한국통계학회논문집 제2권 2호
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Properties of the Poisson-power Function Distribution[1)]

Joo-Hwan Kim[2)]

Abstract

When a neutral particle beam(NPB) aimed at the object and receive a small number of neutron signals at the detector without any errors, it obeys Poisson law. Under the two assumptions that neutral particle scattering distribution and aiming errors have a circular Gaussian distribution respectively, an exact probability distribution of neutral particles becomes a Poisson-power function distribution. We study and prove some properties, such as limiting distribution, unimodality, stochastical ordering, computational recursion formula, of this distribution. We also prove monotone likelihood ratio(MLR) property of this distribution. Its MLR property can be used to find a criteria for the hypothesis testing problem.

1. Introduction

A beam of neutral particles can be used to estimate the density or mass of an object (Feller (1970)). A method of discrimination proposed here is to use a neutral particle beam aimed at the object, and a small number of neutron signals are counted at the detector. Beyer and Qualls (1987) showed that the return neutron particles from an object interrogation for a given dwell time obeys Poisson statistics.

The mean neutron signal λ for the Poisson distribution parameter is computed by the *bistatic radar formula*:

$$\lambda = [I\tau] \cdot \left[\frac{A_t}{\pi(R\sqrt{2}\sigma_1)^2} \right] \cdot K(E,\theta) \cdot \left[\frac{A\varepsilon}{4\pi r^2} \right] \tag{1.1}$$

where I is the probe current in amperes divided by 1.602×10^{-19} coulombs, τ is the dwell time in seconds, A_t is the object area in m^2, R is the probe to object distance in m, $\sqrt{2}\sigma_1$ is the beam half divergence angle, $K(E,\theta)$ is the mean number of neutrons leaked from the object per incident particle and it depends on the mass of the object, E is the probe particle energy in election volts, θ is the scattering angle, A is the detector area in m^2, ε is the detector efficiency, and r is the object to detector distance in m. A detailed description of

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2) Department of Computer Science & Statistics, Dongguk University, Kyong-Ju, 780-714, KOREA.

this formula is given in the report of the American Physical Society report (1987).

Let

$$S = \frac{A\varepsilon\tau I}{(Rr\sigma_1)^2} \quad \text{and} \quad f(E,\theta) = \frac{K(E,\theta)A_t}{4\pi} \tag{1.2}$$

where $f(E,\theta)$ combines parameters specific to the object design. Note that the mean return signal in (1.1) becomes $\lambda = S \cdot f(E,\theta)/(2\pi)$.

2. Exact Neutron Counts Distribution with Aiming Errors : Poisson-power Function Distribution

From the result of Beyer and Qualls (1987), we assume that the count of return neutron particles from an object interrogation during the given dwell time obeys Poisson distributions. The interrogation requires the true value of the parameters in (1.1) to compute the mean of the Poisson statistics. One source of errors in measurement is aiming errors (or tracking and pointing errors) which is the uncertainty about the location of the axis of the beam relative to the object. Wehner (1987) studied the aiming error distribution of NPB.

In this paper we consider aiming errors of the beam for an object interrogation and make the following two assumptions about aiming errors.

(i) The beam has a circular Gaussian distribution of intensity with standard deviation σ_1. This distribution is on a plane perpendicular to the beam axis.

(ii) Aiming errors yield a circular Gaussian distribution of the beam axis relative to the object center. The standard deviation of the distribution is σ_2.

Beckman and Johnson (1987) give evidence from an experiment that the beam has a Pearson Type VII distribution of intensity instead of a circular Gaussian distribution of intensity in assumption (i). This distribution is much heavier in the tails than is the Gaussian. Kim (1994) compared a circular Gaussian distribution with a Pearson Type VII distribution for scattering distribution of the NPB. Kim (1994) also studied the probability distribution of exactly x neutron particles, $x = 0, 1, 2, \ldots,$ are received by the single detector in presence of aiming errors.

The probability of exactly x neutron particles being counted, under the assumption of a Poisson distribution of counts and aiming errors, is

$$P(x|\lambda) = \frac{1}{x!}\int_{\omega_2=-\infty}^{\infty}\int_{\omega_1=-\infty}^{\infty} e^{-\lambda}\lambda^x e^{-(\omega_1^2+\omega_2^2)/(2\sigma_2^2)} \frac{d\omega_1 d\omega_2}{2\pi\sigma_2^2} \tag{2.1}$$

where λ is defined by

$$\lambda = ke^{-(\omega_1^2+\omega_2^2)/(2\sigma_1^2)} \tag{2.2}$$

where $k = (2\pi)^{-1} Sf(E,\theta)$ and it represents the mean return neutron counts without aiming errors, and S and f are defined in (1.2). In (2.2) σ_1 is a standard deviation of the circular Gaussian intensity distribution of the beam at the object, and (ω_1, ω_2) are coordinates of points on beam cross section. σ_2 in (2.1) is a standard deviation of the circular Gaussian aiming error distribution of the beam relative to the object. We average over the aiming error distribution in (2.1) to modify discrimination for this uncertainty. In repeated sequential interrogation, the probability in (2.1) leads to a reasonable and correct modification. Using the polar coordinates transformation, and letting

$$\ell = \left(\frac{\sigma_1}{\sigma_2}\right)^2 \tag{2.3}$$

we obtain

$$P(x|\lambda) = \frac{\ell}{k^{\ell} x!} \gamma(x+\ell; k), \tag{2.4}$$

where

$$\gamma(\nu; k) = \int_0^k t^{\nu-1} e^{-t} dt$$

is the incomplete gamma function.

We have defined in (2.2) that k be the mean number of return neutron signals counted with the assumption that no aiming errors are made in the measurement of the parameters and that the beam is perfectly centered on the object.

Consider the probability distribution in (2.4) by $P(x;k,\ell)$

$$\begin{aligned} P(x;k,\ell) &= \frac{\ell}{k^{\ell} x!} \int_0^k e^{-\omega} \omega^{x+\ell-1} d\omega \\ &= \frac{1}{x!} \int_0^k e^{-\omega} \omega^x \ell \left(\frac{1}{k}\right)^{\ell} \omega^{\ell-1} d\omega \\ &= \frac{1}{x!} E_{\omega}(e^{-\omega}\omega^x) \end{aligned}$$

where E_{ω} represents expected value of ω, and ω has a probability distribution

$$f(\omega) = \ell k^{-\ell} \omega^{\ell-1}, \qquad \ell > 1,\ 0 \leq \omega \leq k \tag{2.5}$$

The distribution in (2.5) is called the power-function distribution. From the above expression, the distribution we have derived in (2.4) is a special case of a *compound Poisson distribution* where ω has a power-function distribution, and ω is a mean of the Poisson distribution. See Johnson and Kotz (1970) for the definition of compound Poisson distribution. Thus the probability distribution represented by (2.4) may be reasonably called a *Poisson-power function distribution.*

The mean and variance of the Poisson-power function random variable, by an elementary reasoning often used in Bayesian statistics, are

$$E(X) = E_{\omega}(E(X|\omega)) = E_{\omega}(\omega) = k\left(\frac{\ell}{\ell+1}\right)$$

and

$$\begin{aligned} Var(X) &= E_{\omega}[Var(X|\omega)] + Var_{\omega}[E(X|\omega)] \\ &= E_{\omega}(\omega) + Var_{\omega}(\omega) \\ &= k\left(\frac{\ell}{\ell+1}\right) + k^2\left\{\frac{\ell}{\ell+2} - \left(\frac{\ell}{\ell+1}\right)^2\right\}. \end{aligned}$$

The moment generating function is

$$M_X(t) = E(e^{tX}) = E(E(e^{tX}|\omega)) = E_{\omega}(e^{\omega(e^t-1)}), \tag{2.6}$$

which gives

$$M_X(t) = M(\ell,\ \ell+1,\ k(e^t-1)), \tag{2.7}$$

where M is a Kummer's function (see Abramowitz (1964)) and is defined by

$$M(a,b,x) = 1 + \frac{ax}{b} + \frac{(a)_2 x^2}{(b)_2 2!} + \cdots + \frac{(a)_n x^n}{(b)_n n!} + \cdots$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, and $(a)_0 = 1$. From the moment generating function the moments of any order can easily be evaluated by the usual differentiation procedure.

3. Some Properties of the Distribution

Some properties of the Poisson-power function distribution which can be applied to the distribution of the neutral particles with aiming errors follow.

Property 1. The Poisson-power function distribution $P(x;k,\ell)$ converges to Poisson probability distribution $P(x;k)$ as $\ell \to \infty$.

Proof. We use the continuity theorem. Now from (2.6) the moment generating function of Poisson-power function random variable X is

$$M_X(t) = E_{\omega}(e^{\omega(e^t-1)})$$

where ω has a power-function distribution. Since ω has a degenerate probability distribution at $\omega = k$ for $\ell \to \infty$, and by the Helly theorem (see Lamperti (1966) p. 51),

$$\lim_{\ell \to \infty} \int_0^k e^{\omega(e^t-1)} \ell\, k^{-\ell}\, \omega^{\ell-1} d\omega = e^{k(e^t-1)}$$

Therefore

$$\lim_{\ell \to \infty} M_X(t) = e^{k(e^t-1)} \tag{3.1}$$

for real values t. The right-hand side of (3.1) is the moment generating function of the Poisson random variable with mean k. That is, the Poisson-power function distribution has a limiting Poisson distribution as $\ell \to \infty$. ▣

Property 2. The Poisson-power function distribution has the recursion formula:

$$P(x;k,\ell) = \frac{x+\ell}{x+1} P(x;k,\ell) - \frac{1}{x+1} \frac{k^x e^{-k}}{x!} \tag{3.2}$$

for integer $x \geq 0$.

Proof. The integration by parts for the incomplete gamma function:

$$\gamma(x+\ell+1;k) = (x+\ell)\gamma(x+\ell;k) - k^{x+\ell} e^{-k} \tag{3.3}$$

yields (3.2). ▣

Property 3. A probability distribution P_n, n is an integer, is called *unimodal* about a mode M if $P_n \geq P_{n-1}$ for $n \leq M$ and $P_n \leq P_{n-1}$ for $n \geq M+1$. The Poisson-power function distribution is a unimodal probability distribution.

Proof. The Poisson-power function distribution is

$$P(x;k,\ell) = \frac{1}{x!} \int_0^k e^{-\omega} \omega^x f(\omega) d\omega \qquad x = 0,1,2,\cdots \tag{3.4}$$

where $f(\omega)$ appears in (2.5). Note that $f(\omega)$ is differentiable on $(0,k)$. Then integration by parts in (3.4) yields

$$P(x;k,\ell) = P(x-1;k,\ell) - \frac{\ell}{x!} e^{-k} k^{x-1} + \frac{1}{x!} \int_0^k e^{-\omega} \omega^x f(\omega)\ d\omega. \tag{3.5}$$

where x is an integer and $x \geq 1$. Write

$$\Delta P(x;k,\ell) = P(x+1;k,\ell) - P(x;k,\ell).$$

Then (3.5) may be written as

$$\Delta P(x-1;k,\ell) = \frac{1}{x!} \int_0^k e^{-\omega} \omega^x f(\omega)\ d\omega - \frac{\ell}{x!} e^{-k} k^{x-1}. \tag{3.6}$$

The integrand in the first term on the right side of (3.6) is nonnegative, and the second term with negative sign is negative. Now

$$\begin{aligned}\Delta P(x;k,\ell) &= \frac{1}{(x+1)!}\int_0^k \omega^{x+1}e^{-\omega}f(\omega)\,d\omega - \frac{\ell}{(x+1)!}e^{-k}k^x \\ &= \frac{k^{x+1}}{x+1}\left\{\frac{1}{x!}\int_0^k e^{-\omega}\left(\frac{\omega}{k}\right)^{x+1}f(\omega)\,d\omega - \frac{\ell}{x!}e^{-k}\frac{1}{k}\right\} \\ &\le \frac{k^{x+1}}{x+1}\left\{\frac{1}{x!}\int_0^k e^{(-\omega)}\left(\frac{\omega}{k}\right)^{x}f(\omega)\,d\omega - \frac{\ell}{x!}e^{(-k)}\frac{1}{k}\right\} \\ &= \frac{k}{x+1}\left\{\frac{1}{x!}\int_0^k e^{-\omega}\omega^x f(\omega)\,d\omega - \frac{\ell}{x!}e^{-k}k^{x-1}\right\} \\ &= \frac{k}{x+1}\Delta P(x-1;k,\ell).\end{aligned}$$

Thus

$$(x+1)\Delta P(x;k,\ell) \le k\Delta P(x-1;k,\ell).$$

It follows that $\Delta P(x;k,\ell) \ge 0$ implies $\Delta P(x-1;k,\ell) \ge 0$, and while $\Delta P(x-1;k,\ell) \le 0$ implies $\Delta P(x;k,\ell) \le 0$. Hence the Poisson-power function distribution is a unimodal distribution. ▣

Property 4. The Poisson-power function distribution is stochastically ordered in k, in fact, stochastically decreasing.

Proof. We know that Poisson distribution has a stochastical ordering property. That is

$$\sum_{x=0}^{c}\frac{e^{-k}k^x}{x!} = P[X\le c] = P[\chi_{2c}^2 > k]$$

where X is a Poisson random variable with mean k and χ_{2c}^2 is a chi-square random variable with degrees of freedom $2c$. The cumulative probability is decreasing as k increases. This implies, for $d<t$

$$\sum_{x=0}^{c}\frac{e^{-d}d^x}{x!} > \sum_{x=0}^{c}\frac{e^{-t}t^x}{x!}, \qquad \text{for all } c.$$

Then, for $d<t$, by interchanging sum and integral for the Poisson-power function distribution

$$\begin{aligned}\sum_{x=0}^{c}P(x;d,\ell) &= \int_0^1\sum_{x=0}^{c}\frac{e^{-dz}(dz)^x}{x!}f(z)dz \\ &> \int_0^1\sum_{x=0}^{c}\frac{e^{-tz}(tz)^x}{x!}f(z)dz \\ &= \sum_{x=0}^{c}P(x;t,\ell) \qquad \text{for all } c.\end{aligned}$$

$f(z)$ is defined in (2.5) except for $k=1$. ▣

Property 5. A useful computation formula for the Poisson-power function cumulative distribution function (cdf) is

$$\sum_{x=0}^{c} P(x;k,\ell) = 1 - \frac{\gamma(c+1;k)}{\Gamma(c+1)} + \frac{k^{-\ell}\gamma(c+\ell+1;k)}{\Gamma(c+1)}. \tag{3.7}$$

Proof. For any integer $c \ge 0$, we obtain the formula (3.8) by the successive difference of (3.7)

$$P(c;k,\ell) = \frac{k^{c}e^{-k}}{c!} + \frac{k^{-\ell}}{\Gamma(c+1)}\{\gamma(c+\ell+1;k) - c\gamma(c+\ell;k)\}. \tag{3.8}$$

which is a Poisson-power function distribution. Note that the first two terms of the right hand side of (3.7) is the Poisson cdf, which explains the first term of the right hand side of (3.8). Now the integration by parts formula (3.3) applied to $\gamma(c+\ell+1;k)$ verifies (3.8), which in turn verifies (3.7). ▣

Now we compare the Poisson-power function distribution $P(x;k,\ell)$ with the Poisson distribution $P(x;k)$.

Property 6. $P(0;k,\ell) \ge P(0;k) = e^{-k}$.

Proof. $P(0;k,\ell) = \int_0^k e^{-\omega} f(\omega)d\omega \ge e^{-k}\int_0^k f(\omega)d\omega \ge e^{-k} = P(0;k).$ ▣

Property 7. $\frac{P(1;k,\ell)}{P(0;k,\ell)} \le \frac{P(1;k)}{P(0;k)} = k.$

Proof. $kP(0;k,\ell) - P(1;k,\ell) = k\int_0^k e^{-\omega} f(\omega)d\omega - \int_0^k e^{-\omega}\omega f(\omega)d\omega$

$$= \int_0^k e^{-\omega}(k-\omega)f(\omega)d\omega \ge 0.$$ ▣

Property 8. $\frac{P(x+1;k,\ell)}{P(x;k,\ell)} \le \frac{P(x+1;k)}{P(x;k)} = \frac{k}{x+1}.$

Proof.

$$\frac{k}{x+1}P(x;k,\ell) - P(x+1;k,\ell)$$

$$= \frac{k}{x+1}\int_0^k \frac{e^{-\omega}\omega^{x}}{x!} f(\omega)d\omega - \int_0^k \frac{e^{-\omega}\omega^{x+1}}{(x+1)!} f(\omega)d\omega$$

$$= \int_0^k \frac{e^{-\omega}\omega^{x}}{(x+1)!}(k-\omega)f(\omega)d\omega \ge 0.$$ ▣

Note that Properties 6 and 7 show that, when the Poisson-power function law applies instead of the Poisson law, there will be too many cases with "no counts signal" and, as compared with the "no counts signal" case, too few with "one count signal."

4. Monotone Likelihood Ratio Property

In this section we will show that the Poisson-power distribution has a monotone likelihood ratio property. First we ne[illegible] lemmas that are properties of the incomplete gamma function.

Lemma 4.1. Let γ be the incomplete gamma function defined by

$$\gamma(\nu;k) = \int_0^k t^{\nu-1} e^{-t} dt \quad .$$

Then $\gamma(\nu+a+b;t)\,\gamma(\nu;t) - \gamma(\nu+a;t)\,\gamma(\nu+b;t) > 0$ for all $\nu>0$, $t>0$, $a>0$, and $b>0$.

Proof. Writing this quantity as an integral, we have

$$\int_0^t \int_0^t (x^{\nu+a+b-1} y^{\nu-1} - x^{\nu+a-1} y^{\nu+b-1}) e^{-x} e^{-y} dxdy$$

$$= \int_0^t \int_0^t x^a (x^b - y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy$$

$$= \int\int_{(0<y<x<t)} x^a (x^b - y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy + \int\int_{(0<x<y<t)} x^a (x^b - y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy$$

$$= \int\int_{(0<y<x<t)} x^a (x^b - y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy - \int\int_{(0<y<x<t)} y^a (x^b - y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy$$

$$= \int\int_{(0<y<x<t)} (x^a - y^a)\,(x^b - y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy > 0. \quad ∎$$

Lemma 4.2. The function $f(t;\nu) = \nu e^t t^{-\nu} \gamma(\nu;t)$ is monotone decreasing from e^t to 1 as $0 \le \nu \to \infty$ and monotone increasing from 1 to ∞ as $0 \le t \to \infty$.

Proof. First write, by a change of variable, that

$$f(t;\nu) = \int_0^1 \nu z^{\nu-1} e^{t(1-z)} dz. \tag{4.1}$$

From (4.1) it is clear that f is increasing in t and that $\lim_{t\to 0} f(t;\nu) = 1$. Also one finds by computation that

$$\lim_{t\to\infty} f(t;\nu) = \nu \cdot \Gamma(\nu) \lim_{t\to\infty} t^{-\nu} e^t = \infty.$$

Second, by an integration by parts in (4.1), one obtains

$$f(t;\nu) = 1 + t \int_0^1 z^\nu e^{t(1-z)} dz. \tag{4.2}$$

Because z in (4.2) is less than 1, it is clear that f is decreasing in ν, and by the dominated convergence theorem, that the limit is 1 as $\nu\to\infty$ and is e^t as $\nu\to0$. ▣

Lemma 4.3. The function $g(t,\nu) = \gamma(\nu+1;t)/\gamma(\nu;t)$ is monotone increasing from 0 to t as $0\le\nu\to\infty$, and monotone increasing from 0 to ν as $0\le t\to\infty$.

Proof. First to show the monotone property of g in ν, consider the ratio

$$\frac{g(t,\nu+\varepsilon)}{g(t,\nu)} = \frac{\gamma(\nu+\varepsilon+1;t)}{\gamma(\nu+\varepsilon;t)}\,\frac{\gamma(\nu;t)}{\gamma(\nu+1;t)}\ .$$

The ratio is greater than 1 by Lemma 4.1. So g is monotone decreasing in ν. Now compute

$$\lim_{\nu\to\infty} g(t,\nu) = \lim_{\nu\to\infty}\frac{\frac{1}{\nu+1}\,e^{-t}\,t^{\nu+1}\,f(t,\nu+1)}{\frac{1}{\nu}\,e^{-t}\,t^{\nu}\,f(t,\nu)}$$

where function f is defined in Lemma 4.2. Then by Lemma 4.2,

$$\lim_{\nu\to\infty} g(t,\nu) = \lim_{\nu\to\infty}\left(\frac{\nu}{\nu+1}\right) t\left(\frac{1+t\int_0^1 z^{\nu+1}\,e^{t(1-z)}dz}{1+t\int_0^1 z^{\nu}\,e^{t(1-z)}dz}\right) = t,$$

and the limit as $\nu\to0$ is 0. Second, to show the monotone property of g in t, consider the derivative of g with respect to t

$$g'(t,\nu) = \frac{t^{\nu}e^{-t}\gamma(\nu;t) - \gamma(\nu+1;t)\,t^{\nu-1}e^{t}}{\gamma^2(\nu;t)}\ .$$

Then

$$\frac{\gamma^2(\nu;t)\,g'(t,\nu)}{t^{\nu-1}e^{-t}} = t\gamma(\nu;t) - \gamma(\nu+1;t) = \int_0^t (t-y)\,y^{\nu-1}\,e^{-y}dy > 0\ .$$

Thus g is monotone increasing in t. Now

$$\lim_{t\to\infty} g(t,\nu) = \frac{\Gamma(\nu+1)}{\Gamma(\nu)} = \frac{\nu\Gamma(\nu)}{\Gamma(\nu)} = \nu$$

and

$$\lim_{t\to0} g(t,\nu) = \lim_{t\to0}\frac{\int_0^t e^{-x}\,x^{\nu}dx}{\int_0^t e^{-x}\,x^{\nu-1}dx} = \lim_{t\to0}\frac{t^{\nu+1}\int_0^1 e^{-tz}\,z^{\nu}dz}{t^{z}\int_0^1 e^{-tz}\,z^{\nu}dz} = 0\ . \qquad ▣$$

Theorem 4.1. The random variable X of the Poisson-power function distribution in (2.4) has a monotone likelihood ratio; and the Neyman-Pearson test rule for the hypotheses of $H_0 : k = t$ vs. $H_1 : k = d$, when $d < t$, is that reject H_0 if $X \leq c$ which is a left-tail test.

Proof. From (2.4) the likelihood ratio of X for the Poisson-power function distribution is

$$L(x) = \frac{P(x;d,\ell)}{P(x;t,\ell)} = \left(\frac{d}{t}\right)^{-\ell} \frac{\gamma(x+\ell\,;d)}{\gamma(x+\ell\,;t)} .$$

Now by Lemma 4.3, for $d<t$

$$\frac{L(x+1)}{L(x)} = \frac{\gamma(\nu+1;d)}{\gamma(\nu;d)} \Big/ \frac{\gamma(\nu+1;t)}{\gamma(\nu;t)} = \frac{g(d,\nu)}{g(t,\nu)} < 1 .$$

Thus $L(x)$ is monotone decreasing function of x. It implies that the Neyman-Pearson test for an object interrogation with aiming errors is a left-tail test : reject H_0 if $X \leq c$. ▣

References

[1] APS (1987). Report to The American Physical Society of the Study Group on Science and Technology of Directed Energy Weapons, *Reviews of Modern Physics*, Vol. 59.

[2] Abramowitz, M. and Stegun, Irene A. (1964). *Handbook of Mathematical Functions*, p. 504, National Bureau of Standards, Washington, DC.

[3] Beckman, R. J. and Johnson, M. (1987). Fitting the Student-t Distribution to Grouped Data, with Application to a Particle Scattering Experiment, *Technometrics*, Vol. 29, 17-22.

[4] Beyer, W. A. and Qualls, C. R. (1987). Discrimination with Neutral Particle Beams and Neutrons, LA-UR 87-3140, Los Alamos National Laboratory, Los Alamos, New Mexico, U.S.A.

[5] Feller, W. (1970). *An Introduction to Probability Theory and Its Applications: Volume II*, Third Ed., John Wiley & Sons.

[6] Johnson, N. L., and Kotz, S. (1970). *Distributions in Statistics : Continuous Univariate Distributions-I*, John Wiley & Sons.

[7] Kim, J. H. (1994). Monotone Bayes Classification Rule on the Sample Space, *Dongguk Journal : Natural Sciences*, Vol. 33, 345-359.

[8] Kim, J. H. (1994). The Distribution of Neutral Particles with Aiming Errors, *Dongguk Journal*, Vol. 13, 231-248.

[9] Lamperti, J. (1966). *Probability*, The Benjamin & Cummings Publishing Company.

[10] Wehner, T. R. (1987). NPB Aiming Error and Its Effect on Discrimination, *LA-UR 18-20, Los Alamos National Laboratory*, Los Alamos, New Mexico.