Journal of the Korean
Statistical Society
Vol. 24, No. 1, 1995

Estimators with Nondecreasing Risk in a Multivariate Normal Distribution

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ABSTRACT

Consider a p-variate ($p \geq 4$) normal distribution with mean $\underline{\theta}$ and identity covariance matrix. For estimating $\underline{\theta}$ under a quadratic loss we investigate the behavior of risks of Stein-type estimators which shrink the usual estimator toward the mean of observations. By using concavity of the function appearing in the shrinkage factor together with new expectation identities for noncentral chi-squared random variables, a characterization of estimators with nondecreasing risk is obtained.

KEYWORDS: p-variate normal distribution, Quadratic loss, Stein-type estimator, Noncentral chi-squared distribution, Concavity.

1. INTRODUCTION

Let $\underline{X}$ have a p-variate ($p \geq 3$) normal distribution with mean $\underline{\theta}$ and identity covariance matrix. It is desired to estimate $\underline{\theta}$ using an estimator $\underline{d}(\underline{X})$ under a quadratic loss $L(\underline{\theta}, \underline{d}) = |\underline{\theta} - \underline{d}(\underline{X})|^2$ where $|\cdot|$ denotes Euclidean distance. The risk of $\underline{d}(\underline{X})$, $R(\underline{\theta}, \underline{d})$, is given by $R(\underline{\theta}, \underline{d}) = E_{\underline{\theta}}|\underline{\theta} - \underline{d}(\underline{X})|^2$.

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Blyth(1951) for $p = 1$ and Stein(1956) for $p = 2$ proved that the usual estimator (maximum liklihood, uniformly minimum variance, or best invariant) $\underline{X}$ is admissible. However, $\underline{X}$ ceases to be an admissible estimator for $p \geq 3$. This was first proved by Stein(1956). An explicit estimator dominating $\underline{X}$ was produced by James and Stein(1961), which shrinks $\underline{X}$ toward the origin. Subsequently a number of authors provided classes of Stein-type estimators dominating $\underline{X}$ (see, for examples Efron and Morris (1976), Ghosh, Hwang, and Tsui(1984) where other references are cited). One common feature of the above classes of estimators dominating $\underline{X}$ is that they are all spherically symmetric shrinking $\underline{X}$ toward some particular point, not necessarily the origin.

Casella(1990) considered Stein-type estimators of the form $\underline{d}(\underline{X}) = (1 - \frac{r(|\underline{X}|^2)}{|\underline{X}|^2})\underline{X}$, $p \geq 3$, and gave sufficient conditions on $r(|\underline{X}|^2)$ for the risk of the estimator to be nondecreasing in $|\underline{\theta}|^2$. In particular, this result generalizes a result of Efron and Morris(1973), who showed that the risk of ordinary James-Stein estimator with $r(|\underline{X}|^2) = p - 2$ is nondecreasing in $|\underline{\theta}|^2$.

Consider Stein-type estimators of the form

$$\underline{d}(\underline{X}) = \bar{X}\underline{1} + (\underline{X} - \bar{X}\underline{1})(1 - r(S^2)/S^2), \ p \geq 4, \tag{1.1}$$

where $S^2 = |\underline{X} - \bar{X}\underline{1}|^2$ and $\underline{1}$ is the column vector of ones. The estimator in (1.1) shrinks $\underline{X}$ toward $\bar{X}\underline{1}$. Lindley (1962) proposed an estimator with $r(t) \equiv p - 3$ in the discussion of Stein's(1962) paper. In a very interesting article Lindley and Smith(1972) presented a class of biased linear estimators with $r(t) = \frac{1}{2}t$ motivated from a Bayesian view-point. Efron and Morris(1973) suggested that Lindley's estimator is good compared to the ordinary James-Stein estimator in the sense of the relative savings loss. Leonard (1976) proposed generalized Bayes estimators with $r(t) = (p-3) - 2[\exp(-\frac{t}{2})]/\{\int_0^1 \lambda^{[(p-3)/2]-1}[\exp(-\lambda t/2)]d\lambda\}$ in a hierarchical Bayesian set-up. He demonstrated that Lindley's estimator shrinks more rapidly toward $\bar{X}\underline{1}$ than does his estimator. Also, Berger(1980) provided generalized Bayes estimators with $r(t) = 2n - 2[\exp(-\frac{t}{2})]/\{\int_0^1 \lambda^{n-1}[\exp(-\lambda t/2)]d\lambda\}, n > 0$, in a hierarchical Bayesian model which contains Leonard's estimator as a special case with $n = \frac{p-3}{2}$.

In this paper we investigate the risk behavior of the estimator in (1.1). In Section 2 we give several chi-squared identities which are crucial in our analysis. In Section 3 we provide sufficient conditions on $r(t)$ for the risk of the estimator in (1.1) to be nondecreasing in $|\underline{\theta} - \bar{\theta}\underline{1}|^2$.

2. CHI-SQUARED IDENTITIES

This section deals with some chi-squared identities which may seem new. In the following $\chi^2_{k;\delta^2}$ denotes a noncentral chi-squared random variable with k degrees of freedom and noncentrality parameter $\delta^2 = \mid \underline{\theta} - \bar{\theta}\underline{1} \mid^2$. The value of $\underline{\theta}$ is indicated by the subscript on the expectation operator.

Lemma 2.1. Let $h : [0,\infty] \longrightarrow (-\infty,\infty)$. Then, provided the expectations exist,

$$E_{\underline{\theta}}\{\underline{\theta}'(\underline{X} - \bar{X}\underline{1})h(S^2)/S^2\} = \delta^2 E_{\underline{\theta}}\{h(\chi^2_{p+1;\delta^2})/\chi^2_{p+1;\delta^2}\}, \tag{2.1}$$

$$\begin{aligned}\delta^2 E_{\underline{\theta}}\{h(\chi^2_{p+1;\delta^2})/\chi^2_{p+1;\delta^2}\} \\ = E_{\underline{\theta}}\{h(\chi^2_{p-3;\delta^2})\} - (p-3)E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})/\chi^2_{p-1;\delta^2}\},\end{aligned} \tag{2.2}$$

$$\frac{\partial}{\partial\delta^2}E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})\} = \frac{1}{2}\{E_{\underline{\theta}}h(\chi^2_{p+1;\delta^2}) - E_{\underline{\theta}}h(\chi^2_{p-1;\delta^2})\}. \tag{2.3}$$

Proof. See Appendix.

Using Lemma 2.1 we have the following result:

Lemma 2.2. Let $h : [0,\infty) \to (-\infty,\infty)$ be differentiable. Then, provided both sides exist,

$$\frac{\partial}{\partial\delta^2}E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})\} = E_{\underline{\theta}}\{\frac{\partial}{\partial\chi^2_{p+1;\delta^2}}h(\chi^2_{p+1;\delta^2})\}. \tag{2.4}$$

Proof. The lemma is established by equating the result of the well-known integraton by parts technique with the result of Lemma 2.1. We will proceed by evaluating the risk of the estimator $\underline{d}(\underline{X}) = \bar{X}\underline{1} + (\underline{X} - \bar{X}\underline{1})(1 - \frac{h(S^2)}{S^2})$ where $\underline{X} \sim N_p(\underline{\theta}, I_p)$. First,

$$\begin{aligned}E_{\underline{\theta}}\mid \underline{\theta} - \underline{d}(\underline{X}) \mid^2 &= E_{\underline{\theta}}\mid \underline{\theta} - \bar{X}\underline{1} - (\underline{X} - \bar{X}\underline{1})(1 - \frac{h(S^2)}{S^2}) \mid^2 \\ &= p + 2E_{\underline{\theta}}\{(\underline{\theta} - \underline{X})'(\underline{X} - \bar{X}\underline{1})\frac{h(S^2)}{S^2}\} + E_{\underline{\theta}}\{\frac{h^2(S^2)}{S^2}\}.\end{aligned} \tag{2.5}$$

Now, consider the first expectation in the right-hand side of (2.5). By the usual integration by parts

$$
\begin{aligned}
& E_{\underline{\theta}}\{(\underline{\theta}-\underline{X})'(\underline{X}-\bar{X}\underline{1})\frac{h(S^2)}{S^2}\} \\
= & \sum_{i=1}^{p}\int_{R^p}(x_i-\bar{x})[(\theta_i-x_i)\frac{h(s^2)}{s^2}(2\pi)^{-p/2}e^{-\frac{1}{2}|\underline{x}-\underline{\theta}|^2}]\,d\underline{x} \\
= & -\sum_{i=1}^{p}\int_{R^p}[(1-\frac{1}{p})\frac{h(s^2)}{s^2}+2(x_i-\bar{x})^2\{\frac{h'(s^2)}{s^2}-\frac{h(s^2)}{s^4}\}](2\pi)^{-p/2}e^{-\frac{1}{2}|\underline{x}-\underline{\theta}|^2}]\,d\underline{x} \\
= & -\int_{R^p}[(p-1)\frac{h(s^2)}{s^2}+2\{h'(s^2)-\frac{h(s^2)}{s^2}\}](2\pi)^{-p/2}e^{-\frac{1}{2}|\underline{x}-\underline{\theta}|^2}]\,d\underline{x} \\
= & -(p-3)E_{\underline{\theta}}\{\frac{h(S^2)}{S^2}\}-2E_{\underline{\theta}}\{h'(S^2)\}, \qquad (2.6)
\end{aligned}
$$

where $h'(S^2)=\partial h(S^2)/\partial S^2$. Hence, from (2.5) and (2.6), we have

$$
E_{\underline{\theta}}|\ \underline{\theta}-\underline{d}(\underline{X})\ |^2 = p-4E_{\underline{\theta}}[h'(S^2)]+E_{\underline{\theta}}\{\frac{h(S^2)}{S^2}[h(S^2)-2(p-3)]\}. \qquad (2.7)
$$

Next, applying (2.1) and (2.2) of Lemma 2.1 to the first expectation in the right-hand side of (2.5) and rearranging terms yield

$$
\begin{aligned}
& E_{\underline{\theta}}|\ \underline{\theta}-\underline{d}(\underline{X})\ |^2 \\
= & p+2E_{\underline{\theta}}\{\underline{\theta}'(\underline{X}-\bar{X}\underline{1})\frac{h(S^2)}{S^2}-\underline{X}'(\underline{X}-\bar{X}\underline{1})\frac{h(S^2)}{S^2}\}+E_{\underline{\theta}}\{\frac{h^2(S^2)}{S^2}\} \\
= & p+2E_{\underline{\theta}}\{\underline{\theta}'(\underline{X}-\bar{X}\underline{1})\frac{h(S^2)}{S^2}\}-2E_{\underline{\theta}}\{h(S^2)\}+E_{\underline{\theta}}\{\frac{h^2(S^2)}{S^2}\} \\
= & p+2\delta^2E_{\underline{\theta}}\{h(\chi^2_{p+1;\delta^2})/\chi^2_{p+1;\delta^2}\}-2E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})\}+E_{\underline{\theta}}(\frac{h^2(S^2)}{S^2}) \\
= & p+2\{E_{\underline{\theta}}[h(\chi^2_{p-3;\delta^2})]-(p-3)E_{\underline{\theta}}[\frac{h(S^2)}{S^2}]\}-2E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})\}+E_{\underline{\theta}}\{\frac{h^2(S^2)}{S^2}\} \\
= & p-2\{E_{\underline{\theta}}h(\chi^2_{p-1;\delta^2})-E_{\underline{\theta}}h(\chi^2_{p-3;\delta^2})\}+E_{\underline{\theta}}\{\frac{h(S^2)}{S^2}[h(S^2)-2(p-3)]\}. \qquad (2.8)
\end{aligned}
$$

Now, equating (2.7) and (2.8), cancelling common terms, and using (2.3) yield

$$
\begin{aligned}
\frac{\partial}{\partial\delta^2}E_{\underline{\theta}}\{h(\chi^2_{p-3;\delta^2})\} &= \frac{1}{2}\{E_{\underline{\theta}}h(\chi^2_{p-1;\delta^2})-E_{\underline{\theta}}h(\chi^2_{p-3;\delta^2})\} \\
&= E_{\underline{\theta}}\{h'(S^2)\}
\end{aligned}
$$

$$= E_{\underline{\theta}}\left\{\frac{\partial h(\chi^2_{p-1;\delta^2})}{\partial \chi^2_{p-1;\delta^2}}\right\}.$$

3. ESTIMATORS WITH NONDECREASING RISK.

Before providing a main result we need a result due to Casella(1990) which relates the property of concavity of $r(t)$ to usual conditions on $r(t)$ about the risk behavior of the estimator in (1.1), namely, $r(t)$ is nondecreasing and $r(t)/t$ is nonincreasing.

Lemma 3.1. Let $r(\cdot):[0,\infty)\rightarrow[0,\infty)$ be concave. Then

(i) $r(t)$ is nondecreasing in t;

(ii) $r(t)/t$ is nonincreasing in t.

Proof. See Casella(1990).

Remark 3.1. Note that the converse of Lemma 3.1 is false. An example is given by

$$r(t)=\begin{cases} 2\sqrt{e}t & , \quad 0\le t<\frac{1}{2} \\ e^t & , \quad \frac{1}{2}\le t<1 \\ \ln t+e & , \quad 1\le t, \end{cases}$$

which satisfies (i) and (ii) of Lemma 3.1, but is not concave.

Now, we give a main result which characterizes a class of minimax estimators whose minimum risk is attained at $\delta^2=\mid\underline{\theta}-\bar{\theta}\underline{1}\mid^2=0$.

Theorem 3.1. Let $\underline{d}(\underline{X})$ be as in (1.1) where $r(\cdot):[0,\infty)\rightarrow[0,\infty)$ is concave. If $0\le r(t)\le 2(p-3)$, $p\ge 4$, then $R(\underline{\theta},\underline{d})$ is a nondecreasing risk function of $\delta^2=\mid\underline{\theta}-\bar{\theta}\underline{1}\mid^2$.

Proof. Assume, for the moment, that r is twice differentiable. It follows from (2.7) that

$$R(\underline{\theta}, \underline{d}) = p - 4E_{\underline{\theta}}[r'(S^2)] + E_{\underline{\theta}}\{\frac{r(S^2)}{S^2}[r(S^2) - 2(p-3)]\}, \quad (3.1)$$

where $r'(S^2) = \partial r(S^2)/\partial S^2$. The concavity of $r(t)$, together with Lemma 3.1, insure that the function inside the second expectation in (3.1) is nondecreasing in S^2, and hence the expectation is nondecreasing in $\delta^2 = |\underline{\theta} - \bar{\theta}\underline{1}|^2$. Thus, we only need show that $E_{\underline{\theta}}[r'(S^2)]$ is nonincreasing in $\delta^2 = |\underline{\theta} - \bar{\theta}\underline{1}|^2$. Using Lemma 2.2 yields

$$\frac{\partial}{\partial \delta^2} E_{\underline{\theta}}[r'(S^2)] = \frac{\partial}{\partial \delta^2} E_{\underline{\theta}}[r'(\chi^2_{p-1;\delta^2})] = E_{\underline{\theta}}[r''(\chi^2_{p+1;\delta^2})] \leq 0$$

by the fact that r is concave, where $r''(\chi^2_{p+1;\delta^2}) = \partial r'(\chi^2_{p+1;\delta^2})/\partial \chi^2_{p+1;\delta^2}$. If r is not twice-differentiable, we can take a sequence $\{r_n\}$ of twice-differentiable concave functions which uniformly converges to r. Then, for $n = 1, 2, \ldots,$

$$\frac{\partial}{\partial \delta^2} E_{\underline{\theta}}[r'_n(S^2)] = \frac{\partial}{\partial \delta^2} E_{\underline{\theta}}[r'_n(\chi^2_{p-1;\delta^2})] = E_{\underline{\theta}}[r''_n(\chi^2_{p+1;\delta^2})] \leq 0.$$

and hence

$$\lim_{n\to\infty} \frac{\partial}{\partial \delta^2} E_{\underline{\theta}}[r'_n(S^2)] \leq 0.$$

Now , using Bounded Convergence Theorem yield

$$\frac{\partial}{\partial \delta^2} E_{\underline{\theta}}[r'(S^2)] \leq 0.$$

Hence, the theorem is established.

Remark 3.2. From Lemma 3.1, it also follows that $\underline{d}(\underline{X})$ of Theorem 3.1 is minimax. This can be seen from (3.1) . Although the concavity condition seems rather strong, all estimators introduced in Section 1 and the positive-part Lindley-Efron- Morris-Stein estimator $\underline{d}(\underline{X}) = \bar{X}\underline{1} + (1-(p-3)/S^2)^+(\underline{X} - \bar{X}\underline{1})$ satisfy the condition.

APPENDIX

Proof of (2.1) :

With $\underline{X} \sim N_p(\underline{\theta}, I_p)$ we make an orthogonal transformation $\underline{Y} = A\underline{X}$ where $A = (a_{ij})$ is an $p \times p$ orthogonal matrix with $a_{1j} = (\theta_j - \bar{\theta})/\sqrt{\sum_{i=1}^{p}(\theta_i - \bar{\theta})^2}, a_{pj} = 1/\sqrt{p}, j = 1, 2, \cdots, p$, and $\sum_{j=1}^{p} a_{ij} = 0, i = 2, 3, \cdots, p-1$. Then $E_{\underline{\theta}}(Y_1) = \sqrt{\sum_{i=1}^{p}(\theta_i - \bar{\theta})^2}$, $E_{\underline{\theta}}(Y_i) = 0, i = 2, 3, \cdots, p-1, E_{\underline{\theta}}(Y_p) = \sqrt{p}\bar{\theta}, Var_{\underline{\theta}}\,(Y_i) = 1, \ i = 1, 2, \cdots, p$, and the Y_i's are independent normal variables. Now,

$$\begin{aligned}
S^2 &= \sum_{i=1}^{p}(X_i - \bar{X})^2 \\
&= \underline{X}'(I_p - \frac{1}{p}\underline{1}\underline{1}')\underline{X} \\
&= (A^{-1}\underline{Y})'(I_p - \frac{1}{p}\underline{1}\underline{1}')A^{-1}\underline{Y} \\
&= \underline{Y}'A(I_p - \frac{1}{p}\underline{1}\underline{1}')A'\underline{Y} \\
&= \underline{Y}'\underline{Y} - \underline{Y}'(\frac{A\underline{1}\underline{1}'A'}{p})\underline{Y} \\
&= \underline{Y}'\underline{Y} - Y_p^2 = \sum_{i=1}^{p-1} Y_i^2.
\end{aligned}$$

Hence, with $g(S^2) = h(S^2)/S^2$ and $g(\chi^2_{p+1;\delta^2}) = h(\chi^2_{p+1;\delta^2})/\chi^2_{p+1;\delta^2}$,

$$\begin{aligned}
&E_{\underline{\theta}}\{\underline{\theta}'(\underline{X} - \bar{X}\underline{1})g(S^2)\} \\
&= E_{\underline{\theta}}\{((\underline{\theta} - \bar{\theta}\underline{1})'\underline{X})g(S^2)\} \\
&= \delta E_{\underline{\theta}}\{Y_1 g(Y_1^2 + \sum_{i=2}^{p-1} Y_i^2)\} \\
&= \delta E_{\underline{\theta}}\{e^{-\delta^2/2}[\int_{-\infty}^{\infty} g(x^2 + \sum_{i=2}^{p-1} Y_i^2)\frac{xe^{-x^2/2}e^{x\delta}}{\sqrt{2\pi}}\, dx]\} \\
&= \delta E_{\underline{\theta}}\{\frac{e^{-\delta^2/2}}{\sqrt{2\pi}}[\int_{0}^{\infty} g(y + \sum_{i=2}^{p-1} Y_i^2)e^{-\frac{y}{2}}(e^{\delta\sqrt{y}} - e^{-\delta\sqrt{y}})\,\frac{dy}{2}]\}
\end{aligned}$$

$$
\begin{aligned}
&= \delta E_{\underline{\theta}}\{\frac{e^{-\delta^2/2}}{2\sqrt{2\pi}}[\int_0^\infty g(y+\sum_{i=2}^{p-1}Y_i^2)e^{-\frac{y}{2}}(\sum_{k=0}^{\infty}\frac{2(\delta\sqrt{y})^{2k+1}}{(2k+1)!})\,dy]\} \\
&= \delta^2 E_{\underline{\theta}}\{\int_0^\infty g(y+\sum_{i=2}^{p-1}Y_i^2)e^{-\frac{\delta^2}{2}}(\sum_{k=0}^{\infty}\frac{(\frac{\delta^2}{2})^k y^{\frac{2k+3}{2}-1}e^{-y/2}}{k!\Gamma(\frac{2k+3}{2})2^{\frac{2k+3}{2}}})\,dy\} \\
&\qquad (\,(2k+1)! = \Gamma(2k+2) = \Gamma(k+1)\Gamma(k+\frac{3}{2})2^{2k+1}/\sqrt{\pi}) \\
&= \delta^2 E_{\underline{\theta}}\{g(\chi^2_{3;\delta^2}+\sum_{i=2}^{p-1}Y_i^2)\} \\
&= \delta^2 E_{\underline{\theta}}\{g(\chi^2_{3;\delta^2}+\chi^2_{p-2})\} \\
&= \delta^2 E_{\underline{\theta}}\{g(\chi^2_{p+1;\delta^2})\}.
\end{aligned}
$$

Proof of (2.2) :

$$
\begin{aligned}
&E_{\underline{\theta}}\{h(\chi^2_{p-3;\delta^2})\} \\
&= \sum_{k=0}^{\infty}\frac{e^{-\delta^2/2}(\frac{\delta^2}{2})^k}{k!}\int_0^\infty \frac{v^{\frac{2k+p-3}{2}-1}e^{-\frac{v}{2}}}{\Gamma(\frac{2k+p-3}{2})2^{\frac{2k+p-3}{2}}}h(v)\,dv \\
&= \sum_{k=0}^{\infty}\frac{e^{-\delta^2/2}(\frac{\delta^2}{2})^k}{k!}\int_0^\infty \frac{h(v)}{v}\frac{v^{\frac{2k+p-1}{2}-1}e^{-\frac{v}{2}}}{\frac{1}{2k+p-3}\Gamma(\frac{2k+p-1}{2})2^{\frac{2k+p-1}{2}}}\,dv \\
&= \sum_{k=0}^{\infty}\frac{e^{-\delta^2/2}(\frac{\delta^2}{2})^k}{k!}(2k+p-3)\int_0^\infty \frac{h(v)}{v}\frac{v^{\frac{2k+p-1}{2}-1}e^{-\frac{v}{2}}}{\Gamma(\frac{2k+p-1}{2})2^{\frac{2k+p-1}{2}}}\,dv \\
&= (p-3)E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})/\chi^2_{p-1;\delta^2}\} \\
&\quad +\sum_{k=1}^{\infty}\frac{e^{-\delta^2/2}(\frac{\delta^2}{2})^k}{(k-1)!}\cdot 2\cdot\int_0^\infty \frac{h(v)}{v}\frac{v^{\frac{2k+p-1}{2}-1}e^{-\frac{v}{2}}}{\Gamma(\frac{2k+p-1}{2})2^{\frac{2k+p-1}{2}}}\,dv \\
&= (p-3)E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})/\chi^2_{p-1;\delta^2}\} \\
&\quad +(\frac{\delta^2}{2})\sum_{k=0}^{\infty}\frac{e^{-\delta^2/2}(\frac{\delta^2}{2})^k}{k!}\cdot 2\cdot\int_0^\infty \frac{h(v)}{v}\frac{v^{\frac{2k+p+1}{2}-1}e^{-\frac{v}{2}}}{\Gamma(\frac{2k+p+1}{2})2^{\frac{2k+p+1}{2}}}\,dv \\
&= (p-3)E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})/\chi^2_{p-1;\delta^2}\} \\
&\quad +\delta^2 E_{\underline{\theta}}\{h(\chi^2_{p+1;\delta^2})/\chi^2_{p+1;\delta^2}\}.
\end{aligned}
$$

Proof of (2.3) :

$$
\begin{aligned}
&\frac{\partial}{\partial \delta^2} E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})\} \\
=\ & \frac{\partial}{\partial \delta^2} \sum_{k=0}^{\infty} \frac{e^{-\delta^2/2}(\frac{\delta^2}{2})^k}{k!} \int_0^{\infty} h(v) \frac{v^{\frac{2k+p-1}{2}-1} e^{-\frac{v}{2}}}{\Gamma(\frac{2k+p-1}{2}) 2^{\frac{2k+p-1}{2}}}\, dv \\
=\ & -\frac{1}{2} E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})\} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{e^{-\delta^2/2}(\frac{\delta^2}{2})^k}{k!} \int_0^{\infty} h(v) \frac{v^{\frac{2k+p+1}{2}-1} e^{-\frac{v}{2}}}{\Gamma(\frac{2k+p+1}{2}) 2^{\frac{2k+p+1}{2}}}\, dv \\
=\ & -\frac{1}{2} E_{\underline{\theta}}\{h(\chi^2_{p-1;\delta^2})\} + \frac{1}{2} E_{\underline{\theta}}\{h(\chi^2_{p+1;\delta^2})\} \\
=\ & \frac{1}{2}\{E_{\underline{\theta}} h(\chi^2_{p+1;\delta^2}) - E_{\underline{\theta}} h(\chi^2_{p-1;\delta^2})\}.
\end{aligned}
$$

ACKNOWLEDGMENTS

The authors are grateful to all the referees and the Associate Editor for helpful suggestions which led to an improved version of the paper.

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