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Applications of Saddlepoint Method to Stress-Strength Model[1)]

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Abstract

In many problems concerned with statistical inferences, it will be of interest to compute tail areas rather than densities. But, it is often hard to calculate the exact tail probability. Saddlepoint approximation formula to the tail probability of a smooth function of random vector is developed by DiCiccio and Martin(1991). Applications of this method to stress- strength model are considered in this paper. To obtain the generalized p-values suggested by Tsui and Weerahandi(1989), we need to calculate complicated multiple integration. However, DiCiccio and Martin's(1991) results offer a convenient method to approximate these very accurately. For many artificial data sets, we access the accuracy of DiCiccio and Martin's by comparing the approximate value with the exact one.

1. Introduction

Since Daniels(1954) introduced the saddlepoint method into statistical problem, many approximation formulae to the *density* of the various type of statistics, including sample mean and maximum likelihood estimator, etc, has been developed so far. Barndorff- Nielsen and Cox(1979) reviewed the method of approximations related to the *density*. These approximation formulae to the *density* can not be used directly to the problems concerned with statistical inferences.

To construct confidence intervals or p-values for testing, we need to approximate the *tail* area (or cumulative distribution function), rather than the *density*. It can be obtained by integrating the approximation of the *density*. However, since the probability is often expressed in the form of multiple integration, we need many computation to solve it numerically. In addition, as the dimension is larger the results from numerical integration are not reliable. To get around this problem, many statisticians are devoted to develop the accurate approximation formulae to the *tail* probability.

Among many related papers, Lugannani- Rice(1980) and Daniels(1987) offer very accurate saddlepoint approximations to the *tail* probability of sample mean. Also, Barndorff-

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Nielsen(1990) and Fraser(1990), etc, studied to the approximations of the *tail* probability of maximum likelihood estimator. Unfortunatly, most of the development of saddlepoint theories related to the *tail* probabilty have been restricted to the univariate problems. Recently, Wang(1990) has derived the saddlepoint approximation to the *tail* probability for the sample mean of n independent bivariate random variables.

Generalized p-values suggested by Tsui and Weerahandi(1989) are a kind of significant probability for testing parameters in stress-strength model. But, the generalized p-values are often expressed in the form containing multiple integral, we have difficulty in calculating these numerically. In this paper, we will use the approximation formula developed by DiCiccio and Martin(1991) to avoid the complicated multiple integration needed to calculate the generalized p-values. Their results are very accurate and offer a convenient method to approximate the generalized p-values. In Section 2, we briefly reviewed the approximation suggested by DiCiccio and Martin(1991) to the *tail* probability of a smooth function of random vector. Section 3 devoted to some application problems concerned with stress- strength model in reliability theory. For many artificial data set, we access the accuracy of the formula by comparing the approximate value with the exact one.

2. Saddlepoint approximations to marginal tail probability

In many situations, inference for a scalar parameter in the presence of nuisance parameters . requires integration of either a joint density of pivotal quantities or a joint posterior density. DiCiccio and Martin(1991) give the approximation to marginal tail probability for a real-valued function of a random vector, where the function has continuous gradient that does not vanish at the mode of the joint density of the random vector.

Consider a continuous random vector $Y=(Y^1,\cdots,Y^p)$ having density of the form

$$f_Y(y) \propto b(y)\exp\{\ell(y)\}, \quad y=(y^1,\cdots,y^p). \tag{2.1}$$

Suppose that the function ℓ attains its maximum value at $\hat{y}=(\hat{y}^1,\cdots,\hat{y}^p)$ and $Y-\hat{y}$ is $O_p(n^{-1/2})$ as sample size n increases indefinitely. For each fixed y, assume that $\ell(y)$ and its partial derivatives are $O(n)$ and that $b(y)$ is $O(1)$. Now, consider a real-valued variable $Z=g(Y)$, where the function g has continuous gradient that is nonzero at $\hat{y}$. We will discuss an accurate approximation for marginal tail probability of Z that is easy to compute and can avoid numerical integration.

Let $\tilde{y}=\tilde{y}(z)$ be the value of y that maximizes $\ell(y)$ subject to the constraint $Z=z$. Moreover, let $\hat{z}=g(\hat{y})$, so that $Z-\hat{z}$ is $O_p(n^{-1/2})$ and $\tilde{y}(\hat{z})=\hat{y}$. Consider the function

$$r(z)=sgn(z-\hat{z})[2\{\ell(\hat{y})-\ell(\tilde{y}(z))\}]^{\frac{1}{2}}, \tag{2.2}$$

which is assumed to be monotonically increasing. Let us define some notations as follows : For $i,j=1,\cdots,p$,

$$\ell_i(y)=\partial\ell(y)/\partial y^i, \quad \ell_{ij}(y)=\partial^2\ell(y)/\partial y^i\partial y^j, \tag{2.3}$$

$$g_i(y)=\partial g(y)/\partial y^i, \quad g_{ij}(y)=\partial^2 g(y)/\partial y^i\partial y^j, \tag{2.4}$$

$$H_{ij}(z)=-\ell_{ij}\{\tilde{y}(z)\}+\frac{\ell_k\{\tilde{y}(z)\}}{g_k\{\tilde{y}(z)\}}g_{ij}\{\tilde{y}(z)\}, \tag{2.5}$$

where k is any index such that $g_k\{\tilde{y}(z)\}$ does not vanish and

$$H(z)=\{H_{ij}(z)\}, \quad \{H(z)\}^{-1}=\{H^{ij}(z)\}. \tag{2.6}$$

Note that $H(z)$ is a $p\times p$ matrix and $H(\hat{z})=\{-\ell_{ij}(\hat{y})\}$.

DiCiccio and Martin(1991) gives the tail probability of Z as follows.

$$\Pr(Z\geq z)=\bar{\Phi}(r)-\phi(r)\left[\frac{1}{r}+D(z)\frac{g_j\{\tilde{y}(z)\}}{\ell_j\{\tilde{y}(z)\}}\frac{b\{\tilde{y}(z)\}}{b(\hat{y})}\right]+O(n^{-\frac{3}{2}}), \tag{2.7}$$

where $r=r(z)$ and j is any index such that $g_j\{\tilde{y}(z)\}$ is nonzero and

$$D(z)=\left\{H^{ij}(z)g_i\{\tilde{y}(z)\}g_j\{\tilde{y}(z)\}\frac{|H(z)|}{|H(\hat{z})|}\right\}^{-\frac{1}{2}} \tag{2.8}$$

It is noted that the expression of summation convension is used in (2.8).

3. Applications of saddlepoint method to stress- strength model

An important problem in stress-strength model concerns testing hypotheses about the reliability parameter $R=\Pr\{X>Y\}$. Suppose that the reliability of a unit is to be tested using independent samples $X_1,X_2,\cdots,X_m$ and $Y_1,Y_2,\cdots,Y_n$ obtained from the normal populations $N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$, respectively. Then the problem of testing

$$H_0: R\leq R_0 \quad \text{versus} \quad H_1: R>R_0 \tag{3.1}$$

is equivalent to testing

$$H_0:\theta\leq\theta_0 \quad \text{versus} \quad H_1:\theta>\theta_0, \tag{3.2}$$

where $\theta=(\mu_1-\mu_2)/\sqrt{\sigma_1^2+\sigma_2^2}$ and $\theta_0=\Phi^{-1}(R_0)$.

Now consider the testing problem of (3.2) in case of $\theta_0=0$ for convenience. In this case, $\theta=(\mu_1-\mu_2)/\sqrt{\sigma_1^2+\sigma_2^2}$ is parameter of interest and $\nu=(\sigma_1^2,\sigma_2^2)$ is nuisance parameter. Tsui and Weerahandi(1989) suggest *the generalized p-value* so that it serve to measure how strongly the observed data support the null hypothesis. It is a kind of significant probability for testing problem. The smaller the p-value, the stronger the evidence against the null hypothesis. The generalized p-value for testing (3.2) is given by

$$p=\Pr(W \geq w \mid \theta=0), \tag{3.3}$$

where

$$W=(\overline{X}-\overline{Y})\left(\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}\right)^{-\frac{1}{2}}\left\{\frac{\sigma_1^2}{m}\frac{s_1^2}{S_1^2}+\frac{\sigma_2^2}{n}\frac{s_2^2}{S_2^2}\right\}^{\frac{1}{2}}. \tag{3.4}$$

Here, four statistics $\overline{X},\overline{Y},S_1^2,$ and S_2^2 are the sample means and the variances of random samples and are mutually independent. Note that the observed value of W is given by $w=\overline{x}-\overline{y}$.

By letting

$$c_1=\frac{s_1^2}{s_1^2+s_2^2},\quad c_2=\frac{s_2^2}{s_1^2+s_2^2}=1-c_1,\quad c_3=\frac{w}{\sqrt{s_1^2+s_2^2}}, \tag{3.5}$$

it can be easily shown that the expression (3.3) is equivalent to

$$\Pr\left\{\frac{T}{\sqrt{m+n-2}}\left(\frac{c_1}{B}+\frac{c_2}{1-B}\right)^{\frac{1}{2}} \geq c_3\right\}, \tag{3.6}$$

where T has a Student's t distribution with $(m+n-2)$ degrees of freedom and is independent of B which is beta distribution with parameters $(m-1)/2$, $(n-1)/2$. We can obtain the *exact* value of (3.6) by the numerical computation of

$$E_B[\psi\{-c_3\sqrt{m+n-2}\ (c_1/B+c_2/(1-B))^{-1/2}\}], \tag{3.7}$$

where $\psi(\cdot)$ is the *CDF* of Student's t distribution with $(m+n-2)$ degree of freedom, and E_B denotes the expectation with respect to B.

To avoid the complicated numerical integration of (3.7), we will consider the method to approximate the values of (3.6). By using the DiCiccio and Martin's(1991) results (2.7) in Section 2, we can approximate (3.6) as follows :

Let $\gamma=m+n-2$ and $U=T/\sqrt{\gamma}$. Then the joint density of U and B is given by, from the independence of U and B,

$$\ell(u,b) = -\frac{\gamma+1}{2}\log(1+u^2) + \frac{m-3}{2}\log b + \frac{n-3}{2}\log(1-b). \tag{3.8}$$

The maximized values of (3.8) are

$$\hat{u}=0, \quad \hat{b}=(m-3)/(m+n-6), \tag{3.9}$$

and it can be easily shown that the regularity conditions, by letting, $m=\tau n$, $\tau \in [0,1]$,

$$U=\hat{u}+O_p(n^{-\frac{1}{2}}), \quad B=\hat{b}+O_p(n^{-\frac{1}{2}}) \tag{3.10}$$

are satisfied.

Suppose $Z=g(U,B)$, where $g(U,B)=U\sqrt{c_1/B+c_2/(1-B)}$. Then, under $g(u,b)=c_3$, the maximized values $\tilde{u}(c_3),\tilde{b}(c_3)$ of (3.8) can be obtained by numerical or theoretical method. The corresponding values of (2.3) and (2.4) are given by

$$\ell_u=-u(\nu+1)/(1+u^2), \quad \ell_b=\{(m-3)/b-(n-3)/(1-b)\}/2,$$

$$\ell_{uu}=(\nu+1)(u^2-1)/(1+u^2)^2,$$

$$\ell_{ub}=0, \quad \ell_{bb}=\{-(m-3)/b^2+(n-3)/(1-b)^2\}/2$$

and

$$g_u=\{c_1/b+c_2/(1-b)\}^{1/2}, \quad g_b=ua/(2g_u),$$

$$g_{uu}=0, \quad g_{ub}=a/(2g_u), \quad g_{bb}=u\,(2a'g_u^2-a^2)/(4g_u^3),$$

$$a=-c_1/b^2+c_2/(1-b)^2, \quad a'=2\,\{c_1/b^3+c_2/(1-b)^3\},$$

where g_α, ℓ_α and $g_{\alpha\beta}, \ell_{\alpha\beta}$ are the first- and second-derivatives with respect to its subscripts, respectively. For a given values of c_1,c_3, we can obtain the values of $\tilde{y}=(\tilde{u},\tilde{b})$ and (2.8) which are needed to calculate (2.7).

Figure 1 shows that the approximate results via the saddlepoint approximation formula (2.7) are very close to the exact values. Also, the approximate method requires much less computing time than the case of the exact one. As the approximation (2.7) is easy to use and very accurate, we can avoid the complicate numerical integration of (3.7). For the values of $\tilde{y}=(\tilde{u},\tilde{b})$ in the approximation, we use the Brent's method. The exact values of (3.7) are calculated by using the algorithm of the Romberg integration. Both exact and approximate values used in Figure 1 are listed in Table 1 to access the accuracy of the approximation. FORTRAN programs to obtain the values of the exact and the approximate in Figure 1 are available from me on request.

Figure 1. Generalized p-values in the case of normal distribution

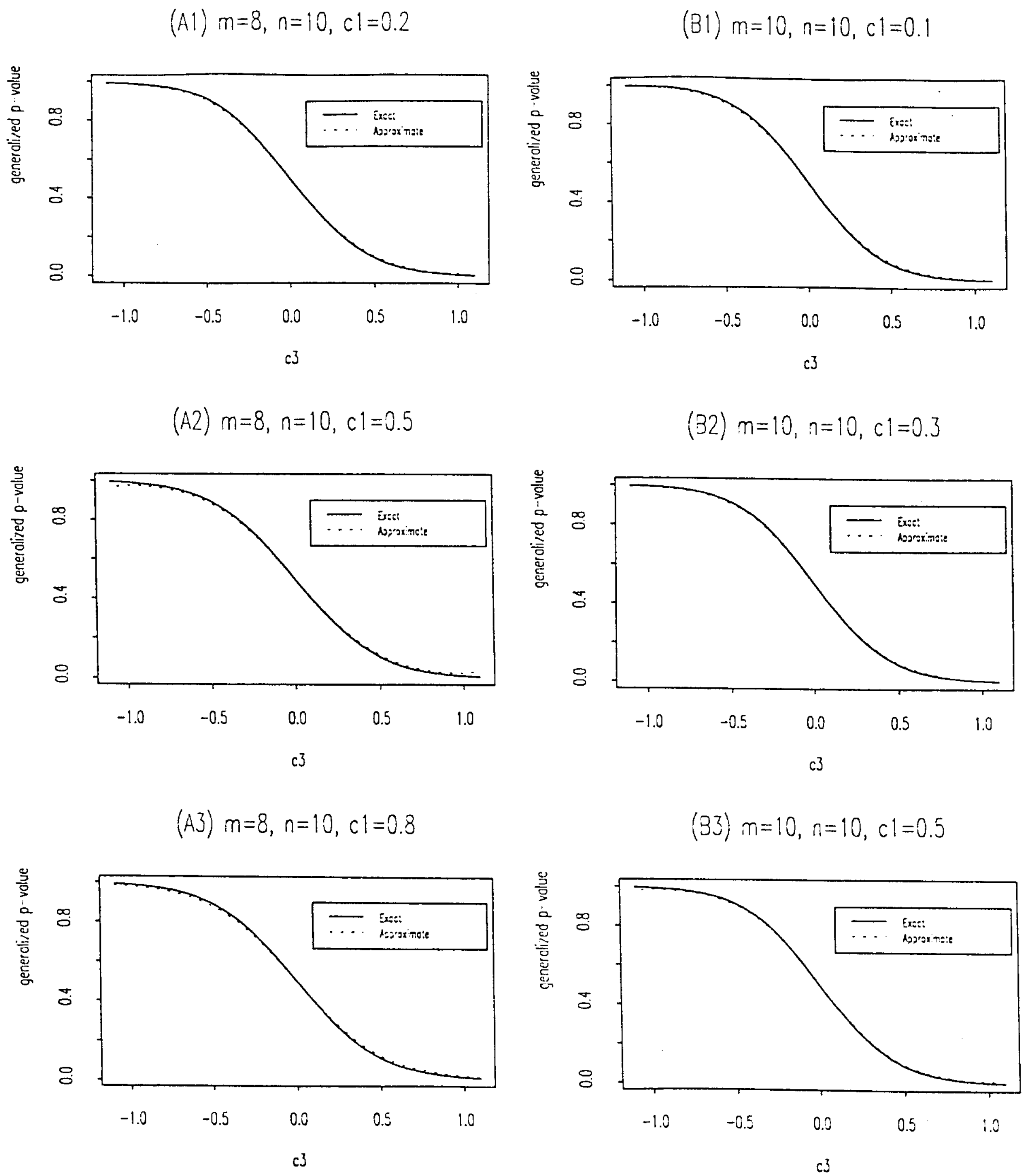

Table 1. Generalized p-values used in Figure 1

Figure 1	c_3	Exact	Approximate	Figure 1	c_3	Exact	Approximate
	-1.0	.9923	.9919		-1.0	.9930	9892
	-.8	.9789	.9774		-.8	.9805	9749
	-.6	.9436	.9401		-.6	.9469	9390
	-.4	.8625	.8567		-.4	.8673	8584
A1 :	-.2	.7112	.7062	**B1** :	-.2	.7154	7089
$c_1 = 0.2$	.2	.2888	.2938	$c_1 = 0.1$	.2	.2846	2911
	.4	.1375	.1433		.4	.1327	1416
	.6	.0564	.0599		.6	.0531	0610
	.8	.0211	.0226		.8	.0195	0251
	1.0	.0077	.0081		1.0	.0070	0108
	-1.0	.9912	.9894		-1.0	.9936	9900
	-.8	.9757	.9723		-.8	.9809	9761
	-.6	.9368	.9312		-.6	.9461	9398
	-.4	.8520	.8450		-.4	.8645	8577
A2 :	-.2	.7023	.6970	**B2** :	-.2	.7122	7075
$c_1 = 0.5$	.2	.2977	.3030	$c_1 = 0.3$	.2	.2878	2925
	.4	.1480	.1550		.4	.1355	1423
	.6	.0632	.0688		.6	.0539	0602
	.8	.0243	.0277		.8	.0191	0239
	1.0	.0088	.0106		1.0	.0064	0100
	-1.0	.9894	.9860		-1.0	.9938	.9868
	-.8	.9723	.9673		-.8	.9810	.9755
	-.6	.9312	.9249		-.6	.9458	.9396
	-.4	.8450	.8389		-.4	.8636	.8573
A3 :	-.2	.6970	.6932	**B3** :	-.2	.7112	.7070
$c_1 = 0.8$	.2	.3030	.3068	$c_1 = 0.5$	.2	.2888	.2930
	.4	.1550	.1611		.4	.1364	.1427
	.6	.0688	.0751		.6	.0542	.0604
	.8	.0277	.0327		.8	.0190	.0245
	1.0	.0106	.0140		1.0	.0062	.0131

Now, we will consider another similar application. Let $X_1, \cdots, X_m$ be random samples from exponential distribution $\Gamma(1,\mu_1)$ and let $Y_1, \cdots, Y_n$ be random samples from $\Gamma(1,\mu_2)$. The X_i's and Y_i's are independent. Consider the problem of hypotheses testing

$$H_0 : \mu_1-\mu_2 \le \delta_0 \quad \text{versus} \quad H_1 : \mu_1-\mu_2 > \delta_0, \tag{3.11}$$

where $\delta_0 \ge 0$. Let $X=\sum_{i=1}^{m} X_i$ and $Y=\sum_{i=1}^{n} Y_i$. Suppose x and y are the observed values of X and Y, respectively. Let $\lambda_i=\mu_i/x$, $i=1,2$, $\theta_0=\delta_0/x$ and $\theta=\lambda_1-\lambda_2$. Then (3.11) is equivalent to testing

$$H_0 : \theta \le \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0. \tag{3.12}$$

Here, θ is the parameter of interest and λ_2 is the nuisance parameter.

Tsui and Weerahandi(1989) suggest the generalized p-value of the test (3.12) is given by

$$p=\Pr\left\{ \frac{y}{xV} - \frac{1}{U} + \theta \ge 0 \mid \theta=\theta_0 \right\}, \tag{3.13}$$

where $U=X/\{x(\theta+\lambda_2)\} \sim \Gamma(m,1)$, $V=Y/(\lambda_2 x) \sim \Gamma(n,1)$ and U and V are independent. For a given data set, the value of p serves as a measure of how the data support H_0. The *exact* value of (3.13) can be obtained by numerical computation of

$$p=E_B[\Gamma_{m+n}(f(B))], \tag{3.14}$$

where

$$\Gamma_{m+n}(f(B)) = \frac{1}{\Gamma(m+n)} \int_0^{f(B)} z^{m+n-1} e^{-z} dz,$$

$$f(B)=\frac{1}{\theta_0}\left(\frac{y}{xB} - \frac{1}{1-B} \right),$$

and E_B is taken with respect to $B \sim \beta(n,m)$.

Now, we consider the approximation of (3.13) by using (2.7). An equivalent expression of (3.13) is given by

$$\Pr\left\{ \frac{y}{xnS} - \frac{1}{mT} \ge \theta_0 \right\}, \tag{3.15}$$

where $S \sim \Gamma(n,1/n)$ and $T \sim \Gamma(m,1/m)$ and S and T are independent.

Let $Z=g(S,T)=y/(xnS)-1/(mT)$. Then the joint density of S and T is given by

$$\ell(s,t)=(n-1)\log s+(m-1)\log t-ns-mt. \tag{3.16}$$

The maximized values of (3.16) are

$$\hat{s} = (n-1)/n, \quad \hat{t} = (m-1)/m, \tag{3.17}$$

and it can be easily shown that the regularity conditions

$$S = \hat{s} + O_p(n^{-\frac{1}{2}}), \quad T = \hat{t} + O_p(m^{-\frac{1}{2}}) \tag{3.18}$$

are satisfied.

Under $Z=\theta_0$, the maximized values $\tilde{s}(\theta_0), \tilde{t}(\theta_0)$ of (3.16) can be obtained by numerical method. The results of (2.3) and (2.4) are given by

$$\ell_s = (n-1)/s - n, \quad \ell_t = (n-1)/t - m,$$

$$\ell_{ss} = -(n-1)/s^2, \quad \ell_{st} = 0, \quad \ell_{tt} = -(m-1)/t^2$$

and

$$g_s = -y/(xns^2), \quad g_t = 1/(mt^2),$$

$$g_{ss} = 2y/(xns^3), \quad g_{st} = 0, \quad g_{tt} = -2/(mt^3).$$

All the values which are needed to calculate the approximation (2.7) are easily obtained from the above summarized results.

Table 2 gives us the values of exact and approximate results obtained from (3.14) and (2.7). The exact values are calculated from numerical integration of (3.14) containing improper integral. The approximate results from (2.7) are very close to the exact values even for small sample sizes. Moreover, the computation of these approximations requires much less computer time than the computation of the exact values.

Finally, we note that the methods given in this section are also applicable to many other statistical problems. For examples, stress-strength model with covariate (Weerahandi and Johnson(1992)), multivariate Behrens- Fisher problem, and Bayesian analysis, etc.

4. Conclusions

DiCiccio and Martin(1991) gives the approximation formula to the marginal tail probability by using saddlepoint techniques. Their approximation is very accurate and gives a convenient way to solve multiple integration arising in statistical problems. In this paper, we considered the applications of their approximation to the problems concerned with reliability theory.

Generalized p-value, which is a kind of significant probability for testing problem in stress-strength model, can be approximated by using the approximation given by DiCiccio and Martin(1991). The *approximate* values are almost coincide with the *exact* values. Also, the approximation is easy to use and require much less computer time than the case of the *exact* values. So, we can avoid the complicated multiple integration which is needed to calculate the *exact* generalized p-value. Applications to many artificial data sets are considered in this paper.

Table 2. Generalized p-values in the case of exponential distribution

		$\theta_0 = 2.0$					$\theta_0 = 3.0$		
m	n	y/x	Exact	Approximate	m	n	y/x	Exact	Approximate
5	10	5	.0001	.0001	5	10	15	.0205	.0202
		10	.0168	.0166			20	.0967	.0960
		15	.1418	.1407			25	.2487	.2470
		20	.4028	.3965			30	.4451	.4440
		25	.6722	.6705			35	.6346	.6332
		30	.8524	.8511			40	.7826	.7814
		35	.9429	.9420			45	.8815	.8806
		40	.9802	.9797			50	.9400	.9394
		45	.9936	.9932			55	.9714	.9711
		50	.9980	.9978			60	.9871	.9868
		55	.9994	.9996			65	.9944	.9942
		60	.9998	.9999			70	.9976	.9975
10	10	5	.0002	.0002	10	10	15	.0259	.0256
		10	.0234	.0231			20	.1168	.1159
		15	.1808	.1797			25	.2881	.2865
		20	.4751	.4740			30	.4968	.4957
		25	.7434	.7426			35	.6860	.6850
		30	.8999	.8994			40	.8244	.8238
		35	.9673	.9672			45	.9107	.9103
		40	.9907	.9907			50	.9581	.9579
		45	.9977	.9976			55	.9816	.9815
		50	.9995	.9995			60	.9924	.9924
		55	.9999	.9999			65	.9970	.9970
		60	1.0000	1.0000			70	.9989	.9989
30	30	40	.0179	.0179	30	30	60	.0191	.0191
		45	.0635	.0634			65	.0461	.0460
		50	.1598	.1598			70	.0940	.0939
		55	.3089	.3088			75	.1670	.1669
		60	.4871	.4858			80	.2639	.2637
		65	.6587	.6583			85	.3782	.3774
		70	.7963	.7962			90	.4994	.4991
		75	.8904	.8903			95	.6164	.6161
		80	.9464	.9464			100	.7201	.7198
		85	.9761	.9761			105	.8054	.8052
		90	.9902	.9902			110	.8708	.8707
		95	.9963	.9962			115	.9180	.9179

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