Journal of the Korean
Statistical Society
Vol. 24, No. 1, 1995

Double Bootstrap Confidence Cones for Spherical Data based on Prepivoting †

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ABSTRACT

For a distribution on the unit sphere, the set of eigenvectors of the second moment matrix is a conventional measure of orientation. Asymptotic confidence cones for eigenvector under the parametric assumptions for the underlying distributions and nonparametric confidence cones for eigenvector based on bootstrapping were proposed. In this paper, to reduce the level error of confidence cones for eigenvector, double bootstrap confidence cones based on prepivoting are considered, and the consistency of this method is discussed. We compare the performances of double bootstrap method with the others by Monte Carlo simulations.

KEYWORDS: Spherical Data, Eigenvector, Confidence Cone, Bootstrap, Prepivoting,

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† This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1993.

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1. INTRODUCTION

An appropriate eigenvector of the second moment matrix of the underlying distribution can be used as a measure of the spatial orientation of the underlying distribution. Kim(1978) and Watson(1983) discuss the usefulness of eigenvector. In section 2, asymptotic confidence cones for eigenvector are reviewed. A nonparametric approach based on bootstrap to construct confidence cones for eigenvector has been conducted [Shin(1987)]. But the level error of confidence cones is substantial, even at relatively large sample sizes. In this paper, to reduce the level error of confidence cones for eigenvector, a double bootstrap method of constructing confidence cones for eigenvector via prepivoting is proposed.

Let X be a random vector on the unit sphere S^2 in R^3 with distribution F. An eigenvector of the second moment matrix $M_F = E(XX')$ of X is denoted by $e(F)$. A consistent estimator of $e(F)$ is $e(\hat{F}_n)$, where $\hat{F}_n$ is the empirical distribution based on observed values of the random sample $x_1, \ldots, x_n$ from F. A confidence region of level $(1-\alpha)$ for $e(F)$ is the cone, C, with axis $e(F)$ and semi-angle $\phi_\alpha(F)$, defined by

$$C = \{v \in S^2 : \mid v'e(\hat{F}_n) \mid \geq \phi_\alpha(F)\} \tag{1}$$

where $\phi_\alpha(F)$ is chosen so that

$$Pr\{e(\hat{F}_n) \in C\} = 1 - \alpha.$$

Solutions for $\phi_\alpha(F)$ have been proposed, but they require imposing certain parametric assumptions on F. With these assumptions, it is shown that a pivotal quantity

$$P(\hat{F}_n, F) = 2n[1 - \{e(F)e'(\hat{F}_n)\}^2] \tag{2}$$

has a nondegenerate asymptotic distribution, usually depending on F in a simple way. Hence, from this it is possible to estimate $\phi_\alpha(F)$ by $\phi_\alpha(\hat{F}_n)$. Examples of such assumptions are

(I) F is the Bingham distribution with parameters $K = (k_1, k_2, k_3) \in R^3$ and orthonormal vectors $\mu_1, \mu_2, \mu_3 \in S^2$ [Mardia(1972)] and density $f(x) = d(K)exp(k_1\{\mu_1'x)^2 + k_2(\mu_2'x)^2 + k_3(\mu_3'x)^2\}$, where $d(K)$ is the normalizing constant.

(II) F has a density of the form $f(x) = g(\mu' x)$ so that F is rotationally symmetric about μ, where $\| \mu \| = 1$. The density in (I) with $k_2 = k_3 = 0$, which are called the Dimroth-Watson distribution, is a special case of (II).

Shin(1987) proposed a nonparametric method of constructing confidence cones for $e(F)$ based on bootstrap. It used the bootstrap in estimating the sampling distribution of the pivot (2) and consequently constructing confidence cones from this sampling distribution. But, the actual level of the approximate confidence cones differs substantially from the intended level, even at relatively large sample sizes.

We propose a pivotal quantity based on prepivoting

$$P(\hat{F}_n^*, \hat{F}_n) = 2n[1 - \{e(\hat{F}_n)e'(\hat{F}_n^*)\}^2] \tag{3}$$

where $\hat{F}_n^*$ be the empirical distribution based on observed values of the random sample from $\hat{F}_n$ and $e(\hat{F}_n^*)$ is a bootstrap version of $e(\hat{F}_n)$. In section 4, the double bootstrap procedure for approximating the sampling distribution of $P(\hat{F}_n^*, \hat{F}_n)$ is presented and shown to be consistent under minimal conditions. Hence from this result, it is possible to estimate $\phi_\alpha(F)$ by $\phi_\alpha(\hat{F}_n^*)$. In section 5, our double bootstrap confidence cones are compared empirically with the asymptotic confidence cones and Shin's simple bootstrap confidence cones in terms of coverage probabilities by Monte Carlo study. It is illustrated that our double bootstrap confidence cones for $e(F)$ have smaller errors in level than the others. Finally, in section 6, a data set is analyzed using the method developed.

2. ASYMPTOTIC CONFIDENCE CONES FOR EIGENVECTOR

In this section, we review the parametric approach of constructing a confidence cone C of the form (1) about $e(F)$. Even though the Dimroth-Watson distribution is the actual underlying distribution we will be working with, we will construct the confidence cone under more general setting of the rotational symmetry.

Viewing a probability distribution on S^2 as a mass distribution on S^2, we may make an analogy with the problem of finding the orientation of a rigid body. We can use the orthonormal set of eigenvectors of the corresponding matrix of a spherical distribution as a measure of orientation of the distribution. When n points with unit masses are placed at $x_1, \ldots, x_n$, the moment of inertia of these masses about direction a, where $\| a \| = 1$, is given by $a'(I - \sum_{i=1}^n x_i x_i')a$. Since the spectral decomposition of $I - \sum_{i=1}^n x_i x_i'$ and $\sum_{i=1}^n x_i x_i'$ are obviously related, given a probability distribution F on S^2 we may correspondingly consider a simpler matrix $M_F = E_F(XX')$, the second moment matrix. For the random sample $x_1, \ldots, x_n$ from F,

$$M_{\hat{F}_n} = E_{\hat{F}_n} xx' = n^{-1} \sum_{i=1}^{n} x_i x_i'$$

which can be served as a consistent estimator for M_F.

Let X be a random vector on S^2 with $f(x) = g(\mu' x)$, where $\| \mu \| = 1$. Put $t = \mu' x$ and assume that $Et > 0$. For all $x \in S^2$, we can write

$$x = t\xi + (1 - t^2)^{\frac{1}{2}} \eta$$

where $\xi = \mu\mu' x$ and $\eta = (I - \mu\mu')x$. The spectral decomposition of M_F is given by

$$M_F = (Et^2)\mu\mu' + \frac{1 - Et^2}{2}(I - \mu\mu'); \quad \lambda_1 = Et^2 \ \textit{and} \ \lambda_2 = \frac{1 - Et^2}{2}.$$

In this equation, λ_1 is the largest eigenvalue of multiplicity one and μ is the normalized eigenvector corresponding to λ_1 which happens to lie in the direction of the axis of symmetry. For the Dimroth-Watson distribution, μ is the non-degenerate eigenvector, and the corresponding sample eigenvector becomes the maximum likelihood estimator for μ.

Applying the perturbation theory with random perturbation [Watson(1983)], it is possible to show that

$$\begin{aligned} \sqrt{n}(\lambda_{1,n} - \lambda_1) &= tr(G_n \mu\mu') + O_p(n^{-\frac{1}{2}}) \\ &\xrightarrow{d} N(0, var(t^2)) \end{aligned}$$

where $G_n = \sqrt{n}(M_{\hat{F}_n} - M_F)$. Now that G_n has the limiting Gaussian distribution by multivariate central limit theorem, we can view $M_{\hat{F}_n}$ as a small linear Gaussian perturbation of M_F. Asymptotic distributions of orthogonal projection $P_{1,n}$ are given by

$$\sqrt{n}(P_{1,n} - P_1) = \sqrt{n}(\hat{\mu}\hat{\mu}' - \mu\mu')$$
$$= \frac{2}{3Et^2 - 1}(W_n + W_n') + O_p(n^{-\frac{1}{2}})$$

where $W_n = \mu\mu' G_n(I - \mu\mu')$, and $\hat{\mu}$ is the eigenvector of $M_{\hat{F}_n}$ corresponding to $\hat{\lambda}_1$.

To describe the accuracy of $\hat{\mu}$ as an estimator for μ, we need to consider pivotal quantity

$$P(\hat{F}_n, F) = 2n\{1 - (\mu'\hat{\mu})^2\}.$$

Theorem 2.1 (Watson(1983))

$$P(\hat{F}_n, F) \xrightarrow{d} \frac{4(Et^2 - Et^4)}{(3Et^2 - 1)^2}\chi_2^2 .$$

This theorem enables us to construct an approximate confidence cone with axis $\hat{\mu}$ and semi angle $\hat{\theta}$ as determined by the asymptotic distribution of the pivot $P(\hat{F}_n, F)$. Since both of Et^2 and Et^4 will be unknown in practice, we can use their consistent estimators;

$$Et^2 = \sum_{i=1}^{n}(\hat{\mu}'x_i)^2/n, \quad Et^4 = \sum_{i=1}^{n}(\hat{\mu}'x_i)^4/n.$$

Hence, under the rotational symmetry, an approximate level $(1-\alpha)$ confidence cone for the nondegenerate eigenvector μ is given by

$$C_{AS} = \{v \in S^2 \mid (v'\hat{\mu})^2 \geq 1 - \frac{2(\sum(\hat{\mu}'x_i)^2 - \sum(\hat{\mu}'x_i)^4)}{(3\sum(\hat{\mu}'x_i)^2 - n)^2}c(\alpha, \chi_2^2)\}$$

where $c(\alpha, \chi_2^2)$ is the upper α - percentile point of χ_2^2.

3. BOOTSTRAP CONFIDENCE CONES FOR EIGENVECTOR

In this section, we review the nonparametric approach of constructing confidence cone based on the bootstrap distribution of the pivot (2).

Let $x_1, \ldots, x_n$ be independent and identically distributed random vectors with unknown cumulative distribution function F. Efron(1993) discusses a bootstrap method for approximating the sampling distribution of a function of the observations and the underlying distribution F. We call the approximation as the bootstrap distribution. Let $P(\hat{F}_n, F)$ be a real valued function of $x_1, \ldots, x_n$ and F, where $\hat{F}_n$ is the empirical cumulative distribution function putting mass $1/n$ at x_i for $i = 1, \ldots, n$. Let $J_n(., F)$ denote the cumulative distribution function of $P(\hat{F}_n, F)$. Then the bootstrap estimator of $J_n(., F)$ is $J_n(., \hat{F}_n)$ which is the cumulative distribution function of $P(\hat{F}_n^*, \hat{F}_n)$ where $\hat{F}_n^*$ is the empirical cumulative distribution function of the random sample drawn from $\hat{F}_n$.

Beran(1991) gives general conditions under which the bootstrap distribution as an estimator for the sampling distribution of the pivot is consistent.

Theorem 3.1 **[Beran(1991)]** Let $\mathcal{C}_F$ be a set of sequences $\{F_n \in \mathcal{F} : n \geq 1\}$ such that $Pr[\{\tilde{F}_n\} \in \mathcal{C}_F] = 1$. Suppose that for every sequence $\{F_n\} \in \mathcal{C}_F, J_n(., F_n) \rightarrow J(., F)$ a limit distribution depending only on $\mathcal{C}_F$. Then $\{J_n(., \tilde{F}_n)\}$ converges weakly to $J(., F)$ w.p. 1.

Let $\mathcal{F}$ be the class of all cumulative distribution functions F on S^2. Let X be a random vector on S^2 with F. We will assume that F is rotationally symmetric. For any $p \times p$ symmetric matrix $A = \{a_{ij}\}$, let $uvec(A)$ denote the $\frac{p(p+1)}{2}$ dimensional column vector $\{\{a_{ij}; 1 \leq i \leq j\} 1 \leq j \leq p\}$ formed from the elements in the upper triangular half of A including diagonal elements. Suppose that $Z_F = \{z_{F,ij}\}$ is a symmetric $p \times p$ random matrix such that the distribution of $uvec(Z_F)$ is normal with mean vector 0 and covariance matrix Ω_F. If F has finite fourth moments, the distribution of $\{\sqrt{n}(M_{F_n} - M_F) \mid F; n \geq 1\}$ converges weakly to $N(0, \Omega_F)$.

We have the following result from Shin(1987) :

Theorem 3.2

$$J_n(., \hat{F}_n) \xrightarrow{w} J(., F) \quad w.p.\ 1$$

where $J(.,F)$ is the distribution of the random variable $\frac{1}{2} \parallel \dot{P}(M_F)uvec(Z_F) \parallel$, and $J_n(.,\hat{F}_n)$ denotes the distribution of (2) under $\hat{F}_n$.

A confidence region for $e(F)$ is the cone C, with axis $e(\hat{F}_n)$ and semi-angle $\phi_\alpha(F)$ defined by

$$C = \{v \in S^2 \mid (v'e(\hat{F}_n))^2 \geq \phi_\alpha(F)\}$$

where $\phi_\alpha(F)$ is chosen in such a way $Pr[e(F) \in C] = 1-\alpha$. The bootstrap confidence cone C_B is given by,

$$C_B = \{v \in S^2 \mid (v'e(\hat{F}_n))^2 \geq 1 - \frac{1}{2n}c(\alpha,\hat{F}_n)\}$$

where $c(\alpha,\hat{F}_n)$ is an upper α-percentile point of $J_n(.,\hat{F}_n)$. The limiting distribution $J(.,F)$ is continuous in X and strictly increasing because $uvec(Z_F)$ is continuous function. Therefore, $\lim_{n\to\infty} Pr[e(F) \in C_B] = 1-\alpha$ for all $F \in \mathcal{F}$.

4. PREPIVOTING

Prepivoting[Beran(1987, 1988)] is the transformation of pivotal quantity by its estimated bootstrap cumulative distribution. The pivotal quantity $P(\hat{F}_n,F)$ is transformed into $P(\hat{F}_n^*,\hat{F}_n) = J_n(P(\hat{F}_n,F),\hat{F}_n)$ whose limiting distribution is uniform $(0,1)$. Let $J_n(.,F)$ denote the cumulative distribution function of $P(\hat{F}_n,F)$, then the bootstrap estimator of $J_n(.,F)$ is $J_n(.,\hat{F}_n)$. Then the prepivoted double bootstrap estimator of $J_n(.,F)$ is $J_n(.,\hat{F}_n^*)$, the cumulative distribution function of $P(\hat{F}_n^{**},\hat{F}_n^*)$, where $\hat{F}_n^{**}$ is the empirical cumulative distribution function of random sample $x_1^{**},\ldots,x_n^{**}$ drawn from $\hat{F}_n^*$. We call $J_n(.,\hat{F}_n^*)$ double bootstrap estimator of $J_n(.,F)$. Bootstrap confidence sets generated from a pivotal quantity prepivoted one or more times have smaller level error than confidence sets based on the original quantity. Further iterations of prepivoting make higher order corrections automatically.

In this section, we proceed to construct double bootstrap confidence cone for $e(F)$. We will assume that F is rotationally symmetric throughout this section. Now we will proceed to show that the double bootstrap distribution

$J_n(.,\hat{F_n}^*)$ of the pivot (3) has the same limiting distribution as $J_n(.,F)$.

Theorem 4.1 If $J_n(.,\hat{F_n}^*)$ denotes the distribution of (3) under $\hat{F_n}^*$, and if $J(.,F)$ is the distribution of the random variable $\frac{1}{2} \parallel \dot{P}(M_F)uvec(Z_F) \parallel$ where $uvec(Z_F) \sim N(0,\Omega_F)$ then

$$J(.,\hat{F_n}^*) \xrightarrow{w} J(.,F) \quad w.p.\ 1.$$

Proof. Define $\mathcal{C}_F$ as the set of all sequences $\{F_n, n = 1,2,\ldots\}$ of cumulative distribution functions in $\mathcal{F}$ such that

(I) $\{F_n\} \xrightarrow{w} F$

(II) $\lim_{n\to\infty} E(XX' \mid F_n) = E(XX' \mid F)$ and $\lim_{n\to\infty} E(XX'XX' \mid F_n) = E(XX'XX' \mid F)$ for the data set $\{x_i, i = 1,\ldots,n\}$.

(III) $trM_{F_n} = 1$.

First, we want to show that $Pr(\{\hat{F_n}^*\} \in \mathcal{C}_F) = 1$. By the Glivencko-Cantelli theorem, $\hat{F_n}^* \to F \quad w.p.\ 1$. Since the sphere is compact, we have $\lim_{n\to\infty} E(XX' \mid \hat{F_n}^*) = E(XX' \mid F)$ and $\lim_{n\to\infty} E(XX'XX' \mid \hat{F_n}^*) = E(XX'XX' \mid F)$ by using the strong law of large numbers. Since $\hat{F_n}^*$ is a distribution on S^2, $trM_{\hat{F_n}^*} = 1$. Hence

$$P_r[\{\hat{F_n}^*, n = 1,2,\ldots\} \in \mathcal{C}_F] = 1. \tag{4}$$

Let $\{F_n\} \in \mathcal{C}_F$ and $x_{1n},\ldots,x_{nn}$ be a sample from F_n, with the empirical distribution $\hat{F}_n$. Since $M_{F_n} = (\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'$,

$$\begin{aligned}
\sqrt{n}(M_{\hat{F}_n} - M_{F_n}) &= \sqrt{n}\{\frac{\sum_{i=1}^n (x_i - \bar{x_n})(x_i - \bar{x_n})'}{n} - (\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'\} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^n \{(x_i - \mu_{F_n})(x_i - \mu_{F_n})' - (\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'\} \\
&\quad - \sqrt{n}(\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'.
\end{aligned}$$

Let $Z_{n,i} = (x_i - \mu_{F_n})(x_i - \mu_{F_n})' - (\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'$, then

$$\sqrt{n}(M_{\hat{F}_n} - M_{F_n}) = \sqrt{n}^{-1} \sum_{i=1}^{n} Z_{n,i} - \sqrt{n}(\bar{x}_n - \mu_{F_n})(\bar{x}_n - \mu_{F_n})'.$$

Let a be any constant column vector of dimensional $\frac{p(p+1)}{2}$. Let G_n be the cumulative distribution function of $a'uvec(Z_{n,i})$ under F_n and let G be the cumulative distribution function of $a'uvec(x_i x_i' - M)$ under F. Since $\{F_n\} \in \mathcal{C}_F$,

$$G_n \rightarrow G$$

and

$$\lim_{n\rightarrow\infty} \int y^2 dG_n(y) = \int y^2 dG(y) = a'\Omega_F a.$$

Thus

$$\lim_{n\rightarrow\infty} E_{F_n}[a'uvec(Z_{n,i})]^2 = a'\Omega_F a < \infty$$

and

$$\lim_{n\rightarrow\infty} E_{F_n}[a'uvec(Z_{n,i})]^2 I[a'uvec(Z_{n,i}) > \sqrt{n}\delta] = 0$$

for every positive δ. By the Linderberg central limit theorem for a triangular array

$$a'uvec(n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_{n,i} \mid F_n) \longrightarrow a'uvec(Z_F) \sim N(0, a'\Omega_F a).$$

Hence

$$[n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_{n,i} \mid F_n] \longrightarrow Z_F \tag{5}$$

where $Z_F \sim N(0, \Omega_F)$.

From $\mathcal{L}[\sqrt{n}(\bar{x}_n - \mu_{F_n}) \mid F_n] \longrightarrow N(0, \Omega_F)$,

$$\sqrt{n}(\bar{x}_n - \mu_{F_n})(\bar{x}_n - \mu_{F_n})' \longrightarrow 0. \tag{6}$$

From (5), (6),

$$\mathcal{L}[\sqrt{n}(M_{\hat{F}_n} - M_{F_n}) \mid F_n] \longrightarrow N(0, \Omega_F). \tag{7}$$

By idempotency and symmetry of eigenprojections,

$$\| P(M_{\hat{F}_n}) - P(M_F) \|^2 = 2[1 - e'(\hat{F}_n)e(F)]$$

where $P(M_F) = e(F)e(F)'$ and $P(M_{\hat{F}_n}) = e(\hat{F}_n)e(\hat{F}_n)'$. Since $P(M)$ is continuously differentiable function of $uvec(M)$, by expanding $P(M_{F_{\hat{n}}})$ about M_{F_n},

$$P(M_{\hat{F}_n}) = P(M_{F_n}) + \dot{P}(M_{F_n})(M_{\hat{F}_n} - M_{F_n}) + O_P(\| M_{\hat{F}_n} - M_{F_n} \|).$$

Thus

$$\begin{aligned} P(\hat{F}_n, F_n) &= n\{1 - [e'(\hat{F}_n)e(F_n)]^2\} \\ &= \frac{1}{2} \| \sqrt{n}(P(M_{\hat{F}_n}) - P(M_{F_n})) \|^2 \\ &= \frac{1}{2} \| \dot{P}(M_{F_n})\sqrt{n}(M_{\hat{F}_n} - M_{F_n}) + O_P(\| \sqrt{n}(M_{\hat{F}_n} - M_{F_n}) \|) \|^2 \end{aligned}$$

By $M_{F_n} \longrightarrow M_F$ and (??), $O_P(\| \sqrt{n}(M_{\hat{F}_n} - M_{F_n}) \|) = O_P(1)$. Hence by Slutsky lemma and (7)

$$P(\hat{F}_n, F_n) \longrightarrow \frac{1}{2} \| P(M_F)uvec(Z_F) \|^2 \qquad (8)$$

where $uvec(Z_F) \sim N(0, \Omega_F)$.

From (4), (8) and Theorem 3.2, w.p. 1

$$J_n(., \hat{F}_n^{\ *}) \xrightarrow{w} \mathcal{L}[\frac{1}{2} \| \dot{P}(M_F)uvec(Z_F) \|^2]$$

where $uvec(Z_F) \sim N(0, \Omega_F)$.

If C_{B^*} is the double bootstrap confidence cone for $e(F)$,

$$C_{B^*} = \{v \in S^2 \mid (v'e(\hat{F}_n^{\ *}))^2 \geq 1 - \frac{1}{2n}c(\alpha, \hat{F}_n^{\ *})\}$$

where $c(\alpha, \hat{F}_n^{\ *})$ is an upper α - percentage point of $J_n(., \hat{F}_n^{\ *})$. Because $uvec(Z_F)$ is continuous function, the limiting distribution $J(., F)$ is continuous in X and strictly increasing. Therefore, $\lim_{n\to\infty} Pr[e(F) \in C_{B^*}] = 1 - \alpha$ for all $F \in \mathcal{F}$.

5. MONTE CARLO APPROXIMATION

A Monte Carlo simulation was performed to study and compare the performance of the procedures introduced above. We choose to look at the case $p = 3$. For the simulation, we took pseudo-random samples from the Dimroth-Watson distribution with special values $\mu(F) = (0,0,1)$ and $k = 0.1$, 1 and 3.

For $\alpha = 0.05$, the coverage probabilities of confidence cones of C_{AS}, C_B, C_{B^*} were computed. Each cone was obtained from 200 bootstrap replications. It was checked if they contained $\mu(F)$. This was repeated 1000 times in order to get an estimate of coverage probability. The results of this simulation are shown in Table 1.

Table 1. Coverage probabilities of the confidence cones ($\alpha = 0.05$) based on n observations from the Dimorth-Watson distribution $D(k,(0,0,1))$.

n	10			20			40			50		
k	0.1	1.0	3.0	0.1	1.0	3.0	0.1	1.0	3.0	0.1	1.0	3.0
C_{B^*}	.623	.878	.913	.744	.890	.931	.778	.905	.936	.792	.910	.937
C_B	.588	.821	.898	.702	.870	.908	.776	.879	.931	.791	.885	.934
C_{AS}	.651	.835	.909	.707	.854	.924	.760	.881	.938	.781	.889	.930

Replications - 200 times, trials - 1000 times.

The double bootstrap confidence cones, C_{B^*}, perform well for sample size greater than 20 and k greater than or equal to 3. And the level error in the double bootstrap confidence cone is more stable than the level error in the simple bootstrap confidence cone. Double bootstrap procedure is less strongly dependent upon F than the distribution of simple bootstrap procedure. Moreover the double bootstrap is easily implemented on a computer. The reliable method in these cases presented here is the double bootstrap method. And prepivoting can be repeated to reduce level error further.

6. EXAMPLE WITH REAL DATA

Table 2 gives the data listed as 33 measurements of L_0^1 axes (intersections between cleavage and bedding planes of F_1 folds) in Ordovician turbidites. The coordinates are plunge and plunge azimuth. The data were recorded by Powell, Cole and Cudahy [Fisher(1987)]. These geological coordinates are mapped into polar coordinates by the relations [Fisher(1987)]

$$\theta = 2\pi(P + 90°)/360°(-90° < P < 90°)]$$

$$\phi = 2\pi(360° - A)/360°(0 \leq A < 360°).$$

Table 2. Measurements, in degrees, of plunge, P, and plunge azimuth, A.

P	A	P	A	P	A	P	A	P	A
-5	12	-1	17	-10	9	8	342	0	12
8	0	-7	4	-13	2	5	4	-7	4
5	15	-1	2	-12	353	11	350	3	9
15	355	3	344	0	2	-9	359	0	12
-11	10	-13	14	0	12	1	13	3	353
10	3	12	4	15	347	-6	2	-13	7
-17	354	25	4	0	4				

All methods discussed above were applied to this data set. The results are shown in Table 3. We have similar results from all methods. But the semi-angles of the double bootstrap are some what larger, particularly for the 0.99 confidence cone.

Table 3. Confidence region C, cosine of the semi-angle.

1-α	C_{B^*}	C_B	C_{AS}
0.90	0.9967	0.9984	0.9981
0.95	0.9958	0.9980	0.9976
0.99	0.9928	0.9970	0.9962

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