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A Sharp Result of Random Upper Functions for Levy Processes

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ABSTRACT

In this paper, we show that the result of random upper functions for Levy processes obtained by Joo(1993) can be sharpened under some additional assumption. This is the continuous analogue of result obtained by Griffin and Kuelbs(1989) for sums of i.i.d. random variables.

1. INTRODUCTION

Let $\{X(t) : t \geq 0\}$ be a real-valued stochastic process with stationary independent increments whose characteristic function is given by

$$Eexp\{iuX(t)\} = exp\{tg(u)\}$$

where

$$g(u) = ibu + \int (e^{iux} - 1 - \frac{iux}{1+x^2})\, d\nu(x)$$

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and ν is a Levy measure on $R - \{0\}$ satisfying

$$\int (x^2 \wedge 1)\, d\nu(x) < \infty.$$

We first note that a Gaussian component is excluded. And as usual, we assume that $X(0) = 0$ and that we are dealing with a version which has almost all sample functions right continuous and having left limits. For $x > 0$, we define

$$\begin{aligned} G(x) &= \int_{|y|>x} d\nu(y), \\ K(x) &= x^{-2} \int_{|y|\le x} y^2\, d\nu(y) \\ Q(x) &= G(x) + K(x). \end{aligned} \tag{1.1}$$

It is well-known that Q is positive, continuous, decreasing and zero at infinity. Also, it is obvious that

$$x^2 Q(x) \text{ is nondecreasing.} \tag{1.2}$$

The above functions G, K and Q play an important role in studying the sample functions of $X(t)$ including limit theorems. Upper functions for Levy process $X(t)$ as a generalization of the law of the iterated logarithm was established by Fristedt(1971), Joo(1993), Kim and Wee(1991) and the others. In particular, Joo(1993) showed that if $\limsup_{x\to 0} G(x)/K(x) < \infty$, then there exist an appropriate subordinator $T(t)$ and ordinary function $\alpha(t)$ such that for some positive finite constant C,

$$\limsup_{t\to 0} \frac{X(t) - \alpha(t)}{(T(t) \log|\log t|)^{1/2}} = C \quad a.s. \;.$$

Similar conclusion holds if $x \to 0$ and $t \to 0$ are replaced by $x \to \infty$ and $t \to \infty$, respectively. This result is the continuous analogue of result obtained by Griffin and Kuelbs (1989) for sums of i.i.d. random variables.
In this paper, we show that the above result obtained by Joo(1993) can be sharpened if $\limsup G(x)/K(x) < 1$. Section 2 consists of the basic facts and some lemmas. In section 3, we prove the main result. We deal mainly with the behavior for small t since the case of large t follows from similar arguments. Throughout this paper, the following notation will be used frequently;

(1) As $x \to 0$ and $x \to \infty$, $f(x) \sim g(x)$ iff $f(x)/g(x) \to 1$.
(2) $L_2 t = \log|\log t|$.

2. PRELIMINARIES

In this section, we give the basic facts and some lemmas which will be used in proving the main result. We start with the useful properties of Q as follows.

Lemma 2.1. (1) If $\limsup_{x\to 0} G(x)/K(x) < \theta < \infty$, then $x^{2/(1+\theta)}Q(x)$ is strictly decreasing for x sufficiently small, and

$$\int_{|x|\le 1} |x|^{\varepsilon}\, d\nu(x) = \infty$$

for each $\varepsilon \le 2/(1+\theta)$.

(2) If $\limsup_{x\to\infty} G(x)/K(x) < \theta < \infty$, then $x^{2/(1+\theta)}Q(x)$ is strictly decreasing for x sufficiently large, and

$$\int_{|x|>1} |x|^{\varepsilon}\, d\nu(x) < \infty \qquad \text{for each } \varepsilon < 2/(1+\theta).$$

Proof. First we note that for $0 < \lambda < 2$ and $0 < a < b$,

$$\lambda \int_a^b x^{\lambda-1} G(x)\, dx = \int_{a<|x|\le b} |x|^{\lambda}\, d\nu(x) + b^{\lambda}G(b) - a^{\lambda}G(a), \tag{2.1}$$

and

$$(2-\lambda)\int_a^b x^{\lambda-1}K(x)\, dx = \int_{a<|x|\le b} |x|^{\lambda}\, d\nu(x) + a^{\lambda}K(a) - b^{\lambda}K(b). \tag{2.2}$$

Hence,

$$b^{\lambda}Q(b) - a^{\lambda}Q(a) = \int_a^b x^{\lambda-1}\{\lambda G(x) - (2-\lambda)K(x)\}\, dx.$$

Thus, the first parts follow immediately.
Now, the second parts are consequenses of the first parts since by (2.1) and(2.2)

$$\int_0^1 x^{\lambda-1}Q(x)\, dx < \infty \iff \int_{|x|\le 1} |x|^{\lambda}\, d\nu(x) < \infty$$

and
$$\int_1^\infty x^{\lambda-1}Q(x)\,dx < \infty \iff \int_{|x|>1} |x|^\lambda \,d\nu(x) < \infty. \quad \square$$

If $\limsup G(x)/K(x) < \theta < \infty$, then by lemma 2.1, for x sufficienly small or large,
$$x^p Q(x) \text{ is strictly decreasing,} \tag{2.3}$$

where $p = 2/(1+\theta)$ is used throughout this paper.

If we define, for $\lambda > 0$,

$$\begin{aligned} Q(a(t)) &= 1/t, \\ b_\lambda(t) &= a(\lambda t/L_2 t), \\ \beta_\lambda(t) &= b_\lambda^2(t) L_2 t, \end{aligned} \tag{2.4}$$

then $b_\lambda(t)$ and $\beta_\lambda(t)$ increase for small or large t.

Lemma 2.2. If $\limsup_{x\to 0} G(x)/K(x) < \theta < \infty$, then for $y \le x$ and all x sufficiently small,

$$(y/x)^{1/p} \le a(y)/a(x) \le 1. \tag{2.5}$$

Proof. See lemma 2.3 of Joo(1993). $\square$

We now introduce a process $T(t)$ as follows;

$$T(t) = \lim_{\varepsilon\to 0} \sum_{s\le t} \{X(s) - X(s^-)\}^2 I\{|X(s) - X(s^-)| > \varepsilon\}, \tag{2.6}$$

where $I(A)$ denotes the indicator function of A.
It follows immediately that $T(t)$ is an increasing Levy process whose characteristic function is given by
$$Eexp\{iuT(t)\} = exp\{t\int (e^{iux} - 1)\,d\mu(x)\}, \tag{2.7}$$

where μ is a Borel measure on $(0,\infty)$ defined by $\mu(E) = \nu(E^{1/2}) + \nu(-E^{1/2})$ admitting the notation $E^{1/2} = \{x^{1/2} : x \in E\}$.

Lemma 2.3. Suppose that (2.3) holds as $x \to 0$, and for $\lambda > 0$, let

$$\eta(\lambda) = 2^{-1}\lambda^{1/2}e^{\lambda^{1/2}} + 2\lambda^{1/2}. \tag{2.8}$$

If λ_0 is sufficiently small such that $\eta(\lambda_0) < (1+\theta)^{-1}$, then for all $\lambda \geq \lambda_0$,

$$\liminf_{t \to 0} \frac{T(t)}{\beta_\lambda(t)} \geq \kappa(\lambda, \lambda_0) > 0, \tag{2.9}$$

where $\kappa(\lambda, \lambda_0) = \{(1+\theta)^{-1} - \eta(\lambda_0)\}(\lambda_0/\lambda)^{2/p}\lambda_0^{-1}$.

Proof. See lemma 2.6. of Joo(1993). □

Lemma 2.4. If $t_k = exp(-k^q)$ with $q > 1$, then

$$\lim_{k \to \infty} T(t_{k+1})/T(t_k) = 0. \quad a.s. \ .$$

Proof. See lemma 4.6. of Joo(1993). □

3. MAIN RESULT

In this section, we assume that

$$\limsup G(x)/K(x) < \theta < 1 \tag{3.1}$$

as $x \to 0$ or $x \to \infty$.

It follows immediately from (1.2) and (2.3) that if (3.1) holds, for x sufficiently small or large,

$$\xi^{-2}Q(x) \leq Q(\xi x) \leq \xi^{-p}Q(x) \qquad \text{if } \ \xi > 1, \tag{3.2}$$

where $p = 2/(1+\theta) \in (1,2)$. Also, it is a consequence of lemma 2.1 that the condition (3.1) as $x \to \infty$ implies

$$\int_{|x|>1} |x| \, d\nu(x) < \infty$$

which is equivalent to $E|X(t)| < \infty$.

In this case, if we define, for $x > 0$,

$$H(x) = x^{-1} \int_{|y|>x} |y| \, d\nu(y),$$

then (2.2) with $\lambda = 1$ shows

$$\int_x^\infty K(y) \, dy = xK(x) + xH(x). \tag{3.3}$$

On the other hand, the condition (3.1) as $x \to 0$ implies

$$\int_{|x| \le 1} |x| \, d\nu(x) = \infty,$$

and by (2.3),

$$xQ(x) \to \infty \qquad \text{as} \quad x \to 0,$$

which means

$$t^{-1}a(t) \to \infty \qquad \text{as} \quad t \to 0.$$

Thus, (2.9) implies that

$$t/(T(t)L_2 t)^{1/2} \to 0 \qquad \text{as} \quad t \to 0. \tag{3.4}$$

Now we decompose $X(t)$ as follows; for $k = 1, 2$, let

$$Eexp\{iuX_k(t;a)\} = exp\{tg_k(u)\},$$

where $g_1(u) = \int_{|x| \le a} (e^{iux} - 1 - iux) \, d\nu(x)$ and $g_2(u) = \int_{|x|>a} (e^{iux} - 1) \, d\nu(x)$.

Then $X(t) = X_1(t : a) + X_2(t : a) + tM(a)$,

where $M(a) = b + \int_{|x| \le a} x^3/(1 + x^2) \, d\nu(x) - \int_{|x|>a} x/(1 + x^2) \, d\nu(x)$.

Since $X_2(t : a) = 0$ *a.s.* for t sufficiently small and (3.4) holds, we may assume

that $X(t) = X_1(t : a)$ in order to obtain the result that $\limsup_{t\to 0} X(t) / (T(t)L_2 t)^{1/2}$ is a.s. positive finite constant. Hence whenever we consider the behavior for small t, we restrict ourselves to the case where $G(1) = 0, G(x) \leq \theta K(x)$ for $x \leq 1$ and $X(t) = X_1(t, 1)$.
On the other hand, whenever we deal with the behavior for large t, we assume that $EX(1) = 0$.
In each setting, we have

$$\int_{|x|>1} |x| d\nu(x) < \infty,$$

and

$$Eexp\{iuX(t)\} = exp\{t \int (e^{iux} - 1 - iux)\, d\nu(x)\}.$$

Lemma 3.1. (a) If $G(x)/K(x) \leq \theta < 1$ for all $x \leq 1$ and $G(1) = 0$, then

$$H(x) \leq Q(x)(2 + \theta - p)/((p-1)(1+\theta)) \text{ for all } x \leq 1 .$$

(b) Similar statement for large x holds

Proof. By (2.3), for $x \leq 1$,

$$\int_x^\infty K(y)\, dy \leq x^p Q(x) \int_x^\infty y^{-p} dy = xQ(x)/(p-1).$$

Thus we obtain that by (3.4), for $x \leq 1$,

$$\begin{aligned} H(x) &= x^{-1} \int_x^\infty K(y)\, dy - K(x) \\ &\leq Q(x)/(p-1) - Q(x)/(1+\theta) \\ &= Q(x)(2 + \theta - p)/(p-1)(1+\theta). \ \square \end{aligned}$$

Lemma 3.2. If (3.1) holds, then $\{T(t)/a^2(t)\}$ is tight for t sufficiently small or large.

Proof. For $\xi > 0$, let $\{\overline{T}(t)\}$ be a process whose characteristic function is given by

$$Eexp\{iu\overline{T}(t)\} = exp\{t \textstyle\int_{x \leq \xi^2 a^2(t)} (e^{iux} - 1)\, d\mu(x)\}.$$

Then

$$\begin{aligned} P\{T(t) \geq Na^2(t)\} &\leq P\{\overline{T}(t) \geq Na^2(t)\} + 1 - exp\{-tG_\mu(\xi^2a^2(t))\} \\ &\leq E\overline{T}(t)/Na^2(t) + tG_\mu(\xi^2a^2(t)) \\ &= t\xi^2K(\xi a(t))/N + tG(\xi a(t)) \\ &\leq t(\xi^2/N + 1)Q(\xi a(t)) \end{aligned}$$

where G_μ is given as in (1.1) by replacing ν with μ.
It follows from (3.2) that for $\xi > 1$ and for t small or large

$$Q(\xi a(t)) \leq \xi^{-p}Q(a(t)) = \xi^{-p}t^{-1}.$$

Hence

$$P\{T(t) \geq Na^2(t)\} \leq \xi^{2-p}/N + \xi^{-p}.$$

Thus given $\varepsilon > 0$, choose ξ large enough that $\xi^{-p} < \varepsilon/2$ and then N large enough that $\xi^{2-p}/N < \varepsilon/2$. For this choice of N and for t small or large,

$$P\{T(t) \geq Na^2(t)\} < \varepsilon,$$

which proves the lemma. □

Lemma 3.3. (a) If $\limsup_{x\to 0} G(x)/K(x) < 1$, then there exists $C_0 > 0$ such that for all t sufficiently small,

$$P\{X(t) \geq a(t)\} \geq C_0 \quad . \tag{3.5}$$

(b) Similar statement for large t holds if $EX(1) = 0$.

Proof. See Theorem 4.6 of Wee(1988). □

Lemma 3.4. (a) If $\limsup_{x\to 0} G(x)/K(x) < 1$, then for $\delta > 0$ sufficiently small and t small,

$$P\{X(t) \geq \delta(T(t)L_2t)^{1/2}\} \geq exp(-2^{-1}L_2t). \tag{3.6}$$

(b) If $\limsup\limits_{x\to\infty} G(x)/K(x) < \theta$ and $EX(1) = 0$,then for $\delta > 0$ sufficiently small and n large integer,

$$P\{X(n) \geq \delta(T(n)L_2n)^{1/2}\} \geq exp(-2^{-1}L_2n).$$

Proof. Let $\xi = 4\log(2/C_0)$ where $C_0 > 0$ is such that (3.5) holds. Let $\rho(t) = [L_2t/\xi] + 1$, $\tau(t) = \xi t/L_2t$ and $k(t) = (\rho(t) - 1)\tau(t)$, where [ ] is the greatest integer function.
Then $\rho(t) \sim L_2t/\xi$ and $k(t) \leq t$.
Now we define for $i = 1, 2, \cdots, \rho(t)$,

$$\begin{aligned} V_i &= X(i\tau(t)) - X((i-1)\tau(t)), \\ W_i &= T(i\tau(t)) - T((i-1)\tau(t)). \end{aligned} \tag{3.7}$$

If we write $b(t)$ for $a(t/L_2t)$ then for small t,

$$P\{X(t) \geq \delta(T(t)L_2t)^{1/2}\}$$

$$\geq P\{X(t) \geq \delta(T(t)L_2t)^{1/2}, W_i \leq Nb^2(t), 1 \leq i \leq \rho(t)\}$$

$$\geq P\{X(t) \geq \delta(\rho(t)Nb^2(t)L_2t)^{1/2}, W_i \leq Nb^2(t), 1 \leq i \leq \rho(t)\} \tag{3.8}$$

$$\geq P\{V_i \geq \delta(2N\xi)^{1/2}b(t), W_i \leq Nb^2(t), 1 \leq i < \rho(t), X(t) - X(k(t)) \geq 0, W_{\rho(t)} \leq Nb^2(t)\}$$

$$= [P\{V_1 \geq \delta(2N\xi)^{1/2}b(t), W_1 \leq Nb^2(t)\}]^{\rho(t)-1}P\{X(t) - X(k(t)) \geq 0, W_{\rho(t)} \leq Nb^2(t)\}.$$

Now, we apply (2.5) and Lemma 3.2 to choose N large enough so that

$$P\{W_1 \leq Nb^2(t)\} = P\{T(\xi t/L_2t) \leq Nb^2(t)\} \geq 1 - C_0/2. \tag{3.9}$$

For this choice of N we take $\delta = (2N\xi)^{-1/2}$. Then by (3.5),

$$P\{V_1 \geq \delta(2N\xi)^{1/2}b(t)\} \geq P\{X(\tau(t)) \geq a(\tau(t))\} \geq C_0 \tag{3.10}$$

for all small t since $\xi > 1$.

Furthermore, $P\{X(t) - X(k(t)) \geq 0, W_{\rho(t)} \leq Nb^2(t)\} \geq C_0/2$ for all small t.
Thus for t sufficiently small, (3.8),(3.9) and (3.10) combine to yield

$$P\{X(t) \geq \delta(T(t)L_2t)^{1/2}\} \;\geq\; (C_0/2)^{\rho(t)}$$

$$\begin{aligned} &= exp(-\xi\rho(t)/4) \\ &\geq exp(-2^{-1}L_2 t). \end{aligned}$$

The proof for (b) run in a similar way except that,in (3.9), we choose N large enough so that

$$P\{W_1 \leq Nb^2(t)\} \geq \max(1 - C_0/2, 1 - C_1/2), \tag{3.9'}$$

where C_1 is a positive constant satisfying

$$\inf_n P\{X(n) \geq 0\} > C_1.$$

The reason for which we consider only integer is that for large t

$$P\{X(t) - X(k(t)) \geq 0\} \geq C_0$$

does not hold in general. □

Theorem 3.5. (a) If $\limsup_{x\to 0} G(x)/K(x) < \theta < 1$, then for some positive finite constant C,

$$\limsup_{t\to 0} \frac{X(t)}{(T(t)L_2t)^{1/2}} = C \ \ a.s..$$

(b) Similar statement for large t holds if $EX(1) = 0$.

Proof. We give the proof for $t \to 0$. Recalling first that we assume $G(1) = 0$ and

$$Eexp\{iuX(t)\} = exp\{t\int (e^{iux} - 1 - iux)\, d\nu(x)\},$$

we have

$$M(a) = -\int_{|x|>a} x\, d\nu(x).$$

Now (2.9) and Lemma 3.1 imply

$$\begin{aligned} \limsup_{t\to 0} \frac{|M(b_\lambda(t))|}{(T(t)L_2t)^{1/2}} &\leq \limsup_{t\to 0} \frac{tb_\lambda(t)H(b_\lambda(t))}{(\kappa(\lambda,\lambda_0)\beta_\lambda(t)L_2t)^{1/2}} \\ &\leq \frac{2+\theta-p}{(p-1)(1+\theta)\lambda\kappa^{1/2}(\lambda,\lambda_0)}. \end{aligned}$$

Combining this estimate and Theorem 1 of [6], it follows that C is a finite constant.
Now to prove that C is positive, let $t_k = exp(-k^q)$ with $1 < q < 2$ and write

$$X(t_k) = X(t_k) - X(t_{k+1}) + X(t_{k+1}). \tag{3.11}$$

By the argument above and Lemma 2.4, we obtain

$$\limsup_{k\to\infty} \frac{|X(t_{k+1})|}{(T(t_k)L_2 t_k)^{1/2}} = 0 \ \ a.s.. \tag{3.12}$$

Furthermore, if $\delta > 0$ is as in Lemma 3.4, and

$$E_k = \{X(t_k) - X(t_{k+1}) > \delta[(T(t_k) - T(t_{k+1}))L_2(t_k - t_{k+1})]^{1/2}\}$$

then for all large k,

$$P(E_k) \geq (k+1)^{-q/2}. \tag{3.13}$$

Since the events $\{E_k\}$ are independent and $1 < q < 2$, (3.13) and the Borel-Cantelli lemma imply

$$\limsup_{k\to\infty} \frac{X(t_k) - X(t_{k+1})}{[(T(t_k) - T(t_{k+1}))L_2(t_k - t_{k+1})]^{1/2}} \geq \delta \ \ a.s.$$

which,together with (3.11),(3.12) and Lemma 2.4, yields the desired result.
The proof of (b) can be proceeded in a similar way except we consider $n_k = [exp(k^q)]$ with $1 < q < 2$ to prove the lower bound. □

Corollary 3.6. (a) If $X(t)$ is a stable process of exponent $\alpha \in (1,2)$, then $T(t)$ is a stable subordinator of exponent $\alpha/2$ and

$$\limsup_{t\to 0} \frac{X(t)}{(T(t)L_2(t))^{1/2}} = C \quad a.s.$$

for some positive constant C.
(b) Similar conclusion holds for large t if we consider the process $X(t) - EX(t)$.

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