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Relations between the Lagrange Multiplier Tests and t-statistics for Seasonal Unit Roots

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ABSTRACT

The Lagrange multiplier test statistics for seasonal unit roots are derived and the asymptotic distributions of the derived statistics are obtained. The relationship between the derived LM test statistics and the "DHF" type regression t statistics are shown.

KEYWORDS : Seasonal unit root, Deterministic trend, Brownian motions, Lagrange multiplier test.

1. INTRODUCTION

Consider a seasonal time series model of the form

$$Y_t = \rho Y_{t-d} + \varepsilon_t, \tag{1.1}$$

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where the ε_t are independent random variables with mean 0 and variance σ^2. Under the null hypothesis $H_0 : \rho = 1$ in model (1.1), we first consider the alternative $H_a : |\rho| < 1$. In general, we also consider the following models with the possibility of a mean and seasonal trends.

$$Y_t = \alpha + \rho Y_{t-d} + \varepsilon_t, \tag{1.2}$$

$$Y_t = \sum_{j=1}^{d} \alpha_j \delta_{jt} + \rho Y_{t-d} + \varepsilon_t, \tag{1.3}$$

$$Y_t = \sum_{j=1}^{d} (\alpha_j + \beta_j \tau)\delta_{jt} + \rho Y_{t-d} + \varepsilon_t, \tag{1.4}$$

where $\tau = [(t-1)/d + 1]$ with $[x]$ denoting the largest integer no larger than x, and δ_{jt} are seasonal dummy variables such that

$$\delta_{jt} = \begin{cases} 1 & \text{if } j \equiv t \pmod{d} \\ 0 & \text{otherwise.} \end{cases}$$

We shall study the Lagrange Multiplier (LM) test of the hypothesis that $\rho = 1$ for model (1.1) and the LM tests of the hypothesis that $\rho = 1$ and the nuisance parameters are equal to zeros for models (1.2)-(1.4). For the test of hypothesis $H_0 : \rho = 1$, Dickey *et al.* (1984) suggested regression t statistics, $\hat{\tau}_d$, $\hat{\tau}^*_{\mu d}$, and $\hat{\tau}_{\mu d}$, for models (1.1)-(1.3), respectively and Cho *et al.* (1995) suggested T for model (1.4).

Since Dickey and Fuller (1979) proposed regression t statistics for testing the nonseasonal unit root, Solo (1984) and Guilkey and Schmidt (1989) obtained the corresponding *LM* test statistics and showed the relationship between the regression t test statistics and the *LM* test statistics. In this paper we derive the *LM* test statistics for models (1.1)-(1.4) and obtain the asymptotic distributions in terms of the functionals of Brownian motions. The relationship between the derived *LM* test statistics and the corresponding regression t statistics are shown.

2. *LM* TESTS FOR SEASONAL UNIT ROOTS

The *LM* test statistics for the null hypothesis $\rho = 1$ for models (1.1)-(1.4) can be derived following Solo (1984).

Let the parameter sets $\Theta_1' = (\rho)$, $\Theta_2' = (\alpha, \rho)$, $\Theta_3' = (\alpha_1, \ldots, \alpha_d, \rho)$ and $\Theta_4' = (\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d, \rho)$ correspond to models (1.1)-(1.4), respectively. The LM calculation is based on the approximate log-likelihood function

$$L(\Theta_i) = \frac{1}{2\sigma^2}\sum_{t=1}^{n}\varepsilon_t^2$$

and LM_i statistics for models (1.1)-(1.4) are defined by

$$LM_i = (\frac{\partial L(\Theta_i)}{\partial \Theta_i})' J_i^{-1} (\frac{\partial L(\Theta_i)}{\partial \Theta_i}), \quad \text{for } i = 1, \ldots, 4 \tag{2.1}$$

where $J_i = -\partial^2 L(\Theta_i)/\partial\Theta_i\partial\Theta_i'$.

Now we derive the LM test statistics and their limiting distributions for models (1.1)-(1.4).

Since $\partial L(\Theta_i)/\partial\rho = \sum Y_{t-d}(Y_t - \rho Y_{t-d}) = \sum Y_{t-d}\varepsilon_t$ and $\partial L^2(\Theta_i)/\partial\rho^2 = \sum Y_{t-d}^2$, under the null hypothesis $\rho = 1$, the LM statistic for model (1.1) is

$$LM_1 = \{\sum_{t=1}^{n}\varepsilon_t Y_{t-d}\}^2 / \sum_{t=1}^{n} Y_{t-d}^2. \tag{2.2}$$

The limiting distribution of LM_1 can be easily obtained using Chan and Wei (1988) as follows with $n = md$,

$$LM_1 \xrightarrow{\mathcal{L}} \{\sum_{j=1}^{d}(W_j(1)^2 - 1)/2\}^2 / \sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr, \tag{2.3}$$

where $W_j(r)$ are independent standard Brownian motions.

For model (1.2) we obtain

$$LM_2 = A_2/B_2, \tag{2.4}$$

where

$$A_2 = n(\sum_{t=1}^{n}\varepsilon_t Y_{t-d})^2 - 2\sum_{t=1}^{n}\varepsilon_t \sum_{t=1}^{n} Y_{t-d}\sum_{t=1}^{n}\varepsilon_t Y_{t-d} + (\sum_{t=1}^{n}\varepsilon_t)^2 \sum_{t=1}^{n} Y_{t-d}^2,$$

$$B_2 = n\sum_{t=1}^{n} Y_{t-d}^2 - (\sum_{t=1}^{n} Y_{t-d})^2 ,$$

and its limiting distribution

$$LM_2 \xrightarrow{\mathcal{L}} C_2/D_2, \tag{2.5}$$

where

$$C_2 = d\{\sum_{j=1}^{d} \frac{W_j(1)^2 - 1}{2}\}^2 - 2\sum_{j=1}^{d} W_j(1) \sum_{j=1}^{d} \int_0^1 W_j(r)dr \sum_{j=1}^{d} \frac{W_j(1)^2 - 1}{2}$$
$$+\{\sum_{j=1}^{d} W_j(1)\}^2 \sum_{j=1}^{d} \int_0^1 W_j(r)^2 dr,$$
$$D_2 = d\sum_{j=1}^{d} \int_0^1 W_j(r)^2 dr - \{\sum_{j=1}^{d} \int_0^1 W_j(r)dr\}^2.$$

Details of the derivations of (2.4) and (2.5) are given in the Appendix.

Similarly for model (1.3),

$$LM_3 = \sum_{j=1}^{d} \frac{1}{m} \sum_{t=1}^{m-1} \varepsilon_{(t-1)d+j}^2 + A_3/B_3, \tag{2.6}$$

where

$$A_3 = \{\sum_{j=1}^{d} \sum_{t=1}^{m-1} \varepsilon_{(t-1)d+j} \sum_{t=1}^{m-1} Y_{(t-1)d+j}\}^2$$
$$-2\sum_{j=1}^{d} (\sum_{t=1}^{m-1} \varepsilon_{(t-1)d+j} \sum_{t=1}^{m-1} Y_{(t-1)d+j}) \sum_{t=1}^{n} \varepsilon_t Y_{t-d} + (\sum_{t=1}^{n} \varepsilon_t Y_{t-d})^2,$$
$$B_3 = \sum_{t=1}^{n} Y_{t-d}^2 - \sum_{j=1}^{d} \frac{1}{m} (\sum_{t=1}^{m-1} Y_{(t-1)d+j})^2,$$

and its limiting distribution

$$LM_3 \xrightarrow{\mathcal{L}} \sum_{j=1}^{d} W_j(1)^2 + C_3/D_3, \tag{2.7}$$

where

$$C_3 = \{\sum_{j=1}^{d} W_j(1) \int_0^1 W_j(r)dr\}^2$$
$$-2\sum_{j=1}^{d} \{W_j(1) \int_0^1 W_j(r)dr\} \sum_{j=1}^{d} \frac{W_j(1)^2 - 1}{2} + (\sum_{j=1}^{d} \frac{W_j(1)^2 - 1}{2})^2,$$
$$D_3 = \sum_{j=1}^{d} \int_0^1 W_j(r)^2 dr - \sum_{j=1}^{d} \{\int_0^1 W_j(r)dr\}^2.$$

Details of the derivations of (2.6) and (2.7) are also given in the Appendix.

Finally for model (1.4), which is the most general one, since it allows different mean and deterministic trend for each season, we derive the test statistic

LM_4 and its limiting distribution,

$$\begin{aligned}LM_4 &= \sum_{j=1}^{d} a(\sum_{t=1}^{m-1} \varepsilon_{(t-1)d+j})^2 + \sum_{j=1}^{d} c(\sum_{t=1}^{m-1} t\varepsilon_{(t-1)d+j})^2 \\ &\quad - \sum_{j=1}^{d} 2b \sum_{t=1}^{m-1} \varepsilon_{(t-1)d+j} \sum_{t=1}^{m-1} t\varepsilon_{(t-1)d+j} + A_4/B_4, \qquad (2.8)\end{aligned}$$

and

$$\begin{aligned}LM_4 &\xrightarrow{\mathcal{L}} \sum_{j=1}^{d} 4W_j(1)^2 + \sum_{j=1}^{d} 12\{W_j(1) - \int_0^1 W_j(r)dr\}^2 \\ &\quad - \sum_{j=1}^{d} 12W_j(1)\{W_j(1) - \int_0^1 W_j(r)dr\} + C_4/D_4, \qquad (2.9)\end{aligned}$$

where

$$\begin{aligned}A_4 &= \{\sum_{j=1}^{d}\sum_{t=1}^{m-1} \varepsilon_{(t-1)d+j}(-a\sum_{t=1}^{m-1} Y_{(t-1)d+j} + b\sum_{t=1}^{m-1} tY_{(t-1)d+j})\}^2 \\ &\quad + \{\sum_{j=1}^{d}\sum_{t=1}^{m-1} t\varepsilon_{(t-1)d+j}(b\sum_{t=1}^{m-1} Y_{(t-1)d+j} - c\sum_{t=1}^{m-1} tY_{(t-1)d+j})\}^2 \\ &\quad + (\sum_{t=1}^{n} \varepsilon_t Y_{t-d})^2 \\ &\quad + 2\sum_{j=1}^{d}\{\sum_{t=1}^{m-1} \varepsilon_{(t-1)d+j}(-a\sum_{t=1}^{m-1} Y_{(t-1)d+j} + b\sum_{t=1}^{m-1} tY_{(t-1)d+j})\} \\ &\quad\quad \sum_{j=1}^{d}\{\sum_{t=1}^{m-1} t\varepsilon_{(t-1)d+j}(b\sum_{t=1}^{m-1} Y_{(t-1)d+j} - c\sum_{t=1}^{m-1} tY_{(t-1)d+j})\} \\ &\quad + 2\sum_{j=1}^{d}\{\sum_{t=1}^{m-1} \varepsilon_{(t-1)d+j}(-a\sum_{t=1}^{m-1} Y_{(t-1)d+j} + b\sum_{t=1}^{m-1} tY_{(t-1)d+j})\}\sum_{t=1}^{n} \varepsilon_t Y_{t-d} \\ &\quad + 2\sum_{j=1}^{d}\{\sum_{t=1}^{m-1} t\varepsilon_{(t-1)d+j}(b\sum_{t=1}^{m-1} Y_{(t-1)d+j} - c\sum_{t=1}^{m-1} tY_{(t-1)d+j})\}\sum_{t=1}^{n} \varepsilon_t Y_{t-d}, \\ B_4 &= \sum_{j=1}^{d}\sum_{t=1}^{m-1} Y^2_{(t-1)d+j} - \sum_{j=1}^{d} a(\sum_{t=1}^{m-1} Y_{(t-1)d+j})^2 \\ &\quad + \sum_{j=1}^{d} 2b\sum_{t=1}^{m-1} tY_{(t-1)d+j}\sum_{t=1}^{m-1} Y_{(t-1)d+j} - \sum_{j=1}^{d} c(\sum_{t=1}^{m-1} tY_{(t-1)d+j})^2, \\ a &= 2(2m+1)/(m+1)(m+2), \quad b = 6/(m+1)(m+2),\end{aligned}$$

$$
\begin{aligned}
c = \quad & 12/m(m+1)(m+2), \\
C_4 = \quad & \{ \sum_{j=1}^{d} W_j(1)(-4\int_0^1 W_j(r)dr + 6\int_0^1 rW_j(r)dr) \}^2 \\
& +\{ \sum_{j=1}^{d} (W_j(1) - \int_0^1 W_j(r)dr)(6\int_0^1 W_j(r)dr - 12\int_0^1 rW_j(r)dr) \}^2 \\
& +\{ \sum_{j=1}^{d} \frac{W_j(1)^2 - 1}{2} \}^2 \\
& +2\{ \sum_{j=1}^{d} W_j(1)\ (-4\int_0^1 W_j(r)dr + 6\int_0^1 rW_j(r)dr) \} \\
& \quad \{\sum_{j=1}^{d}(W_j(1) - \int_0^1 W_j(r)dr)(6\int_0^1 W_j(r)dr - 12\int_0^1 rW_j(r)dr)\} \\
& +2\{\sum_{j=1}^{d} W_j(1)(-4\int_0^1 W_j(r)dr + 6\int_0^1 rW_j(r)dr)\}\{\sum_{j=1}^{d} \frac{W_j(1)^2 - 1}{2}\} \\
& +2\{\sum_{j=1}^{d}(W_j(1) - \int_0^1 W_j(r)dr)(6\int_0^1 W_j(r)dr - 12\int_0^1 rW_j(r)dr)\} \\
& \quad \{\sum_{j=1}^{d} \frac{W_j(1)^2 - 1}{2}\}^2,
\end{aligned}
$$

and

$$
D_4 = \sum_{j=1}^{d} \int_0^1 W_j(r)^2 dr - \sum_{j=1}^{d}\{\int_0^1 W_j(r)dr\}^2 - 3\sum_{j=1}^{d}\{2\int_0^1 rW_j(r)dr - \int_0^1 W_j(r)dr\}^2.
$$

Details of the derivation for LM_4 are also given in the Appendix.

3. THE RELATIONS BETWEEN "DHF" TYPES AND LM TYPES

For the nonseasonal case, Solo (1984) showed the relationship between the LM test statistic and the regression t statistic derived by Dickey and Fuller

(1979). In this section we obtain the relationship between the LM test statistics derived in section 2 for models (1.1)-(1.4) and the regression t statistics by Dickey *et al.* (1984) for models (1.1)-(1.3) and T statistic for model (1.4) by Cho *et al.* (1993).

Theorem 3.1. Let LM_i satisfy (2.2), (2.4), (2.6) and (2.8), respectively. Then we have

$$LM_1 = \hat{\tau}_d^2, \tag{3.1}$$

$$LM_2 = \hat{\tau}_{\mu d}^{*2} + d^{-1}\{\sum_{j=1}^{d} W_j(1)\}^2, \tag{3.2}$$

$$LM_3 = \hat{\tau}_{\mu d}^2 + \sum_{j=1}^{d} W_j(1)^2, \tag{3.3}$$

and

$$LM_4 = T^2 + 4\sum_{j=1}^{d}[W_j(1)^2 - 3W_j(1)\int_0^1 W_j(r)dr + 3\{\int_0^1 W_j(r)dr\}^2], \tag{3.4}$$

where $\hat{\tau}_d$, $\hat{\tau}_{\mu d}^*$, and $\hat{\tau}_{\mu d}$ are the regression t statistics for models (1.1)-(1.3), respectively and T is in Cho *et al.* (1995).

Proof. Details of the derivations are given in the Appendix.

It can be shown that, for $d = 1$ the results of Theorem 3.1, (3.1) and (3.2), are the same as those of Solo (1984) in the regular case.

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APPENDIX

We choose an appropriate diagonal matrix D_{in} to derive the LM_i for $i = 1, \ldots, 4$. Then LM_i defined in section 2 can be constructed as follows.

$$LM_i = (D_{in}\frac{\partial L}{\partial \Theta_i})'(D_{in}J_iD_{in})^{-1}(D_{in}\frac{\partial L}{\partial \Theta_i}) = P_i'Q_i^{-1}P_i.$$

For model (1.1) if we choose $D_{1n} = n^{-1}$ then the derivation of (2.3) and (3.1) is straightforward.

Model (1.2)

Since $\partial L/\partial\alpha = \sum_{t=1}^{n}\varepsilon_t$, $\partial^2 L/\partial\alpha^2 = n$, $\partial L/\partial\rho = \sum_{t=1}^{n}\varepsilon_t Y_{t-d}$, $\partial^2 L/\partial\rho^2 = \sum_{t=1}^{n} Y_{t-d}^2$ and $\partial^2 L/\partial\rho\partial\alpha = \sum_{t=1}^{n} Y_{t-d}$, if we choose $D_{2n} = diag(\ n^{-1/2}, n^{-1}\ )$, then

$$LM_2 = P_2' Q_2^{-1} P_2,$$

where

$$P_2' = (\ n^{-1/2}\sum_{t=1}^{n}\varepsilon_t \quad n^{-1}\sum_{t=1}^{n}\varepsilon_t Y_{t-d}\ ),$$

and

$$Q_2 = \begin{pmatrix} 1 & n^{-3/2}\sum_{t=1}^{n} Y_{t-d} \\ n^{-3/2}\sum_{t=1}^{n} Y_{t-d} & n^{-2}\sum_{t=1}^{n} Y_{t-d}^2 \end{pmatrix}.$$

Using Chan and Wei (1988), we can show that

$$LM_2 = \frac{(\sum_{t=1}^{n}\varepsilon_t Y_{t-d})^2/n^2 - 2\sum_{t=1}^{n}\varepsilon_t\sum_{t=1}^{n} Y_{t-d}\sum_{t=1}^{n}\varepsilon_t Y_{t-d}/n^3 + (\sum_{t=1}^{n}\varepsilon_t)^2\sum_{t=1}^{n} Y_{t-d}^2/n^3}{\sum_{t=1}^{n} Y_{t-d}^2/n^2 - (\sum_{t=1}^{n} Y_{t-d})^2/n^3}$$

$$\xrightarrow{\mathcal{L}} C_2/D_2,$$

where

$$C_2 = d\{\sum_{j=1}^{d}\frac{W_j(1)^2-1}{2}\}^2 - 2\sum_{j=1}^{d} W_j(1)\sum_{j=1}^{d}\int_0^1 W_j(r)dr\sum_{j=1}^{d}\frac{W_j(1)^2-1}{2} \qquad (A.1)$$

$$+\{\sum_{j=1}^{d} W_j(1)\}^2\sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr$$

and

$$D_2 = d\sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr - \{\sum_{j=1}^{d}\int_0^1 W_j(r)dr\}^2. \qquad (A.2)$$

To derive the relationship between LM_2 and the studentized regression statistic $\hat{\tau}_{\mu d}^*$ of Dickey *et al.* (1984), from (A.1) and (A.2) we obtain

$$C_2/D_2 = \frac{\{d\sum_{j=1}^{d}(W_j(1)^2-1)/2 - \sum_{j=1}^{d} W_j(1)\sum_{j=1}^{d}\int_0^1 W_j(r)dr\}^2}{d[d\sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr - \{\sum_{j=1}^{d}\int_0^1 W_j(r)dr\}^2]}$$
$$+\frac{\{\sum_{j=1}^{d} W_j(1)\}^2\sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr - d^{-1}\{\sum_{j=1}^{d} W_j(1)\}^2\{\sum_{j=1}^{d}\int_0^1 W_j(r)dr\}^2}{d\sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr - \{\sum_{j=1}^{d}\int_0^1 W_j(r)dr\}^2}$$

$$= \quad \hat{\tau}^{*2}_{\mu d} + \frac{\{\sum_{j=1}^d W_j(1)\}^2[\sum_{j=1}^d \int_0^1 W_j(r)^2 dr - d^{-1}\{\sum_{j=1}^d \int_0^1 W_j(r)dr\}^2]}{d\sum_{j=1}^d \int_0^1 W_j(r)^2 dr - \{\sum_{j=1}^d \int_0^1 W_j(r)dr\}^2}$$

$$= \quad \hat{\tau}^{*2}_{\mu d} + d^{-1}\{\sum_{j=1}^d W_j(1)\}^2,$$

where $\hat{\tau}^*_{\mu d}$ is defined in Dickey *et al.* (1984),

$$\hat{\tau}^*_{\mu d} = \frac{d\sum_{j=1}^d (W_j(1)^2 - 1)/2 - \sum_{j=1}^d W_j(1)\sum_{j=1}^d \int_0^1 W_j(r)dr}{d^{1/2}[d\sum_{j=1}^d \int_0^1 W_j(r)^2 dr - \{\sum_{j=1}^d \int_0^1 W_j(r)dr\}^2]^{1/2}}.$$

Model (1.3)

For the derivation of (2.6), (2.7) and (3.3) we use $\partial L/\partial\alpha_j = \sum_{t=1}^m \varepsilon_{(t-1)d+j}$, $\partial^2 L/\partial\alpha_j^2 = m$, $\partial L/\partial\rho = \sum_{t=1}^n \varepsilon_t Y_{t-d}$, $\partial^2 L/\partial\rho^2 = \sum_{t=1}^n Y_{t-d}^2$, $\partial^2 L/\partial\alpha_j\partial\alpha_k = 0$, for $j \neq k$, and $\partial^2 L/\partial\rho\partial\alpha_j = \sum_{t=1}^m Y_{(t-1)d+j}$, and choose $D_{3n} = diag(\ n^{-1/2}, \ldots, \ n^{-1/2}, \ n^{-1}\ )$. Then

$$LM_3 = P_3' Q_3^{-1} P_3,$$

where

$$P_3' = (\ n^{-1/2}\sum_{t=1}^m \varepsilon_{(t-1)d+1}, \ \ldots, \ n^{-1/2}\sum_{t=1}^m \varepsilon_{(t-1)d+d}, \ n^{-1}\sum_{t=1}^n \varepsilon_t Y_{t-d}\ ),$$

and

$$Q_3 = \begin{pmatrix} n^{-1}m & & & n^{-3/2}\sum_{t=1}^n Y_{(t-1)d+1} \\ 0 & \ddots & 0 & \vdots \\ & & n^{-1}m & n^{-3/2}\sum_{t=1}^n Y_{(t-1)d+d} \\ n^{-3/2}\sum_{t=1}^n Y_{(t-1)d+1} & \ldots & n^{-3/2}\sum_{t=1}^n Y_{(t-1)d+d} & n^{-2}\sum_{t=1}^n Y_{t-d}^2 \end{pmatrix}.$$

Thus

$$LM_3 = \sum_{j=1}^d \frac{1}{m}(\sum_{t=1}^m \varepsilon_{(t-1)d+j})^2 + [\frac{1}{n^2}\sum_{t=1}^n Y_{t-d}^2 - \frac{1}{n^2 m}\sum_{j=1}^d(\sum_{t=1}^m Y_{(t-1)d+j})^2]^{-1}$$

$$[\ \frac{1}{n^2 m^2}\{\ \sum_{j=1}^d\ (\sum_{t=1}^m \varepsilon_{(t-1)d+j}\sum_{t=1}^m Y_{(t-1)d+j}\ )\ \}^2$$

$$-2\frac{1}{n^2m}\sum_{j=1}^{d}(\sum_{t=1}^{m}\varepsilon_{(t-1)d+j}\sum_{t=1}^{m}Y_{(t-1)d+j})\sum_{t=1}^{n}\varepsilon_t Y_{t-d}+\frac{1}{n^2}(\sum_{t=1}^{n}\varepsilon_t Y_{t-d})^2]$$

$$\xrightarrow{\mathcal{L}} \sum_{j=1}^{d}W_j(1)^2+[\sum_{j=1}^{d}\int_0^1 W_j(r)^2dr-\sum_{j=1}^{d}\{\int_0^1 W_j(r)dr\}^2]^{-1}$$

$$[\ \{\ \sum_{j=1}^{d}W_j(1)\int_0^1 W_j(r)dr\ \}^2$$

$$-2\sum_{j=1}^{d}W_j(1)\int_0^1 W_j(r)dr\sum_{j=1}^{d}\frac{W_j(1)^2-1}{2}+\{\sum_{j=1}^{d}\frac{W_j(1)^2-1}{2}\}^2]$$

$$=\sum_{j=1}^{d}W_j(1)^2+\frac{\{\ \sum_{j=1}^{d}W_j(1)\int_0^1 W_j(r)dr-\sum_{j=1}^{d}(W_j(1)^2-1)/2\ \}^2}{\sum_{j=1}^{d}\int_0^1 W_j(r)^2dr-\sum_{j=1}^{d}\{\int_0^1 W_j(r)dr\}^2}$$

$$=\hat{\tau}_{\mu d}^2+\sum_{j=1}^{d}W_j(1)^2,$$

where $\hat{\tau}_{\mu d}$ is defined in Dickey *et al.* (1984),

$$\hat{\tau}_{\mu d}=\frac{\sum_{j=1}^{d}W_j(1)\int_0^1 W_j(r)dr-\sum_{j=1}^{d}(W_j(1)^2-1)/2}{[\sum_{j=1}^{d}\int_0^1 W_j(r)^2dr-\sum_{j=1}^{d}\{\int_0^1 W_j(r)dr\}^2]^{1/2}}.$$

Model (1.4)

We have to show that (2.8), (2.9) and (3.4) hold. Using $\partial L/\partial\alpha_j=\sum_{t=1}^{m}\varepsilon_{(t-1)d+j}$, $\partial^2 L/\partial\alpha_j^2=m$, $\partial L/\partial\beta_j=\sum_{t=1}^{m}t\varepsilon_{(t-1)d+j}$, $\partial^2L/\partial\beta_j^2=m(m+1)(2m+1)/6$, $\partial L/\partial\rho=\sum_{t=1}^{n}\varepsilon_tY_{t-d}$, $\partial^2L/\partial\rho^2=\sum_{t=1}^{n}Y_{t-d}^2$, $\partial^2L/\partial\alpha_j\partial\alpha_k=0$, $\partial^2L/\partial\beta_j\partial\beta_k=0$, for $j\neq k$, $\partial^2L/\partial\alpha_j\partial\beta_j=m(m+1)/2$, $\partial^2L/\partial\rho\partial\alpha_j=\sum_{t=1}^{m}Y_{(t-1)d+j}$, and $\partial^2L/\partial\rho\partial\beta_j=\sum_{t=1}^{m}tY_{(t-1)d+j}$, if we choose $D_{4n}=diag(\ n^{-1/2},\ \ldots,\ n^{-1/2},\ n^{-3/2},\ \ldots,\ n^{-3/2},\ n^{-1}\ )$, we obtain

$$LM_4=P_4'(D_{4n}J_4D_{4n})^{-1}P_4,$$

where

$$P_4'=(\ n^{-1/2}\sum_{t=1}^{m}\varepsilon_{(t-1)d+1},\ \ \ldots,\ \ n^{-1/2}\sum_{t=1}^{m}\varepsilon_{(t-1)d+d},\ \ n^{-3/2}\sum_{t=1}^{m}t\varepsilon_{(t-1)d+1},\ \ \ldots,$$

$$n^{-1/2}\sum_{t=1}^{m} t\varepsilon_{(t-1)d+d},\quad n^{-1}\sum_{t=1}^{n}\varepsilon_t Y_{t-d}\ ),$$

and

$$J_4 = \begin{pmatrix} A & B & D \\ B & C & E \\ D' & E' & F \end{pmatrix},$$

where $A = mI$, $B = (m+1)/2I$, $C = m(m+1)(2m+1)/6I$, I is an identity matrix with dimension d, $D' = (\ \sum_{t=1}^{m-1} Y_{(t-1)d+1},\ \ldots,\ \sum_{t=1}^{m-1} Y_{(t-1)d+d}\ )$, $E' = (\ \sum_{t=1}^{m-1} tY_{(t-1)d+1},\ \ldots,\ \sum_{t=1}^{m-1} tY_{(t-1)d+d}\ )$, and $F = \sum_{t=1}^{n} Y_{t-d}^2$.

A computation of J_4^{-1} is given in the Appendix of Cho *et al.* (1993). Thus LM_4 is evaluated as $P_4'(D_{4n}J_4D_{4n})^{-1}P_4$. That is

$$\begin{aligned} LM_4 &= \sum_{j=1}^{d} a(\sum_{t=1}^{m-1}\varepsilon_{(t-1)d+j})^2 + \sum_{j=1}^{d} c(\sum_{t=1}^{m-1} t\varepsilon_{(t-1)d+j})^2 \\ &\quad - \sum_{j=1}^{d} 2b\sum_{t=1}^{m-1}\varepsilon_{(t-1)d+j}\sum_{t=1}^{m-1} t\varepsilon_{(t-1)d+j} + A_4/B_4, \end{aligned}$$

where

$$\begin{aligned} A_4 = \ & n^{-2}\{\sum_{j=1}^{d}\sum_{t=1}^{m-1}\varepsilon_{(t-1)d+j}(-a\sum_{t=1}^{m-1}Y_{(t-1)d+j} + b\sum_{t=1}^{m-1}tY_{(t-1)d+j})\}^2 \\ & +n^{-2}\{\sum_{j=1}^{d}\sum_{t=1}^{m-1}t\varepsilon_{(t-1)d+j}(b\sum_{t=1}^{m-1}Y_{(t-1)d+j} - c\sum_{t=1}^{m-1}tY_{(t-1)d+j})\}^2 \\ & +n^{-2}(\sum_{t=1}^{n}\varepsilon_t Y_{t-d})^2 \\ & +2n^{-2}\sum_{j=1}^{d}\{\sum_{t=1}^{m-1}\varepsilon_{(t-1)d+j}(-a\sum_{t=1}^{m-1}Y_{(t-1)d+j} + b\sum_{t=1}^{m-1}tY_{(t-1)d+j})\} \\ & \qquad \sum_{j=1}^{d}\{\sum_{t=1}^{m-1}t\varepsilon_{(t-1)d+j}(b\sum_{t=1}^{m-1}Y_{(t-1)d+j} - c\sum_{t=1}^{m-1}tY_{(t-1)d+j})\} \\ & +2n^{-2}\sum_{j=1}^{d}\{\sum_{t=1}^{m-1}\varepsilon_{(t-1)d+j}(-a\sum_{t=1}^{m-1}Y_{(t-1)d+j} + b\sum_{t=1}^{m-1}tY_{(t-1)d+j})\}\sum_{t=1}^{n}\varepsilon_t Y_{t-d} \\ & +2n^{-2}\sum_{j=1}^{d}\{\sum_{t=1}^{m-1}t\varepsilon_{(t-1)d+j}(b\sum_{t=1}^{m-1}Y_{(t-1)d+j} - c\sum_{t=1}^{m-1}tY_{(t-1)d+j})\}\sum_{t=1}^{n}\varepsilon_t Y_{t-d}, \end{aligned}$$

$$B_4 = n^{-2}\sum_{j=1}^{d}\sum_{t=1}^{m-1} Y^2_{(t-1)d+j} - n^{-2}\sum_{j=1}^{d} a(\sum_{t=1}^{m-1} Y_{(t-1)d+j})^2$$
$$+n^{-2}\sum_{j=1}^{d} 2b\sum_{t=1}^{m-1} tY_{(t-1)d+j}\sum_{t=1}^{m-1} Y_{(t-1)d+j} - n^{-2}\sum_{j=1}^{d} c(\sum_{t=1}^{m-1} tY_{(t-1)d+j})^2,$$

where $a = 2(2m+1)/(m+1)(m+2)$, $b = 6/(m+1)(m+2)$ and $c = 12/m(m+1)(m+2)$.

The limiting distribution for the LM_4 is is given by

$$LM_4 \xrightarrow{\mathcal{L}} \sum_{j=1}^{d} 4W_j(1)^2 + \sum_{j=1}^{d} 12\{W_j(1) - \int_0^1 W_j(r)dr\}^2$$
$$-\sum_{j=1}^{d} 12W_j(1)\{W_j(1) - \int_0^1 W_j(r)dr\} + C_4/D_4,$$

where

$$C_4 = \{ \sum_{j=1}^{d} W_j(1)(-4\int_0^1 W_j(r)dr + 6\int_0^1 rW_j(r)dr) \}^2$$
$$+\{ \sum_{j=1}^{d} (W_j(1) - \int_0^1 W_j(r)dr)(6\int_0^1 W_j(r)dr - 12\int_0^1 rW_j(r)dr) \}^2$$
$$+\{ \sum_{j=1}^{d} \frac{W_j(1)^2-1}{2} \}^2$$
$$+2\{ \sum_{j=1}^{d} W_j(1)\ (-4\int_0^1 W_j(r)dr + 6\int_0^1 rW_j(r)dr) \}$$
$$\{\sum_{j=1}^{d}(W_j(1) - \int_0^1 W_j(r)dr)(6\int_0^1 W_j(r)dr - 12\int_0^1 rW_j(r)dr)\}$$
$$+2\{\sum_{j=1}^{d} W_j(1)(-4\int_0^1 W_j(r)dr + 6\int_0^1 rW_j(r)dr)\}\{\sum_{j=1}^{d}\frac{W_j(1)^2-1}{2}\}$$
$$+2\{\sum_{j=1}^{d}(W_j(1) - \int_0^1 W_j(r)dr)(6\int_0^1 W_j(r)dr - 12\int_0^1 rW_j(r)dr)\}$$
$$\{\sum_{j=1}^{d}\frac{W_j(1)^2-1}{2}\}^2$$

$$= [\sum_{j=1}^{d} W_j(1)(-4\int_0^1 W_j(r)dr + 6\int_0^1 rW_j(r)dr) + \sum_{j=1}^{d} \frac{W_j(1)^2 - 1}{2}$$
$$+ \sum_{j=1}^{d}(W_j(1) - \int_0^1 W_j(r)dr)(6\int_0^1 W_j(r)dr - 12\int_0^1 rW_j(r)dr)\ ]^2$$
$$= [\ \frac{1}{2}\sum_{j=1}^{d}[\ \{W_j(1) - 2\int_0^1 W_j(r)dr\}$$
$$\{W_j(1) - 6(2\int_0^1 rW_j(r)dr - \int_0^1 W_j(r)dr)\} - 1]\ \ ]^2$$

and

$$D_4 = \sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr - \sum_{j=1}^{d}\{\int_0^1 W_j(r)dr\}^2 - 3\sum_{j=1}^{d}\{2\int_0^1 rW_j(r)dr - \int_0^1 W_j(r)dr\}^2.$$

Hence

$$LM_4 = T^2 + 4\sum_{j=1}^{d}[\ W_j(1)^2 - 3W_j(1)\int_0^1 W_j(r)dr + 3\{\int_0^1 W_j(r)dr\}^2\ ],$$

where T is defined in Cho *et al.* (1993), and its explicit form is

$$T = \frac{(1/2)\sum_{j=1}^{d}[\ \{W_j(1) - 2\int_0^1 W_j(r)dr\}\{W_j(1) - 6(2\int_0^1 rW_j(r)dr - \int_0^1 W_j(r)dr)\} - 1]}{[\sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr - \sum_{j=1}^{d}\{\int_0^1 W_j(r)dr\}^2 - 3\sum_{j=1}^{d}\{2\int_0^1 rW_j(r)dr - \int_0^1 W_j(r)dr\}^2]^{1/2}}.$$