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Signed Linear Rank Statistics for Autoregressive Processes

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Abstract

This study provides a nonparametric procedure for the statistical inference of the parameters in stationary autoregressive processes. A confidence region and a hypothesis testing procedure based on a class of signed linear rank statistics are proposed and the asymptotic distributions of the test statistic both under the null hypothesis and under a sequence of local alternatives are investigated. Some desirable asymptotic properties including the asymptotic relative efficiency are discussed for various score functions.

1. Introduction

Let X_t, $t=0,\pm 1,\ldots$, be a p-order autoregressive process with mean zero, that is

$$X_t = \sum_{j=1}^{p} \beta_j X_{t-j} + V_t \tag{1.1}$$

where the V_t are independent identically distributed (i.i.d.) random variables with a common distribution function G. Assume that the processes is stationary, that is, all the zeros of polynomial $\phi(z)$ lie outside the unit cicle where $\phi(z)=1-\sum_{j=1}^{p}\beta_j z^j$. Suppose G possesses absolutely continous and symmetric (about zero) probability density function g with a finite Fisher information

$$\int_{-\infty}^{\infty} \left\{ \frac{g'(x)}{g(x)} \right\}^2 dG(x) \quad (< \infty) \tag{1.2}$$

where $g'(x) = \dfrac{dg(x)}{dx}$ exists a.e..

The parameter $\beta=(\beta_1,\ldots,\beta_p)'$ is unknown and the problem of the time series analysis is to make inference about β in some optimal way, on the basis of observations $X_t, t=1,\ldots,n$. In the classical parametric approach, there are several procedures one can use for estimating or

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hypothesis testing. Especially, the method of maximum likelihood is widely used and has been shown to have certain optimum properties Mann and Wald (1943) developed the maximum likelihood estimators as well as the asymptotic theory. It is implicitly assumed in performing this method that the errors V_t are normally distributed and this assumption sometimes may be unrealistic. Under the assumption of normally distributed errors the least squares method and the (conditional) maximum likelihood method are equivalent.

In this paper we propose a nonparametric procedure for the estimation and testing hypotheses about β, based on signed linear rank statistics when the distribution of errors in the model (1.1) is not necessarily normal. In Section 2 we define a class of signed linear rank statistics suitable for deriving a confidence region and test statistics, and investigate the asymptotic behavior of the statistics. An asymptotic confidence region and test procedure for the hypothesis about β are proposed in Section 3. In order to study the asymptotic power properties of the proposed test, the limiting distribution of the test statistic is derived in Section 4, under a sequence of local alternatives tending to the null hypothesis at a suitable rate. Finally the asymptotic relative efficiency (ARE) of the proposed test with respect to the classical chi-square test based on the least squares estimators is derived in Section 5.

2. Signed Linear Rank Statistics and Their Asymptotic Normality

In this section we shall propose a class of signed linear rank statistics and prove that under appropriate conditions the statistic has a normal distribution in the limit. Now, we note that we can observe only $X_1,\ldots,X_n$. On the basis of these observations we shall like to obtain an efficient estimator of the unknown parameter. For $t=0,\pm 1,\ldots \pm n$, define $Y_t=I_tX_t$ where $I_t=1$ or 0 according as $t \in \{1,\ldots,n\}$ or otherwise. Let $D_i(\beta)=Y_i-\sum_{j=1}^{p}\beta_jY_{t-j}$, $i=1,\ldots,n$, and $R_i(\beta)$ be the rank of $|D_i(\beta)|$ among $|D_1(\beta)|,\ldots,|D_n(\beta)|$. Let $\phi(u) =\phi_1(u)-\phi_2(u)$, $0<u<1$, where $\phi_i(u)$, $i=1,2$, are both nondecreasing square integrable functions on (0,1). Let $x=(X_1,\ldots,X_n)'$. Define the statistic

$$S(x,\beta)=(S_1(x,\beta),\ldots,S_p(x,\beta))' \tag{2.1}$$

where

$$S_u(x,\beta) = \frac{1}{n}\sum_{t=1}^{n}Y_{t-u}\, Sgn(D_t(\beta))\, \phi\left[\frac{R_t(\beta)}{n+1}\right], \quad u=1,\ldots,p, \tag{2.2}$$

and $Sgn(z)=1$ or -1 according as $z \geq 0$ or $z < 0$.

The regression constants, $\frac{1}{n}\sum_{t=1}^{n} Y_{t-u}, u=1,\dots,p,$ are random, and the choice of these constants are inspired by the cofficients in the normal equations of the method of least squares in the model (1.1). The wide choice of score function ϕ is available. Typically the score functions may be an inverse distribution function. Various score functions allied to the expression of the ARE will be discussed in the last section.

The following theorem gives us the asymptotic normality of $S(x,\beta)$.

Theorem 2.1. Let the model (1.1) holds and let the foregoing assumptions on V_t hold. Then $\sqrt{n}S(x,\beta)$ has asymptotically a p-variate normal distribution with mean vector zero and variance-covariance matrix $C_X\int_0^1 \phi^2(\omega)d\omega$, where C_X is the $p\times p$ matrix with $(u,v)th$ elements, $C(u,v)=Cov\,[X_t, X_{t+|u-v|}]$.

To prove this theorem we need the following lemma.

Let us define further notations: Let $D_i^*(\beta)=X_i-\sum_{j=1}^{p}\beta_j X_{i-j}$, $i=1,\dots,n$ and $R_i^*(\beta)$ be the rank of $|D_i^*(\beta)|$ among $|D_t^*(\beta)|$, $1\le t\le n$. Let $x^*=(X_{1-p},\dots,X_o,X_1,\dots,X_n)'$, Define $S^*(x^*,\beta)$ as $S(x,\beta)$ in (2.1) except that Y_t, $D_t(\beta)$ and $R_t(\beta)$ are replaced by X_t, $D_t^*(\beta)$ and $R_t^*(\beta)$ in (2.2), respectively. Let E_β and Var_β denote the expected value and variance when β is the true parameter.

Lemma 2.2. Under the assumptions of Theorem 2.1, $\sqrt{n}S^*(x^*,\beta)$ has asymptotically a p -variate normal distribution with mean vector zero and variance-covariance matrix $C_X\int_0^1\phi^2(\omega)d\omega$.

Proof. Given in the Appendix.

Proof of Theorem 2.1. By virtue of Lemma 2.2, it suffices to show that for every $u=1,\dots,p$,

$$\sqrt{n}\ (S_u^*(x^*,\beta)\ -\ S_u(x,\beta)\ )\ \xrightarrow{P}\ 0.$$

Note that for every μ

$$E_\beta\{\sqrt{n}\ [S_u^*(x^*,\beta)\ -\ S_u(x,\beta)\ ]\}^2\ =\ Var_\beta\ \{\sqrt{n}\ [S_u^*(x^*,\beta)\ -\ S_u(x,\beta)\ ]\} \qquad (2.3)$$

$$= \frac{1}{n} E_\beta \left\{ \sum_{t=1}^{n} \left[X_{t-u}\, Sgn(D_t^*(\beta))\ \phi\left(\frac{R_t^*(\beta)}{n+1} \right) - Y_{t-u}\, Sgn(D_t(\beta))\ \phi\left(\frac{R_t(\beta)}{n+1} \right) \right] \right\}^2$$

$$\leq \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} \left\{ E_\beta \left[X_{t-u}\, Sgn(D_t^*(\beta))\ \phi\left(\frac{R_t^*(\beta)}{n+1} \right) - Y_{t-u}\, Sgn(D_t(\beta))\ \phi\left(\frac{R_t(\beta)}{n+1} \right) \right]^2 \right.$$

$$\left. \cdot\ E_\beta \left[X_{s-u}\, Sgn(D_s^*(\beta))\ \phi\left(\frac{R_s^*(\beta)}{n+1} \right) - Y_{s-u}\, Sgn(D_s(\beta))\ \phi\left(\frac{R_s(\beta)}{n+1} \right) \right]^2 \right\}^{\frac{1}{2}} .$$

Furthermore, note that for all t

$$\lim_{n\to\infty} E_\beta \phi^2\left(\frac{R_t^*(\beta)}{n+1} \right) = \int_o^1 \phi^2(w)dw, \quad \lim_{n\to\infty} E_\beta \phi^2\left(\frac{R_t(\beta)}{n+1} \right) \leq \lim_{n\to\infty} np_{(n)} \int_o^1 \phi^2(w)dw,$$

and

$$\lim_{n\to\infty} E_\beta \left\{ \phi\left(\frac{R_t^*(\beta)}{n+1} \right) - \phi\left(\frac{R_t(\beta)}{n+1} \right) \right\}^2 \leq \lim_{n\to\infty} \left(\frac{p}{n+1} \right)^2 \int_o^1 \phi'^{\,2}(w)dw,$$

where $p_{(n)} = \max_{\{p+1\leq k\leq n\}} P\,[R_t(\beta)=k\,]$ which tends to zero as $n\to\infty$, and ϕ' is the derivative of ϕ. The last inequality follows from the fact that $0\leq |R_t^*(\beta)-R_t(\beta)|\leq p$ for all t, and the mean value theorem.

From the above results and the fact that $Y_{t-u}=0$ for $1 \leq t \leq u$, $Y_{t-u}=X_{t-u}$ for $u+1 \leq t \leq n$ and $D_t(\beta)=D_t^*(\beta)$ for $p+1 \leq t \leq n$, where $1 \leq u \leq p$, the (2.3) multipling $C(0)$ is, before taking limit, after simplification, less than or equal to

$$\frac{u^2}{n} I_\phi + 2u(p-u) \left[\frac{1}{n^2} + \frac{p_{(n)}}{n} \right]^{\frac{1}{2}} I_\phi + 2u \left[\frac{p(n-p)}{n(n+1)} \right] (I_\phi\ I_{\phi'})^{\frac{1}{2}}$$

$$+(p-u)^2\left[\frac{1}{n} + p_{(n)} \right] I_\phi + 2p(p-u)\left[\frac{n-p}{n+1} \right]\left[\frac{1}{n^2} + \frac{p_{(n)}}{n} \right]^{\frac{1}{2}} (I_\phi\ I_{\phi'})^{\frac{1}{2}}$$

$$+ \left[\frac{p^2(n-p)^2}{n(n+1)^2} \right] I_{\phi'}$$

which converges to zero as $n \to \infty$, where $I_\phi = \int_o^1 \phi^2(w)dw$ and $I_{\phi'} = \int_o^1 \phi'^{\,2}(w)dw$. This completes the proof.

3. Confidence Region and Test Statistic

In this section we shall consider asymptotic confidence region for the parameter β in the model (1.1), and test procedure for the hypothesis about β based on $S(x,\beta)$ defined in Section 2. The asymptotic power properties of the test will be considered in later sections.

The asymptotic normality of $\sqrt{n}S(x,\beta)$, derived in Theorem 2.1, under the regularity conditions suggests the use of the pivotal quantity of the form

$$Q_n(x,\beta) = n\left[\int_o^1 \phi^2(w)dw\right]^{-1} S'(x,\beta)C_n^{-1}S(x,\beta) \tag{3.1}$$

where C_n is the $p \times p$ matrix with $(u,v)th$ elements,

$$\frac{1}{n}\sum_{t=1}^{n-|u-v|} X_t X_{t+|u-v|}.$$

The following theorem gives the large sample distribution of $Q_n(x,\beta)$.

Theorem 3.1. Under the conditions of Theorem 2.1, $Q_n(x,\beta)$ has asymptotically a central chi-square distribution with p degrees of freedom.

Proof. Theorem 3.1 follows immediately from Theorem 2.1 and the fact that C_n converges to C_X.

By reference to the null limiting distribution of $Q_n(x,\beta)$, we define $C_{1-\alpha}(\beta)$ as the set of β such that

$$S'(x,\beta)\ C_n^{-1}\ S(x,\beta) \leq \delta$$

where δ is $\frac{1}{n}\left[\int_o^1 \phi^2(w)dw\right]\chi^2_{1-\alpha}(p)$ and $\chi^2_{1-\alpha}(p)$ is the $(1-\alpha)th$ quantile of the chi-square distribution. Then, for n large $C_{1-\alpha}(\beta)$ provides a $100(1-\alpha)$ percent confidence region for β.

We also propose the following large sample α-size test procedure for hypothesis $H_o: \beta = \beta_o$ against $H_1: \beta \neq \beta_o$, where β_o is a specified vector : Reject or accept H_o according as $Q_n(x,\beta_o) \geq$ or $< \chi^2_{1-\alpha}(p)$.

4. Asymptotic Distribution of Q_n under Contiguous Alternatives

In order to study the asymptotic power properties of the test considered in Section 3, it is necessary to study the limiting distribution of the test statistic under a sequence of local alternatives tending to the null hypothesis at a certain rate. In this section we shall specifically consider following sequence of local alternative hypotheses H_n defined by

$$H_n : \beta = \beta_n \quad \text{where} \quad \beta_n = \beta_o + \frac{\tau}{\sqrt{n}} \tag{4.1}$$

where $\tau \in R^p$ with each component of $\tau' s$ is positive.

Let (P_o, P_n) be a sequence of two probability distributions of $(X_1, \ldots, X_n)$ under $H_o : \beta = \beta_o$ and H_n respectively. It is well known that the contiguity of P_n to P_o provides the asymptotic normality of a statistic $\sqrt{n}\lambda' S(x, \beta_o)$, $\lambda \in R^p$ under the given sequence of probability distributions P_n if the statistic has an asymptotic normal distribution under the sequence of probability distributions P_o.

We shall define some anxiliary statistics. Let $X_{(t-1,t-p)} = (X_{t-1}, \ldots, X_{t-p})'$. Then the likelihood ratio statistic L_n for P_o vs. P_n is

$$\log L_n = \sum_{t=1}^{n} \log \left[\frac{g(X_t - \beta_n' X_{(t-1,t-p)})}{g(X_t - \beta_o' X_{(t-1,t-p)})} \right] \tag{4.2}$$

where $g(x_t - \beta_o' x_{(t-1,t-p)}) > 0$ for all t . Denote

$$W_n = 2 \sum_{t=1}^{n} \left\{ \left[\frac{g(X_t - \beta_n' X_{(t-1,t-p)})}{g(X_t - \beta_o' X_{(t-1,t-p)})} \right]^{\frac{1}{2}} - 1 \right\} \tag{4.3}$$

and

$$\psi(u) = -\frac{g'(G^{-1}(u))}{g(G^{-1}(u))}, \quad 0 < u < 1$$

where $G(x) = \int_{-\infty}^{x} g(y)dy$.

Before we proceed to consider the main theorem of this section, we present the following lemma required subsequently.

Lemma 4.1. Under P_o, $\log L_n$ has asymptotically $N(-\frac{1}{2}\sigma^2, \sigma^2)$, where

$$\sigma^2 = \tau' C_X \tau \int_o^1 \psi^2(w)dw. \tag{4.4}$$

The proof of this lemma is given in the Appendix.
We have the following theorem from Lemma 4.1.

Theorem 4.2. Let the model (1.1) and condition (1.2) hold. Then, under P_n, $\sqrt{n}\lambda' S(x, \beta_o)$, where $\lambda \in R^p$ and $S(x,\beta)$ is defined in (2.1), has asymptotically a normal distribution with mean $\lambda'\mu$ and variance $\lambda' C_X \lambda \int_o^1 \phi^2(w)dw$ where $\mu = C_X \tau \int_o^1 \psi(w)\phi(w)dw$ and C_X is defined in Theorem 2.1.

Proof. By virtue of LeCam's first lemma and Lemma 4.1, $\{P_n\}$ is contiguous to $\{P_o\}$. It therefore suffices to prove, by LeCam's third lemma, that under P_o, $(\log L_n, \sqrt{n}\lambda' S(x,\beta_o))$ has asymptotically a bivariate normal distribution with mean vector $\bar{\mu} = (-\frac{1}{2}\sigma^2, 0)$ and variance-covariance matrix $\Sigma = [\sigma_{ij}]$ where $\sigma_{11} = \sigma^2$, $\sigma_{12} = \sigma_{21} = \lambda'\mu$ and $\sigma_{22} = \lambda' C_X \lambda \int_o^1 \phi^2(w)dw$, where σ^2 is defined in (4.4). Now, recall the asymptotic distributions of $S^{(o)}(x^*,\beta_o)$ and V_n^* which are defined in (A.1) and (A.2) in the Appendix. In the proof of Lemma 2.2, it is shown that $\sqrt{n}\lambda' S(x,\beta_o)$ and $\sqrt{n}\lambda' S^{(o)}(x^*,\beta_o)$ have the same limiting normal distribution with zero mean and variance $\lambda' C_X \lambda \int_o^1 \phi^2(w)dw$. Moreover, in the proof of Lemma 4.2, $\log L_n$ and $\sqrt{n}V_n^* - \frac{1}{2}\sigma^2$ both are asymptotically $N(-\frac{1}{2}\sigma^2, \sigma^2)$ under P_o. Therefore it remains to show the desired asymptotic normality of $(\sqrt{n}V_n^* - \frac{1}{2}\sigma^2, \sqrt{n}\lambda' S^{(o)}(x^*,\beta_o))$, which follows from Lemma B in the Appendix. This completes the proof.

Theorem 4.3. Under the assumptions of Theorem 4.2, and $\{H_n\}$ in (4.1), $\sqrt{n}S(x,\beta_o)$ has asymptotically a p-variate normal distribution with mean μ and variance-covariance matrix $C_X \int_o^1 \phi^2(w)dw$.

Proof. The proof of this theorem follows directly from Theorem 4.2.

Then, we have the following main theorem.

Theorem 4.4. Under the assumptions of Theorem 4.2, and $\{H_n\}$ in (4.1), the statistic $Q_n(y,\beta_o)$, where $Q_n(y,\beta)$ is defined in (3.1), has asymptotically a noncentral chi-square distribution with p degrees of freedom and noncentrality parameter equal to

$$\Delta_{\tilde{Q}} = \frac{\left[\int_o^1 \psi(w)\phi(w)dw\right]^2}{2\int_o^1 \phi^2(w)dw}\tau' C_X \tau \tag{4.5}$$

Proof. The proof follows, by the same argument as in proof of Theorem 3.1, from Theorem 4.3.

5. Asymptotic Efficiency of the Proposed Tests

In this section we shall consider the relative asymptotic power efficiency of the proposed tests ($\tilde{Q}$-tests) based on $Q_n(x,\beta)$ with respect to a classical parametric chi-square test (χ^2 -test) derived from the asymptotic normality of the least squares estimator (LSE).

It is known [Mann and Wald (1943)] that under the assumption that the V_t are i.i.d. (but not necessarily normal), with finite fourth moments, the sequence of the LSE's $\hat{\beta}_n$ has asymptotically normal distribution in the sense that

$$\sqrt{n}(\hat{\beta}_n-\beta_o) \xrightarrow{L} N_p(0, \sigma^2 C_X^{-1})$$

where σ^2 is the common variance of errors in the model (1.1).

For simplicity, we assume that σ^2 is known. The χ^2-test statistic is, therefore,

$$X^2(x,\beta) = \frac{n}{\sigma^2}(\hat{\beta}_n-\beta_o)' C_n (\hat{\beta}_n-\beta_o)$$

where C_n is defined in (3.1), and the large sample size α-test for $H_o: \beta=\beta_o$ has the form: Reject or accept H_o, according as $X^2(x,\beta_o) \geq$ or $<$ $\chi^2_{1-\alpha}(p)$. It is easily seen that under H_n in (4.1), this statistic has asymptotically a noncentral chi-square distribution with p degrees of freedom and the noncentrality parameter

$$\Delta_{\chi^2} = \frac{1}{2\sigma^2}\tau' C_X \tau. \tag{5.1}$$

If under the same sequence of H_n, the two test statistics have non-central chi-square limit distributions, with the same degrees of freedom, it is known that their ARE is given by the ratio of their non-centrality parameters.

Therefore, the ARE of the $\tilde{Q}$-tests with respect to the χ^2-test is equal to

$$e(\tilde{Q},\chi^2) = \frac{\Delta_{\tilde{Q}}}{\Delta_{\chi^2}}$$

where Δ_{χ^2} and $\Delta_{\tilde{Q}}$ are given by (4.5) and (5.1) respectively.

Under the conditions specified in Theorem 4.3, the ARE thus reduces to

$$e(\tilde{Q},\chi^2) = \sigma^2 \frac{\left[\int_o^1 \phi(w)\psi(w)dw\right]^2}{\int_o^1 \phi^2(w)dw},$$

and this implies that the ARE depends only on the unknown distribution function $G(x)$ of errors, through the score function ϕ.

Various interesting results allied to the expression of the ARE are given below for the specific cases : First, let $\phi(u)=u$ on (0,1) (Wilcoxon score). Then, in this case

$$e(\tilde{Q},\chi^2) = 12\sigma^2\left[\int_{-\infty}^{\infty} g^2(x)dx\right]^2.$$

It is known [Hodges and Lehmann (1956)] that $e(\tilde{Q},\chi^2) \geq 0.864$ for all continuous G. Some particular values are $e(\tilde{Q},\chi^2)=3/\pi=0.955$ when g is a normal density, $e(\tilde{Q},\chi^2)=1$ for the case of a uniform, and $e(\tilde{Q},\chi^2)=81/64$ when $g(x)=x^2e^{-x}/\Gamma(3)$ for $x\geq 0$. It is also known that $e(\tilde{Q},\chi^2)$ exceeds one for distributions G with heavy tails (e.g., Cauchy, double-exponential, logistic distributions etc.). Second, let $\phi(u)=\Phi^{-1}(u)$ on (0,1) (Normal score), where Φ is the standard normal distribution function having the density Φ',

$$e(\tilde{Q},\chi^2)=\sigma^2\left\{\int_{-\infty}^{+\infty} \frac{g^2(x)}{\Phi'(\Phi^{-1}[G(x)])}dx\right\}^2.$$

It has been shown [Chernoff and Savage (1958)] that $e(\tilde{Q},\chi^2)\geq 1$ for all G. Mikulski (1963) has shown that $e(\tilde{Q},\chi^2)=1$ only if G is normal. Finally let $\phi(u)=1$ on (0,1) (Sign score), then $e(\tilde{Q},\chi^2) = 4\sigma^2 g^2(0)$.

Thus from the ARE point of view the $\tilde{Q}$ -test using Wilcoxon or normal scores appears to be superior compared with the classical χ^2-test in many situations. In particular, the $\tilde{Q}$-test using Wilcoxon scores is preferable when the tails of the error distribution are heavy.

Appendix

Proof of Lemma 2.2. Let $\lambda = (\lambda_1,...,\lambda_p)'$ be any arbitrary nonzero vector. Define a statistic which is slight diffierent from $S^*(x^*,\beta)$ as

$$S^{(o)}(x^*,\beta) = (S_1^{(o)}(x^*,\beta),...,S_p^{(o)}(x^*,\beta)\ )' \qquad (A.1)$$

where

$$S_u^{(o)}(x^*,\beta) = \frac{1}{n}\sum_{t=1}^{n} X_{t-u}\ Sgn(D_t^*(\beta))\ \phi[G^*(|D_t^*(\beta)|)\ ], \qquad u = 1,...,p,$$

and G^* is the distribution function of $|D_t^*(\beta)|$. For simplicity, we write $S^* = S^*(x^*,\beta)$, $S^{(o)} = S^{(o)}(x^*,\beta)$. We first prove that $\sqrt{n}\lambda' S^*$ and $\sqrt{n}\lambda' S^{(o)}$ are asymptotic equivalent random variables, and then $\sqrt{n}\lambda' S^{(o)}$ converges in law to a normal distribution with mean zero and variance $\lambda' C_X \lambda \int_o^1 \phi^2(w)dw$. Note that the vectors $(R_1^*,...,R_n^*)$, $(|D_1^*(\beta)|,...,|D_n^*(\beta)|)$ and $(Sgn(D^{*1}(\beta)),..., Sgn(D_n^*(\beta)))$ are mutually independent and $E_\beta\, Sgn(D_t^*(\beta)) = 0$ for all t. Hence we obtain

$$E_\beta \sqrt{n}\lambda'\ (S^* - S^{(o)})^2 = Var_\beta \sqrt{n}\lambda'\ (S^* - S^{(o)})$$

$$= \frac{\lambda' C_X \lambda}{n} \sum_{t=1}^{n} E_\beta \left\{\phi\left[\frac{R_t^*(\beta)}{n+1}\right] - \phi[G^*(|D_t^*(\beta)|)]\right\}^2 .$$

Note that

$$E_\beta \left\{\phi\left[\frac{R_t^*(\beta)}{n+1}\right] - \phi[G^*(|D_t^*(\beta)|)\ ]\right\}^2 = E_\beta \left\{\phi\left[\frac{R_1^{**}}{n+1}\right] - \phi(U_1)\right\}^2$$

which converges to zero, where $U_t = G^*(|D_t^*(\beta)|)$ are independent niform (0,1) random variables and R_t^{**} are the ranks of U_t among U_t, $1 \le t \le n$. The above result yields the first part of the proof.

For the asymptotic normality of $\sqrt{n}\lambda' S^{(o)}$, note that $\sqrt{n}\lambda' S^{(o)} = \sum_{t=1}^{n} Z_t^{(o)}$

where

$$Z_t^{(o)} = \frac{1}{\sqrt{n}}[\lambda' X_{(t-1,t-p)}]\, Sgn(D_t^*(\beta))\ \phi(U_t),$$

where $X_{(t-1,t-p)} = (X_{t-1},\ldots,X_{t-p})'$. Then, it is easily shown that for arbitrary λ, $\frac{1}{\sqrt{n}}\lambda' X_{(t-1,t-p)}$, hence $Z_t^{(o)}$ are asymptotically independent, so that Lindeberg's central limit theorem can be adapted to prove $\sqrt{n}\lambda' S^{(o)}$ asymptotically normal, with $E_\beta Z_t^{(o)}=0$ and $\sigma_t^2 = Var_\beta Z_t^{(o)} = \frac{1}{n}\lambda' C_X \lambda \int_0^1 \phi^{2(w)} dw$. Now, let $s_n^2 = \sum_{t=1}^n \sigma_t^2$. Then for arbitrary $\varepsilon > 0$

$$\sum_{t=1}^n \int_{[|z|>\varepsilon s_n]} z^2 dF_t \le \frac{1}{n}\sum_{t=1}^n \int_\Omega [\lambda' x_{(t-1,t-p)}]^2\, dF_x \int_{[|z|>\varepsilon s_n]} \phi^2(w)dw$$

$$= \lambda' C_X \lambda \int_{[|z|>\varepsilon s_n]} \phi^2(w)dw$$

where F_t and F_x are the distribution functions of $Z_t^{(o)}$ and $X_{(t-1,t-p)}$ respectively. Thus,

$$\frac{1}{s_n^2}\sum_{t=1}^n \int_{[|z|>\varepsilon s_n]} z^2 dF_t \le \left[\int_o^1 \phi^2(w)dw\right]^{-1} \int_D \phi^2(w)dw$$

where

$$D = \left[w : \frac{1}{\sqrt{n}}|\lambda' x_{(t-1,t-p)}\ \phi(w)| > \varepsilon s_n\right].$$

The second term on the right-hand side of the above inequality is less than or equal to

$$\int_{[w\ :\ |\phi(w)|\ >\ \varepsilon\delta_n]} \phi^2(w)dw$$

where

$$\delta_n^2 = \left[\int_o^1 \phi^2(w)dw\right] \frac{n\,\lambda' C_X \lambda}{\max_{1\le t\le n} \lambda' x_{(t-1,t-p)}\, x_{(t-1,t-p)}'\ \lambda}.$$

Since $\delta_n \to \infty$ as $n \to \infty$, the Lindeberg condition thus holds.

Furthermore, obviously $E_\beta[\sqrt{n}\ \lambda' S^{(o)}] = 0$, and

$$Var_\beta[\sqrt{n}\ \lambda' S^{(o)}] = s_n^2 = \sum_{t=1}^n \sigma_t^2 = \lambda' C_X \lambda \int_o^1 \phi^2(w)w dw$$

as $n \to \infty$. This completes the proof.

The following lemma is needed to prove Lemma 4.1.

Let E_o and Var_o denote the expected value and variance under P_o respectively.

Lemma A. Suppose g(x) satisfies condition (1.2). Then under P_o

$$\lim_{n\to\infty} \max_{1\le t\le n} P\left\{ \left| \frac{g(X_t-\beta_n' X_{(t-1,t-p)})}{g(X_t-\beta_o' X_{(t-1,t-p)})} - 1 \right| > \varepsilon \right\} = 0$$

for every $\varepsilon > 0$.

Proof. The proof easily follows from the fact that for every t

$$P_o\left\{ \left| \frac{g(X_t-\beta_n' X_{(t-1,t-p)})}{g(X_t-\beta_o' X_{(t-1,t-p)})} - 1 \right| > \varepsilon \right\} \le \frac{1}{\varepsilon} E_o \left| \frac{g(X_t-\beta_n' X_{(t-1,t-p)})}{g(X_t-\beta_o' X_{(t-1,t-p)})} - 1 \right|$$

$$\le \frac{1}{\varepsilon} |(\beta_o-\beta_n)' x_{(t-1,t-p)}|$$

$$\times \int_{-\infty}^{\infty} \left| \frac{g(x_t-\beta_n' x_{(t-1,t-p)}) - g(x_t-\beta_o' x_{(t-1,t-p)})}{(\beta_o-\beta_n)' x_{(t-1,\, t-p)}} \right| d(x_t-\beta_o' x_{(t-1,t-p)})$$

which converges to

$$\left[\lim_{n\to\infty} \frac{1}{\sqrt{n}} \right] \frac{1}{\varepsilon} |\tau' x_{(t-1,t-p)}| \int_{-\infty}^{\infty} |g'(x)| dx = 0,$$

by the condition (1.2).

Proof of Lemma 4.1. It suffices to prove, by virtue of Lecam's second lemma and Lemma A, that under P_o, W_n is asymptotically $N(-\frac{1}{4}\sigma^2, \sigma^2)$. Now, define

$$V_n^* = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} ((\beta_o-\beta_n)' X_{(t-1,t-p)}) \, Sgn(D_t^*(\beta_o)) \, \psi[G^*(|D_t^*(\beta_o)|)] \qquad (A.2)$$

where G^* is the distribution function of $|D_t^*(\beta)|$. First we shall prove that $\sqrt{n}V_n^*$ has asymptotically $N(0, \sigma^2)$. Next, we will show that $\sqrt{n}V_n^*$ and $W_n - E_oW_n$ have the same limiting distribution. Finally we will show that

$$\lim_{n\to\infty} E_oW_n = -\frac{\sigma^2}{4} \qquad (A.3)$$

Denoting $U_t = G^*(|D_t^*(\beta_o)|)$, $1 \le t \le n$, $\sqrt{n}\, V_n^*$ can be rewritten as $\sqrt{n}V_n^* = \sum_{t=1}^{n} Z_t$, where

$$Z_t = \frac{1}{\sqrt{n}}(\tau' X_{(t-1,t-p)})\, Sgn(D_t^*(\beta_o))\, \psi(U_t). \tag{A.4}$$

From the fact that U_t are (i.i.d.) uniform (0,1) random variables, we obtain $E_n Z_t = 0$ and $\sigma_t^2 = Var_o Z_t = \frac{1}{n}\tau' C_X \tau \int_o^1 \psi^2(w)dw$. As the proof of the Lindeberg condition for the asymptotic normalty of $\sqrt{n}V_n^*$ runs parallel to that of $\sqrt{n}\lambda' S^{(o)}$ in Lemma 2.2, we shall be omit.

For the second part of the proof, note that

$$Sgn(D_t^*(\beta_o))\, \phi[G^*(|D_t^*(\beta_o)|)] = Sgn(D_t^*(\beta_o)) \frac{g^{*'}(|D_t^*(\beta_o)|)}{g^*(|D_t^*(\beta_o)|)} = \frac{g'(D_t^*(\beta_o))}{g(D_t^*(\beta_o))} \tag{A.5}$$

where $g^*(x) = G^{*'}(x)$. Now, let $h(x) = [g(x)]^{\frac{1}{2}}$ so that $h'/h = g'/2g$ Using (4.3), (A.2) and (A.5), it follows that, $Var_o(W_n - E_o W_n - \sqrt{n}V_n^*)$ is less than or equal to

$$\frac{4}{n}\tau' C_X \tau \sum_{t=1}^{n} \int_{-\infty}^{+\infty} \left[\frac{h(d_t(\beta_n)) - h(d_t(\beta_o))}{d_t(\beta_n) - d_t(\beta_o)} - h'(d_t(\beta_o)) \right]^2 dd_t(\beta_o),$$

which converges to zero.

Finally, to prove (A.3) first we note that

$$2E_o\left[\frac{h(D_t^*(\beta_n))}{h(D_t^*(\beta_o))} - 1\right] = -E_o\left[\frac{h(D_t^*(\beta_n))}{h(D_t^*(\beta_o))} - 1\right]^2. \tag{A.6}$$

Using (A.6), it follows that

$$E_o W_n = -\frac{1}{n}\sum_{t=1}^{n} E_o\left[(\tau' X_{(t-1,t-p)}) \frac{h(D_t^*(\beta_n)) - h(D_t^*(\beta_o))}{(D_t^*(\beta_n) - D_t^*(\beta_o)) h(D_t^*(\beta_o))}\right]^2$$

which converges to

$$-\tau' C_X \tau \int_{-\infty}^{+\infty} h'^2(x)dx = -\frac{1}{4}\tau' C_X \tau \int_{-\infty}^{+\infty}\left\{\frac{g'(x)}{g(x)}\right\}^2 dG(x)$$

$$= -\frac{1}{4}\tau' C_X \tau \int_o^1 \psi^2(w)dw.$$

The last equality follows by taking squares and expectations on both sides of (A.5). The lemma follows.

The following technical lemma is needed to prove Theorem 4.3.

Lemma B. Under the same conditions of Theorem 4.3, and under P_o, $(\sqrt{n}V_n^* - \frac{1}{2}\sigma^2, \sqrt{n}\lambda' S^{(o)}(x^*, \beta_o)$ where V_n^*, $S^{(o)}(x^*, \beta_o)$ and σ^2 are defined in (A.1), (A.2) and (4.4) respectively, has asymptotically a bivariate normal distribution with mean $\bar{\mu} = (-\frac{1}{2}\sigma^2, 0)$ and variance-covariance matrix

$$\Sigma = \begin{pmatrix} \sigma^2 & \lambda'\mu \\ \lambda'\mu & \lambda' C_X \lambda \int_o^1 \phi^2(w)dw \end{pmatrix}.$$

Proof. We first consider $\sqrt{n}(V_n^*, \lambda' S^{(o)}(x^*, \beta_o))$. Under P_o, $\sqrt{n}\ (V_n^*, \lambda' S^{(o)}(x^*, \beta_o)\ )$ become

$$\frac{1}{\sqrt{n}} Sgn(D_t^*(\beta_o)) \left[\sum_{t=1}^{n} (\tau' X_{(t-1,t-p)}) \psi(U_t),\ \sum_{t=1}^{n} (\lambda' X_{(t-1,t-p)}) \phi(U_t) \right]$$

where U_t are independent uniform (0,1) random variables, so that

$$E_o[\sqrt{n}V_n^*] = E_o[\sqrt{n}\lambda' S^{(o)}(x^*, \beta_o)] = 0.$$

Moreover,

$$s_{11}(n) = Var_o[\sqrt{n}V_n^*] = \tau' C_X \tau \int_o^1 \psi^2(w)dw,$$

$$s_{22}(n) = Var_o[\sqrt{n}\lambda' S^{(o)}(x^*, \beta_o)] = \lambda' C_X \lambda \int_o^1 \phi^2(w)dw,$$

and

$$s_{12}(n) = s_{21}(n) = Cov_o \sqrt{n}\ [V_n^*, \lambda' S^{(o)}(x^* \cdot \beta_o)] = \lambda' C_X \tau \int_o^1 \psi(w)\phi(w)dw,$$

which are finite. From the above results, we $\bar{\mu} = (-\frac{1}{2}\sigma^2, 0)$, $\sigma_{11} = s_{11}(n) = \sigma^2$, $\sigma_{12} = \sigma_{21} = s_{21}(n) = \lambda'\mu$ and $\sigma_{22} = s_{22}(n) = \lambda' C_X \lambda \int_o^1 \phi^2(w)dw$. Now, we have to show that under P_o, $\sqrt{n}\ (V_n^*, \lambda' S^{(o)}(x^*, \beta_o))$ has a limiting bivariate normal distribution. For any reals l_1 and l_2, let

$$J_n = l_1 (\sqrt{n}V_n^*) + l_2 (\sqrt{n}\lambda' S^{(o)}(x^*, \beta_o)).$$

Then $J_n = \sum_{t=1}^{n} T_t$ where

$$T_t = \frac{1}{\sqrt{n}} Sgn(D_t^*(\beta_o)) \left[l_1(\tau' X_{(t-1,t-p)})\psi(U_t) + l_2(\lambda' X_{(t-1,t-p)})\phi(U_t) \right]$$

which are asymptotic independent, $E_o T_t = 0$ and $\sigma_t^2 = Var_o T_t$ is equal to

$$\frac{l_1^2}{n}\tau' C_X \tau \int_o^1 \psi^2(w)\,dw + \frac{l_2^2}{n}\lambda' C_X \lambda \int_o^1 \phi(w)dw + 2\frac{l_1 l_2}{n}\tau' C_X \lambda \int_o^1 \psi(w)\phi(w)dw.$$

It remains to show that the Lindeberg condition holds : for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{t=1}^{n} \int_{[|\eta| > \varepsilon s_n]} \eta^2 dH_t = 0 \qquad (A.7)$$

where $s_n^2 = \sum_{t=1}^{n} \sigma_t^2$ and H_t is the distribution function of T_t . However, the proof of (A.7) is quite analogous to that of $\sqrt{n}\lambda' S^{(o)}$ in Lemma 2.2, we shall be omit. The proof is complete.

References

[1] Mann, H.B. and Wald, A. (1943). On the statistical treatment of linear stochastic difference equations, *Econometrica,* Vol. 11, 173-220.

[2] Chernoff, H. and Savage, I.R. (1958). Asymptotic normality and efficiency of certain non-parametric test statistics, *Annals of Mathematical Statistics,* Vol. 29, 972-994.

[3] Hodges, J.L. and Lehmann, E.L. (1956). The efficiency of some nonparametric competitions of the t-test, *Annals of Mathematical Staisitics,* Vol. 27, 324-335.

[4] Mikulski, P.W. (1963). On the efficiency of optimal nonparametric procedures in the two sample cases, *Annals of Mathematical Statistics,* Vol. 34, 22-32