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An Invariance Principle for Stationary Strong Mixing Random Fields†

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ABSTRACT

In this paper the concept of strong mixing is extended to random fields, and an invariance principle is obtained for stationary strong mixing random fields.

KEYWORDS: Stationary random fields, Strong-mixing, Invariance principle, Uniform integrability.

1. INTRODUCTION

Let T^d be the d-fold product of the closed unit interval, $[0,1]$. Let C_d be the set of all continuous functions on T^d with the uniform metric and, as in Bickel and Wichura(1971), let us denote by D_d the Skorohod function space on T^d. All properties of D_d that we need can be found in Bickel and Wichura(1971). A subset B of T^d is called a block if it is of the form $\prod_{j=1}^{d}(s_j, t_j]$, $(s_j, t_j]'s$ being

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half-closed subintervals of $[0,1]$. If $X = \{X(\underline{t}) : \underline{t} \in T^d\}$ is a stochastic process, then the increment $X(B)$ of X around a block $B = \prod_{j=1}^{d}(s_j, t_j]$ is given by

$$X(B) = \sum_{\varepsilon_1=0,1}\sum_{\varepsilon_2=0,1}\cdots\sum_{\varepsilon_d=0,1}(-1)^{d-\sum\varepsilon_j}$$
$$\times X(s_1+\varepsilon_1(t_1-s_1), s_2+\varepsilon_2(t_2-s_2), \ldots, s_d+\varepsilon_d(t_d-s_d)).$$

On T^d as well as Z^d we use the maxinum norm, i.e., if $\underline{t} = (t_1, t_2, \ldots, t_d) \in T^d$ or $\underline{n} = (n_1, n_2, \ldots, n_d) \in Z^d$, then $\|\underline{t}\| = \max_{1\le j\le d}|t_j|$, and $\|\underline{n}\| = \max_{1\le j\le d}|n_j|$.
The Wiener process $W = \{W(\underline{t}) : \underline{t} \in T^d\}$ on T^d is characterized by
(a) $P\{W \in C_d\} = 1$,
(b) if $B_1, B_2, \ldots, B_k$ are pairwise disjoint blocks in T^d, then the increments $W(B_1)$, $W(B_2), \ldots, W(B_k)$ are independent normal random variables with means zero and variances $|B_1|, |B_2|, \ldots, |B_k|$, $|\quad|$ being the d-dimensional Lebesgue measure on T^d.

Let $\{\xi_{\underline{n}} : \underline{n} \in Z^d\}$ be a random field, i.e., a collection of random variables indexed by time-set Z^d. The random field is said to be stationary if for each finite subset S of Z^d, and each $\underline{m} \in Z^d$, the joint distribution of $\{\xi_{\underline{n}+\underline{m}} : \underline{n} \in S\}$ is the same as that of $\{\xi_{\underline{n}} : \underline{n} \in S\}$. Here $\underline{n}+\underline{m}$ is the usual coordinatewise sum. We suppose throughout that $\{\xi_{\underline{n}} : \underline{n} \in Z^d\}$ is a stationary random field with $E\xi_{\underline{0}} = 0, E{\xi^2}_{\underline{0}} < \infty$. For $\underline{n} \ge \underline{1}$, put

$$S_{\underline{n}} = \sum_{\underline{1}\le\underline{j}\le\underline{n}} \xi_{\underline{j}},$$

and define for $\underline{t} = (t_1, t_2, \cdots, t_d) \in T^d$,

$$W_n(\underline{t}) = (\sigma^2 n^d)^{-\frac{1}{2}} S_{[nt_1],[nt_2],\ldots,[nt_d]},$$

where $\sigma^2 = \sum_{\underline{j}\in Z^d} \mathrm{Cov}(\xi_{\underline{0}}, \xi_{\underline{j}})$.
The random field $\{\xi_{\underline{n}}; \underline{n} \in Z^d\}$ fulfills the invariance principle if W_n converges weakly to Wiener process W on D_d.

In this paper we wish to extend the concept of strong mixing (α-mixing) to random fields and investigate an invariance principle for such random fields. This theorem may be regarded as a generalization of Theorem 1 of Oodaira and Yoshihara (1972) to "multivariate time". Similar results were proved eariler by Deo (1975), for φ-mixing random fields of which the frame work was assumed in [4]. For each i, $1 \le i \le d$, let

$$0 = a_1^{(i)} < b_1^{(i)} < a_2^{(i)} < b_2^{(i)} < \cdots < a_n^{(i)} < b_n^{(i)} = 1$$

be real numbers. Call a collection of blocks in T^d "strongly separated" if it is of the form

$$\Big\{\prod_{i=1}^{d}(a_k^{(i)}, b_k^{(i)}] : 1 \le k \le n, \quad 1 \le i \le d\Big\}$$

or if it is a subfamily of such a family of blocks.

For $x \in D_d$ and $0 < \delta < 1$, define the modules $w(x;\delta)$ by

$$w(x;\delta) = \sup\Big\{|x(\underline{t}) - x(\underline{s})| : \|\underline{t} - \underline{s}\| \le \delta, \underline{s}, \underline{t} \in T^d\Big\}.$$

If $\underline{n} = (n_1, \ldots, n_d)$, let $|\underline{n}|$ stand for the product $n_1 n_2 \cdots n_d$. Define $|\underline{t}|$ similarly for $\underline{t} \in T^d$. Deo (1975) proved the following lemma and used to show an invariance principle for φ-mixing random field.

Lemma 1.1 (Deo 1975). Let $\{Y_n(\underline{t}) : n \in N, \underline{t} \in T^d\}$ be a sequence of stochastic processes in D_d such that,

(i) $EY_n(\underline{t}) \longrightarrow 0$, $EY_n{}^2(\underline{t}) \longrightarrow |\underline{t}|$ as $n \longrightarrow \infty$ for each $\underline{t} \in [0,1]^d$,

(ii) $\{Y_n{}^2(\underline{t}) : n \in N, \underline{t} \in T^d\}$ is uniformly integrable for each $\underline{t}$,

(iii) if $B_1, B_2, \ldots, B_k$ are a collection of strongly separated blocks, then the increments $Y_n(B_1), Y_n(B_2), \ldots, Y_n(B_k)$ are asymptotically independent in the sense that if $H_1, H_2, \ldots, H_k$ are arbitrary linear Borel sets, then the difference

$$\Big|P\{Y_n(B_1) \in H_1, Y_n(B_2) \in H_2, \cdots, Y_n(B_k) \in H_k\} - \prod_{i=1}^{k} P[Y_n(B_i) \in H_i]\Big| \to 0$$

as $n \longrightarrow \infty$ and

(iv) for each $\varepsilon > 0, \eta > 0$, there exists a $\delta > 0$ such that

$$P\{w(Y_n, \delta) > \varepsilon\} < \eta \text{ for all sufficiently large } n.$$

Then the sequence $\{Y_n(\underline{t})\}$ of stochastic processes converges weakly, in D_d, to the d-parameter Wiener process W.

The framework of strong mixing for random fields and the main theorem are stated in Section 2. The proof of our main theorem as well as some lemmas is given in Section 3.

2. PRELIMINARIES AND MAIN THEOREM

As Deo(1975) extended the concept of φ-mixing to the random fields we will extend the concept of strong mixing(α-mixing) to the random fields in this section. For each j $(1 \le j \le d)$ and $r \ge 0$, let $\mathcal{A}^+(j;r)$ be the σ-field generated by $\{\xi_{n_1,n_2,\ldots,n_d} : n_j \ge r,$ other n_i's unrestricted$\}$ and let $\mathcal{A}^-(j;r)$ be the σ-field generated by $\{\xi_{n_1,n_2,\ldots,n_d} : n_j \le r,$ other n_i's unrestricted$\}$. For $r \ge 1$, we write $\alpha(j;r) = \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{A}^-(j;r) \text{ and } B \in \mathcal{A}^+(j;r)\}$ and $\alpha(r) = \max_{1 \le j \le d} \alpha(j;r)$. Also $\alpha(0) = 1$. Clearly $\{\alpha(r)\}$ is a decreasing sequence of positive integers. If $\alpha(r) \longrightarrow 0$ we say that the random field $\{\xi_{n_1,n_2,\ldots,n_d} : (n_1, n_2, \ldots, n_d) \in Z^d\}$ is strong mixing (α-mixing). This is a natural extension to multivariate time parameter of the well known concept of strong mixing for sequences of random variables.

The random field $\{\xi_{\underline{n}}\}$ may be defined only for $\underline{n} \ge \underline{1}$. In this case we define $\alpha(j;r) = \sup|P(A \cap B) - P(A)P(B)|$, where the supremum is taken over all sets A, B such that for some m, A is in the σ-field generated by $\{\xi_{n_1,n_2,\ldots,n_d} : 1 \le n_j \le m,$ other n_i's $\ge 1\}$ and B is in the σ-field generated by $\{\xi_{n_1,n_2,\ldots,n_d} : n_j \ge m + r,$ othern_i's $\ge 1\}$. Also $\alpha(r) = \max_{1 \le j \le d} \alpha(j;r)$. Given such random field with "one-sided" time set, we can construct a new random field with time set all of Z^d and with the same finite-dimensional distributions and the same α-values. This fact may be proved along the same lines as in the case of univariate time. Thus, without loss of generality, we will assume that the ramdom field $\{\xi_{\underline{n}} : \underline{n} \in Z^d\}$ is defined over all of Z^d.

Let us write $R(\underline{j}) = E(\xi_{\underline{0}}\xi_{\underline{j}}), \underline{j} \in Z^d$. In this paper the limit $\underline{n} = (n_1, n_2, \ldots, n_d) \to \infty$ will mean $\min_{1\le i\le d} n_i \longrightarrow \infty$.

Lemma 2.1. Let $\{\xi_{\underline{j}} : \underline{j} \in Z^d\}$ be a stationary strong mixing random field with $E\xi_{\underline{j}} = 0$, $E\xi^2{}_{\underline{j}} < \infty$. Assume

$$\sum_{r=1}^{\infty} r^{d-1}\alpha(r) < \infty, \tag{2.1}$$

the random variables $\xi_{\underline{j}}$ are bounded, i.e., $|\xi_{\underline{j}}| < C < \infty$ w.p. 1. (2.2)

Then (2.3), (2.4) and (2.5) below hold.

$$\sum_{\underline{j}\in Z^d} |R(\underline{j})| < \infty \tag{2.3}$$

$$|\underline{n}|^{-1}E(S_{\underline{n}}{}^2) \longrightarrow \sum_{\underline{j}\in Z^d} R(\underline{j}) = \sigma^2, \text{ as } \underline{n} \longrightarrow \infty \tag{2.4}$$

$$|\underline{n}|^{-1}E(S_{\underline{n}}^2) \le 4A(d;\alpha)C^2, \text{ for all } \underline{n} \ge \underline{1}, \tag{2.5}$$

where $A(d,\alpha) = 1 + 2d\sum_{r=1}^{\infty}(2r+1)^{d-1}\alpha(r)$.

Proof. Note that there are, at most, $2d(2r+1)^{d-1}$ points $\underline{j} \in Z^d$ such that $\|\underline{j}\| = r$, where r is a positive integer and there exists an inequality

$$|R(\underline{j})| = |E(\xi_{\underline{0}}\xi_{\underline{j}})| \le 4C^2\alpha(r). \tag{2.6}$$

(See Theorem 17.2.1 of Ibragimov and Linnik [8]). Using those facts, the proof can be easily completed along the lines of Lemma 3 on page 172 of Billingsley (1968) (see [4]).

Throughout the rest of this paper, σ^2 will be as defined in (2.4), and we will assume $\sigma^2 > 0$.

Theorem 2.2. Let $\{\xi_{\underline{n}} : \underline{n} \in Z^d\}$ be a stationary strong mixing random field with $E\xi_{\underline{j}} = 0$, $E\xi_{\underline{j}}{}^2 < \infty$. Suppose (2.1) and (2.2) hold. Then the sequence $\{W_n(\underline{t}) : n \in N,\ \underline{t} \in T^d\}$ of stochastic processes converges weakly, in D_d, to the d-parameter Wiener process W.

3. PROOF OF THEOREM 2.2

Lemma 3.1. Suppose $\{\xi_{\underline{n}} : \underline{n} \in Z^d\}$ is a stationary strong mixing random field satisfying conditions of Theorem 2.2. Then we can find a constant $A = A(d, \alpha)$ depending only on d and the α-sequence such that, for all $\underline{n} \geq \underline{1}$,

$$ES_{\underline{n}}^4 \leq A|\underline{n}|^2. \tag{3.1}$$

Proof. To alleviate the notational burden we write out the detail, for $d = 2$. For bigger d the proof is similar but more tedious. Let k be a fixed natural number whose exact value will be specified in what follows. Applying Lemma 4.1 of Davydov(1968) we find a constant K such that for $n_1 \leq k, n_2 \leq k$,

$$E|S_{n_1,n_2}|^4 \leq E(n_1 n_2 C)^4 \leq C^4 k^2 n_1^2 n_2^2 = K n_1^2 n_2^2. \tag{3.2}$$

We now show by induction that (3.2) holds for all $n_1 \geq k$ and $n_2 \leq k$:
For $n_1 = 2m_1 (k < 2m_1 \leq 2k)$ and $n_2 \leq k$,

$$\begin{aligned} E|S_{2m_1,n_2}|^4 &= E|S_{m_1,n_2} + \hat{S}_{m_1,n_2}|^4 \\ &= E|S_{m_1,n_2}|^4 + E|\hat{S}_{m_1,n_2}|^4 \\ &\quad +4\{E(S^3_{m_1,n_2}\hat{S}_{m_1,n_2}) + E(S_{m_1,n_2}\hat{S}^3_{m_1,n_2})\} + 6E(S^2_{m_1,n_2}\hat{S}^2_{m_1,n_2}), \end{aligned}$$

where $\hat{S}_{m_1,n_2} = S_{2m_1,n_2} - S_{m_1,n_2}$.

Using Theorem 17.2.1. of Ibragimov and Linnik [8], Hölder's inequality and (3.2) we obtain the estimates

$$ES^4_{m_1,n_2} + E\hat{S}^4_{m_1,n_2} \leq 2Km_1^2 n_2^2 = \frac{K}{2}(2m_1)^2 n_2^2,$$

$$\begin{aligned}
4\{E|S^3_{m_1,n_2}\hat{S}_{m_1,n_2}| &+ E|S_{m_1,n_2}\hat{S}^3_{m_1,n_2}|\} \\
&\leq 4\{(ES^4_{m_1,n_2})^{\frac{3}{4}}(E\hat{S}^4_{m_1,n_2})^{\frac{1}{4}} + (ES^4_{m_1,n_2})^{\frac{1}{4}}(E\hat{S}^4_{m_1,n_2})^{\frac{3}{4}}\} \\
&= 2K(2m_1)^2n_2^2, \\
6E(S^2_{m_1,n_2}\hat{S}^2_{m_1,n_2}) &\leq 6\{(ES^4_{m_1,n_2})^{\frac{1}{2}}(E\hat{S}^4_{m_1,n_2})^{\frac{1}{2}}\} \\
&= \frac{3}{2}K(2m_1)^2n_2^2.
\end{aligned}$$

Thus

$$E|S_{2m_1,n_2}|^4 \leq 4K(2m_1)^2n_2^2. \tag{3.3}$$

Similary, for $n_1 \leq k$ and $n_2 = 2m_2(k < 2m_2 \leq 2k)$,

$$E|S_{n_1,2m_2}|^4 \leq 4Kn_1^2(2m_2)^2. \tag{3.4}$$

Let $n_1 = 2m_1 + 1$ and $n_2 \leq k$. Then

$$\begin{aligned}
E|S_{2m_1+1,n_2}|^4 &= E|S_{2m_1,n_2} + \hat{S}_{1,n_2}|^4 \\
&\leq E|S_{2m_1,n_2}|^4 + E|\hat{S}_{1,n_2}|^4 + 4\{E|S^3_{2m_1,n_2}\hat{S}_{1,n_2}| \\
&\qquad + E|S_{2m_1,n_2}\hat{S}^3_{1,n_2}|\} + 6E(S^2_{2m_1,n_2}S^2_{1,n_2}),
\end{aligned}$$

where $\hat{S}_{1,n_2} = S_{2m_1+1,n_2} - S_{2m_1,n_2}$.

By (3.2),(3.3) and Hölder's inequality

$$E|S_{2m_1,n_2}|^4 \leq 4K(2m_1)^2n_2^2, \quad E|\hat{S}_{1,n_2}|^4 \leq Kn_2^2,$$

$$\begin{aligned}
4\{E|S^3_{2m_1,n_2}\hat{S}_{1,n_2}| &+ E|S_{2m_1,n_2}\hat{S}^3_{1,n_2}|\} \\
&\leq 4\{(ES^4_{2m_1,n_2})^{\frac{3}{4}}(E\hat{S}^4_{1,n_2})^{\frac{1}{4}} + (ES^4_{2m_1,n_2})^{\frac{1}{4}}(E\hat{S}^4_{1,n_2})^{\frac{3}{4}}\} \\
&\leq 4\{(4K(2m_1)^2n_2^2)^{\frac{3}{4}}(Kn_2^2)^{\frac{1}{4}} + (4K(2m_1)^2n_2^2)^{\frac{1}{4}}(Kn_2^2)^{\frac{3}{4}}\} \\
&\leq 16K(2m_1)^2n_2^2 + 8K(2m_1)n_2^2,
\end{aligned}$$

$$\begin{aligned}
6(ES^2_{2m_1,n_2}S^2_{1,n_2}) &\le 6(ES^4_{2m_1,n_2})^{\frac{1}{2}}(ES^4_{1,n_2})^{\frac{1}{2}} \\
&\le 6(4K(2m_1)^2n_2^2)^{\frac{1}{2}}(Kn_2^2)^{\frac{1}{2}} \\
&\le 12K(2m_1)n_2^2, \\
E|S_{2m_1+1,n_2}|^4 &\le 20K(2m_1)^2n_2^2 + Kn_2^2 + 20K(2m_1)n_2^2 \\
&\le 20K(2m_1+1)^2n_2^2. \qquad (3.5)
\end{aligned}$$

Similarly, for $n_1 \le k$ and $n_2 = 2m_2+1$,

$$E|S_{n_1,2m_2+1}|^4 \le 20Kn_1^2(2m_2+1)^2. \qquad (3.6)$$

Next we show that (3.2) holds for all $n_1 \ge 1$ and $n_2 = 2m_2 (k < 2m_2 \le 2k)$: Let $n_1 = 2m_1 (k < 2m_1 \le 2k)$ and $n_2 = 2m_2 (k < 2m_2 \le 2k)$.

$$\begin{aligned}
|S_{2m_1,2m_2}|^4 &= |S_{m_1,2m_2} + \hat{S}_{m_1,2m_2}|^4 \\
&\le E|S_{m_1,2m_2}|^4 + E|\hat{S}_{m_1,2m_2}|^4 + 4\{E|S^3_{m_1,2m_2}\hat{S}_{m_1,2m_2}| \\
&\quad + E|S_{m_1,2m_2}\hat{S}^3_{m_1,2m_2}|\} + 6E|S^2_{m_1,2m_2}\hat{S}_{m_1,2m_2}|,
\end{aligned}$$

where $\hat{S}_{m_1,2m_2} = S_{2m_1,2m_2} - S_{m_1,2m_2}$.

By (3.4) and Hölder's inequality we obtain the estimates

$$\begin{aligned}
E|S_{m_1,2m_2}|^4 + E|\hat{S}_{m_1,2m_2}|^4 &\le 8Km_1^2(2m_2)^2 = 2K(2m_1)^2(2m_2)^2, \\
4\{E|S^3_{m_1,2m_2}\hat{S}_{m_1,2m_2}| &+ E|S_{m_1,2m_2}\hat{S}^3_{m_1,2m_2}|\} \\
&\le 4\{(E|S_{m_1,2m_2}|^4)^{\frac{3}{4}}(E|S_{m_1,2m_2}|^4)^{\frac{1}{4}} \\
&\quad +(E|S_{m_1,2m_2}|^4)^{\frac{1}{4}}(E|S_{m_1,2m_2}|^4)^{\frac{3}{4}}\} \\
&\le 32Km_1^2(2m_2)^2 \\
&= 8K(2m_1)^2(2m_2)^2, \\
6E|S^2_{m_1,2m_2}\hat{S}^2_{m_1,2m_2}| &\le 6E(|S_{m_1,2m_2}|^4E|\hat{S}_{m_1,2m_2}|^4)^{\frac{1}{2}} \\
&= 6K(2m_1)^2(2m_2)^2.
\end{aligned}$$

Thus

$$E|S_{2m_1,2m_2}|^4 \le 16K(2m_1)^2(2m_2)^2. \tag{3.7}$$

Let $n_1 = 2m_1 + 1, n_2 = 2m_2 (k < 2m_1 \le 2k, k < 2m_2 \le 2k)$,

$$\begin{aligned} E|S_{2m_1,2m_2} + \hat{S}_{1,2m_2}|^4 &= E|S_{2m_1,2m_2}|^4 + E|\hat{S}_{1,2m_2}|^4 \\ &\quad +4\{E|S^3_{2m_1,2m_2}\hat{S}_{1,2m_2}| + E|S_{2m_1,2m_2}\hat{S}^3_{1,2m_2}|\} \\ &\quad +6E|S^2_{2m_1,2m_2}\hat{S}^2_{1,2m_2}|, \end{aligned}$$

where $\hat{S}_{1,2m_2} = S_{2m_1+1,2m_2} - S_{2m_1,2m_2}$.

By (3.4),(3.7) and Hölder's inequality, we obtain estimates

$$\begin{aligned} E|S_{2m_1,2m_2}|^4 &\le 16K(2m_1)^2(2m_2)^2, \\ E|\hat{S}_{1,2m_2}|^4 &\le 4K(2m_2)^2, \end{aligned}$$

$$\begin{aligned} &4\{E|S^3_{2m_1,2m_2}\hat{S}_{1,2m_2}| + E|S_{2m_1,2m_2}\hat{S}^3_{1,2m_2}|\} \\ \le\ & 4\{(E|S_{2m_1,2m_2}|^4)^{\frac{3}{4}}(E|\hat{S}_{1,2m_2}|^4)^{\frac{1}{4}} + (E|S_{2m_1,2m_2}|^4)^{\frac{1}{4}}(E|\hat{S}_{1,2m_2}|^4)^{\frac{3}{4}}\} \\ \le\ & 4\{(16K(2m_1)^2(2m_2)^2)^{\frac{3}{4}}(4K(2m_2)^2)^{\frac{1}{4}} + (16K(2m_1)^2(2m_2)^2)^{\frac{1}{4}}(4K(2m_2)^2)^{\frac{3}{4}} \\ \le\ & 48K(2m_1)^2(2m_2)^2 + 24K(2m_1)(2m_2)^2, \end{aligned}$$

$$\begin{aligned} 6E|S^2_{2m_1,2m_2}\hat{S}^2_{1,2m_2}| &\le 6(E|S_{2m_1,2m_2}|^4 E|\hat{S}_{1,2m_2}|^4)^{\frac{1}{2}} \\ &\le 6(16K(2m_1)^2(2m_2)^2 4K(2m_2)^2)^{\frac{1}{2}} \\ &= 48K(2m_1)(2m_2)^2. \end{aligned}$$

Thus

$$\begin{aligned} E|S_{2m_1+1,2m_2}|^4 &\le 64K(2m_1)^2(2m_2)^2 + 72K(2m_1)(2m_2)^2 + 4K(2m_2)^2 \\ &\le 64K(2m_1+1)^2(2m_2)^2. \end{aligned} \tag{3.8}$$

Similarly

$$E|S_{2m_1,2m_2+1}|^4 \leq 64K(2m_1)^2(2m_2+1)^2. \tag{3.9}$$

Finally, let $n_1 = 2m_1+1, n_2 = 2m_2+1 (k < 2m_1 \leq 2k, k < 2m_2 \leq 2k)$.

$$\begin{aligned} E|S_{2m_1+1,2m_2+1}|^4 &\leq E|S_{2m_1,2m_2+1} + \hat{S}_{1,2m_2+1}|^4 \\ &= E|S_{2m_1,2m_2+1}|^4 + E|\hat{S}_{1,2m_2+1}|^4 + 4\{E|S^3_{2m_1,2m_2+1}\hat{S}_{1,2m_2+1}| \\ &\quad +E|S_{2m_1,2m_2+1}\hat{S}^3_{2m_2+1}|\} + 6\{E|S^2_{2m_1,2m_2+1}\hat{S}^2_{1,2m_2+1}|\}, \end{aligned}$$

where $\hat{S}_{1,2m_2+1} = S_{2m_1+1,2m_2+1} - S_{2m_1,2m_2+1}$.

By (3.6),(3.9) and Hölder's inequality

$$\begin{aligned} E|S_{2m_1,2m_2+1}|^4 &\leq 64K(2m_1)^2(2m_2+1)^2 \\ E|\hat{S}_{1,2m_2+1}|^4 &\leq 20K(2m_2+1)^2, \end{aligned}$$

$$\begin{aligned} &4\{E|S^3_{2m_1,2m_2+1}\hat{S}_{1,2m_2+1}| + E|S_{2m_1,2m_2+1}\hat{S}^3_{1,2m_2+1}|\} \\ &\leq 4\{(E|S_{2m_1,2m_2+1}|^4)^{\frac{3}{4}}(E|\hat{S}_{1,2m_2+1}|^4)^{\frac{1}{4}} + (E|S_{2m_1,2m_2+1}|^4)^{\frac{1}{4}}(E|\hat{S}_{1,2m_2+1}|^4)^{\frac{3}{4}}\} \\ &\leq 4\{(64K(2m_1)^2(2m_2+1)^2)^{\frac{3}{4}}(20K(2m_2+1)^2)^{\frac{1}{4}} \\ &\qquad +(64K(2m_1)^2(2m_2+1)^2)^{\frac{1}{4}}(20K(2m_2+1)^2)^{\frac{3}{4}}\} \\ &\leq 4\{64K(2m_1)^2(2m_2+1)^2 + 64K(2m_1)(2m_2+1)^2\} \\ &= 256K(2m_1)^2(2m_2+1)^2 + 256K(2m_1)(2m_2+1)^2, \end{aligned}$$

$$\begin{aligned} &6E|S^2_{2m_1,2m_2+1}\hat{S}^2_{1,2m_2+1}| \\ &\leq (E|S_{2m_1,2m_2+1}|^4 E|\hat{S}_{1,2m_2+1}|^4)^{\frac{1}{2}} \\ &\leq 6(64K(2m_1)^2(2m_2+1)^2)^{\frac{1}{2}}(20K(2m_2+1)^2)^{\frac{1}{2}} \\ &\leq 240K(2m_1)(2m_2+1)^2. \end{aligned}$$

Thus

$$E|S_{2m_1+1,2m_2+1}|^4 \leq 320K(2m_1+1)^2(2m_2+1)^2. \tag{3.10}$$

Therefore the proof is complete.

Lemma 3.2. There exists $B > 0$ such that

$$E|W_n(t_1,t_2)|^4 \leq B(t_1t_2)^2 \text{ for all } \leq t_1, t_2 \leq 1. \tag{3.11}$$

Proof. According to Lemma 3.1

$$\begin{aligned} E|W_n(t_1,t_2)|^4 &= E|(\sigma^2 n^2)^{-\frac{1}{2}} S_{[nt_1][nt_2]}|^4 \\ &\leq \frac{A([nt_1][nt_2])^2}{\sigma^4 n^4} \leq B(t_1t_2)^2. \end{aligned}$$

Proof of the Theorem 2.2. : We will prove this theorem by using Lemma 1.1 (Lemma 4 of Deo[4]). For the sequence $\{W_n\}$ of stochastic processes in Theorem 2.2, the conditions (i) and (iii) of Lemma 1.1 are trivially seen to be satisfied. Using Lemma 3.1, the estimate (2.5) in Lemma 2.1 and the arguments on page 32 of Billingsley[2] we can show that the condition (ii) of Lemma 1.1 is also satisfied by $\{W_n(t) : n \in N, T \in T^d\}$ (cf. [4]). Finally, the equation (1) and Theorem 1 of Bickel and Wichura (1971) yield a natural extension of Theorem 12.3 of Billingley (1968) to multivariate time. Using this extension and Lemma 3.2 proves that the condition (iv) of Lemma 1.1 is satisfied by $\{W_n(t) : n \in N, t \in T^d\}$. (cf. Lemma 5 of Deo [4]) Thus the proof of Theorem 2.2 is complete.

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