equality comes from Lemma 2. Noting that $E_\omega t = \infty$ implies $\lim_{m\to\infty} E_\omega \sum_{k=1}^{\tau(m)} \int X_k^2 dP_\omega = \infty$, we can choose γ and N_1 such that for all $n \geq N_1$

$$0 < \gamma E_\omega \tau \leq E_\omega S_\tau^2 \quad \text{and} \quad \gamma > c^2.$$

Putting $\gamma^2 = \frac{E_\omega X_\tau^2}{E_\omega \tau}$, we have $0 \leq (c^2 - \gamma) + 2c\gamma + \gamma^2$. An examination of this expression as a quadratic form in τ leads to the inequality $(\sqrt{\tau} - c)^2 \leq E_\omega X_\tau / E_\omega \tau$ for $n \geq N_1$. On the other hand

$$\begin{aligned} E_\omega X_\tau^2 &\leq E_\omega \sum_{k=1}^{\tau} X_k^2 \\ &= E_\omega\Big(\sum_{k=1}^{\tau} X_k^2\Big)\Big(I\{\tau \leq M\}\Big) + E_\omega\Big(\sum_{k=1}^{\tau} X_k^2\Big)\Big(I\{\tau > M\}\Big) \\ &\leq K + (1+\epsilon)E_\omega \sum_{k=1}^{\tau} 1 \\ &= K + (1+\epsilon)E_\omega \tau \end{aligned}$$

for some $K < \infty$. Using the fact $\lim_{n\to\infty} E_\omega \tau = \infty$, we can choose β and N_2 such that $E_\omega X_\tau^2 \leq \beta E_\omega \tau$ for all $n \geq N_2$. Taking N to be $\max\{N_1, N_2\}$ completes the proof.

Proof of the Theorem 1. By the ergodic theorem we have for ν-a.e. ω that $\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \int X_k^2 dP_\omega = 1$. Now suppose that $\nu\{\omega | E_\omega t < \infty\} < 1$, then by Lemma 1 and 3, we can choose ω such that $0 < \alpha E_\omega \tau \leq E_\omega X_\tau^2 \leq \beta E_\omega \tau, n = N, N+1, \ldots$, for some α, β and N and $\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \int_{\{X_k^2 > \epsilon n\}} X_k^2 dP_\omega = 0$. Now using the same method as in Theorem 1 of Gundy and Siegmund(1968), the theorem can be proved.

If the condition of ergodicity in Theorem 1 is dropped, then we have some ristriction on $\mathcal{F}$.

Theorem 2. Let $\mathcal{F} = \Big\{F \Big| \int x dF(x) = 0 \text{ and } \int x^2 dF(x) = 1\Big\}$ and let ν be stationary. Then

$$E_\omega(\tau_c) = \int t_c dP_\omega < \infty \quad \nu\text{-a.e. } \omega$$

if $0 \leq c \leq 1$.

Proof. By the same method as in Lemma 1, we also have that for ν-a.e. ω $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\int_{\{X_k^2>\epsilon n\}} X_k^2 dP_\omega = 0$. Now the proof follows immediate from Theorem 1 of Gundy and Siegmund(1968).

Next we consider a result in the case of infinite variance. For each $a > 0$, $n = 1, 2, \ldots$, define

$$Y_n = Y_n(a) = X_n 1\{|X_n| \le an^{\frac{1}{2}}\},\ T_n = Y_1 + \cdots + Y_n$$

$$\beta_n^\omega = (E_\omega T_n)^2,\ B_n^\omega = \text{ variance of } T_n \text{ with respect to } P_\omega.$$

Theorem 3. Let $0 < c < \infty$ and $\mathcal{F} = \{F|F \text{ is symmetric }\}$ and suppose that for some $a > 2c$. If ν is stationary and ergodic with $\int\int x^2 dF(x)\pi(dF) = \infty$, then

$$E_\omega t < \infty \quad \nu\text{-a.e. } \omega.$$

Proof. Since the X_k are symmetrically distributed and independent with respect to P_ω for all ω, $\limsup(\beta_n^\omega / B_n^\omega) = 0$. By Theorem 2 of Gundy and Siegmund(1968) it suffices to show that

$$\lim n^{-1} B_n^\omega = \infty \quad \nu\text{-a.e. } \omega.$$

Let $M(> 0)$ be given. Now choose T such that $\int\int_{\{|x|\le T\}} x^2 dF(x)\pi(dF) \ge M$. Then we have for ν-a.e. ω

$$\begin{aligned}
&n^{-1}\sum_{k=1}^{n}\int_{\{|x|\le a\sqrt{k}\}} x^2 dF_k^\omega(x) \\
&\ge n^{-1}\Big(\sum_{k=1}^{[T^2]-1}\int_{\{|x|\le a\sqrt{k}\}} x^2 dF_k^\omega(x) + \sum_{k=[T^2]}^{n}\int_{\{|x|\le T\}} x^2 dF_k^\omega(x)\Big) \\
&= n^{-1}\sum_{k=1}^{[T^2]-1}\int_{\{|x|\le a\sqrt{k}\}} x^2 dF_k^\omega(x) + \frac{n-[T^2]}{n}\frac{1}{n-[T^2]}\sum_{k=[T^2]}^{n}\int_{\{|x|\le T\}} x^2 dF_k^\omega(x) \\
&\longrightarrow \int\int_{\{|x|\le T\}} x^2 dF(x)\pi(dF) \ge M,
\end{aligned}$$

where the convergence above comes from ergodic theorem and $[\cdot]$ denote the biggest integer less than $\cdot$. Since M is arbitrary, the proof is completed.

Definition 1 (Berman(1962,1965)). Let $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty, \widehat{P})$ be a probability space. Then $\widehat{P}$ is sign invariant if $\widehat{P}\{(x_1, x_2, \ldots) \in B\} = \widehat{P}\{(-1)^{\alpha_1}x_1, (-1)^{\alpha_2}x_2, \ldots) \in B\}$ for all $(\alpha_1, \alpha_2, \ldots) \in \{1, -1\}_1^\infty$ and for all $B \in \mathcal{B}_1^\infty$.

We denote δ_x by the distribution of point mass 1 at x.

Proposition 1. Let $\widehat{P}$ be any sign invariant probability measure on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$. Then one can determine ν on $\mathcal{F}_1^\infty$ where $\mathcal{F} = \{\frac{1}{2}(\delta_y + \delta_{-y}) | y \in \mathcal{R}^+ \cup \{0\}\}$ so that $\widehat{P} = P$, and $\widehat{P}_k$, the k-th marginal of $\widehat{P}$, is given by $\widehat{P}_k = \int F_k^\omega \nu(d\omega), k = 1, 2, \ldots$. Furthermore, if $\widehat{P}$ is stationary (and ergodic) then ν is stationary (and ergodic).

Proof. Let $\mathcal{F} = \{\frac{1}{2}(\delta_y + \delta_{-y}) | y \in \mathcal{R}^+ \cup \{0\}\}$. Define $\phi : \mathcal{R}_1^\infty \to \mathcal{F}_1^\infty$ by $\phi(x) = \omega = (\frac{1}{2}(\delta_{x_1} + \delta_{-x_1}), \frac{1}{2}(\delta_{x_2} + \delta_{-x_2}), \ldots)$ if $x = (x_1, x_2, \ldots)$. Now let $\nu = \widehat{P} \circ \phi^{-1}$ and let $B = \{x | x_1 > t_1, \ldots, x_n > t_n\}, t_i \geq 0, i = 1, 2, \ldots, n$. Note that

$$P_\omega(B) = \begin{cases} \frac{1}{2^n} & \text{if } |x_i| > t_i,\ i = 1, 2, \ldots, n, \\ 0 & \text{if not,} \end{cases}$$

and $\nu\Big\{\omega = (\frac{1}{2}(\delta_{x_1}+\delta_{-x_1}), \frac{1}{2}(\delta_{x_2}+\delta_{-x_2}), \ldots)\Big| |x_i| > t_i, i = 1, 2, \ldots, n\Big\} = 2^n \widehat{P}(B)$. Then

$$P(B) = \int P_\omega(B)\nu(d\omega) = \frac{1}{2^n} 2^n \widehat{P}(B) = \widehat{P}(B).$$

This proves the proposition.

Hence we have the following results as corollaries of Theorem 1 and Theorem 3.

Corollary 1. Let $\{X_n\}$ be sign-invariant and stationary ergodic with $EX_1^2 = 1$. Then for $0 < c < 1$, we have

$$E\Big[t_c \Big| |X_1|, |X_2|, \ldots\Big] < \infty \quad \text{a.s.}$$

Corollary 2. Let $\{X_n\}$ be sign-invariant and stationary ergodic with $EX_1^2 = \infty$. Then for $0 < c < \infty$, we have

$$E\Big[t_c\Big||X_1|, |X_2|, \ldots\Big] < \infty \quad \text{a.s.}$$

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