

## A Note on Disturbance Variance Estimator in Panel Data with Equicorrelated Error Components

Seuck Heun Song<sup>1)</sup>

### Abstract

The ordinary least square estimator of the disturbance variance in the pooled cross-sectional and time series regression model is shown to be asymptotically unbiased without any restrictions on the regressor matrix when the disturbances follow an equicorrelated error component models.

### 1. Introduction

Let  $y_{it}$  be an observation on the dependent variable for the  $i$ -th cross sectional unit (firms, individuals or countries) for the  $t$ -th time period,  $x_{jit}$  be an observation on the  $j$ -th nonstochastic regressor for the  $i$ th cross sectional unit for the  $t$ th time period. Then the model we are concerned with is

$$y_{it} = \sum_{j=1}^k \beta_j x_{jit} + u_{it}, \quad i = 1, 2, \dots, N \quad \text{and} \quad t = 1, 2, \dots, T. \quad (1.1)$$

The model (1.1) can be written in matrix notation as

$$y = X\beta + u, \quad (1.2)$$

where  $y$  is the  $(NT \times 1)$ -observation vector,  $X$  is  $(NT \times k)$ -design matrix, the  $(k \times 1)$ -vector  $\beta$  contains the unknown regression coefficients to be estimated, and  $u$  is a  $(NT \times 1)$ -vector of disturbances. Both,  $N$  and  $T$  are assumed to be larger than  $k$ .

For the pooled cross-sectional and time series regression models with two-way error component disturbances considered by Wallace and Hsiao (1969), Nerlove (1971), Amemiya (1971), Swamy and Arora (1972), Hsiao (1986) and the recent survey by Baltagi and Raj

---

<sup>1)</sup> Dept. of Statistics, Duksung Women's University, Seoul, 132-174, KOREA

(1992), the disturbances in (1.1) take the following form:

$$u_{it} = \mu_i + \lambda_t + v_{it}, \quad i = 1, 2, \dots, N \text{ and } t = 1, 2, \dots, T, \quad (1.3)$$

where the  $\mu_i$  denote the individual specific effects which are assumed to be i.i.d.  $(0, \sigma_\mu^2)$  and  $\lambda_t$  denote the time-period effects which are i.i.d.  $(0, \sigma_\lambda^2)$ . The  $v_{it}$  are the remainder disturbances which are also assumed to be i.i.d.  $(0, \sigma_v^2)$ . The  $\mu_i$ 's and the  $v_{it}$ 's are independent of each other, i.e.  $\sigma_u^2 = \sigma_\mu^2 + \sigma_\lambda^2 + \sigma_v^2$ .

Under these assumptions, the  $(NT \times NT)$ -disturbance covariance matrix can be written as

$$E(uu') = \sigma_u^2 \Omega = \sigma_\mu^2 (I_N \otimes \iota_T \iota_T') + \sigma_\lambda^2 (\iota_N \iota_N' \otimes I_T) + \sigma_v^2 I_{NT}, \quad (1.4)$$

where  $I_N$  is an  $(N \times N)$ -identity matrix,  $\iota_T$  is a  $(T \times 1)$ -vector of ones and  $\otimes$  denotes the kronecker product.

In this model with disturbances in (1.3), it is well known that the generalized least square (GLS)-based estimator of  $\sigma_u^2$ ,

$$\widehat{S}^2 = \frac{1}{NT-k} (\tilde{u}' \tilde{u}) = \frac{1}{NT-k} (y - X\tilde{\beta})' (y - X\tilde{\beta}), \quad (1.5)$$

where  $\tilde{\beta} = (X' V^{-1} X)^{-1} X' V^{-1} y$  with  $V = \sigma_u^2 \Omega$  is a unbiased estimator. However, in the practice  $V$  is usually unknown, so that  $\widehat{S}^2$  cannot be calculated. Taking the ordinary least square (OLS)-based estimator of  $\sigma_u^2$  instead of (1.5),

$$S^2 = \frac{1}{NT-k} (\hat{u}' \hat{u}) = \frac{1}{NT-k} (y - X\hat{\beta})' (y - X\hat{\beta}), \quad (1.6)$$

where  $\hat{\beta} = (X' X)^{-1} X' y$ , is in general a biased estimator of  $\sigma_u^2$ , when  $\Omega \neq I_{NT}$ . See Moulton (1986), for several examples on the extent of this bias in empirical applications. However, Recently Baltagi and Kraemer (1994) and Song (1994, 1995) have shown the asymptotic unbiasedness of  $S^2$  for the several error component disturbances. In this note, I will show the asymptotic unbiasedness of  $S^2$  if  $u$  follows the two-way error component

model with equicorrelated case (see for several examples, Baltagi (1993)).

## 2. Main Results

We can generalize the two-way error component model to the equicorrelated case. It assumes that the remainder disturbances  $v_{it}$  in (1.3) are generated by a equicorrelated process. In this case the disturbance covariance matrix is given by

$$\sigma_u^2 \Omega_{EQ} = \sigma_u^2 (I_N \otimes \mathbf{1}_T \mathbf{1}_T') + \sigma_k^2 (\mathbf{1}_N \mathbf{1}_N' \otimes I_T) + \sigma_v^2 (I_N \otimes V_1), \quad (2.1)$$

where  $V_1$  is the covariance matrix of order  $T$ ,

$$V_1 = \begin{bmatrix} 1 & \rho & \rho & \cdot & \cdots & \cdot & \rho \\ \rho & 1 & \rho & \cdot & \cdots & \cdot & \rho \\ \rho & \rho & 1 & \cdot & \cdots & \cdot & \rho \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \rho & \rho & \rho & \cdot & \cdots & \cdot & 1 \end{bmatrix}, \quad (2.2)$$

where  $\rho$  is an unknown scalar and  $V_1$  nonsingular, which requires  $\frac{-1}{T-1} < \rho < 1$ .

From Watson (1955), Sathe and Vinod (1974), Neudecker (1977, 1978), Dufour (1986, 1988), Kraemer (1991) and Kiviet and Kraemer (1992), we have the inequalities:

$$\begin{array}{ccc} \text{mean of } NT-k & & \text{mean of } NT-k \\ 0 \leq \text{smallest characteristic} \leq E(S^2/\sigma_u^2) \leq \text{greatest characteristic} \leq NT/(NT-k), \\ \text{roots of } \Omega_{EQ} & & \text{roots of } \Omega_{EQ} \end{array} \quad (2.3)$$

which implies that the upper bound for  $\frac{S^2}{\sigma_u^2}$  tends to one as  $N$  and  $T \rightarrow \infty$ . It remains to

show that the lower bound tends to one as well.

To derive the characteristic roots of  $\sigma_u^2 \Omega_{EQ}$  in (2.1) which are given by the following results:

**Lemma (2.1) :** (Horn and Johnson 1985, p. 181)

Let  $A, B$  be  $(T \times T)$ -Hermitian and let the characteristic roots  $\lambda_i(A)$ ,  $\lambda_i(B)$  and  $\lambda_i(A+B)$  be arranged in decreasing order  $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_T = \lambda_{\min}$ . For each  $i = 1, 2, \dots, T$  we have

$$\lambda_i(A+B) \leq \lambda_i(A) + \lambda_{\max}(B). \quad (2.4)$$

We use this lemma with  $A = \sigma_\mu^2(I_N \otimes \iota_T \iota_{T'}) + \sigma_\lambda^2(\iota_N \iota_{N'} \otimes I_T)$  and  $B = \sigma_v^2(I_N \otimes V_1)$ .

According to Nerlove (1971), first we can derive the characteristic roots and vector of  $\sigma_\mu^2(I_N \otimes \iota_T \iota_{T'}) + \sigma_\lambda^2(\iota_N \iota_{N'} \otimes I_T)$ . These characteristic roots turn out to be 0 with multiplicity  $(N-1)(T-1)$ ,  $T\sigma_\mu^2 + N\sigma_\lambda^2$  with multiplicity 1,  $N\sigma_\lambda^2$  with multiplicity  $(T-1)$  and  $T\sigma_\mu^2$  with multiplicity  $(N-1)$ . It can be shown that the largest characteristic root is

$$\lambda_{\max}[\sigma_\mu^2(I_N \otimes \iota_T \iota_{T'}) + \sigma_\lambda^2(\iota_N \iota_{N'} \otimes I_T)] = T\sigma_\mu^2 + N\sigma_\lambda^2. \quad (2.5)$$

Since the characteristic roots of a kronecker product of matrix are given by the product of the characteristic roots of these matrices (see Horn and Johnson (1990), p. 245), we obtain the characteristic roots of  $\sigma_v^2(I_N \otimes V_1)$  as the products of the characteristic roots of  $V_1$  and the characteristic roots of  $I_N$ . Since the characteristic roots of  $I_N$  are 1's and according to Graybill (1983, p. 122), the characteristic roots of  $V_1$  are given by

$$\begin{aligned} \lambda_1 &= (1 + (T-1)\rho) \\ \lambda_2 &= \lambda_3 = \dots = \lambda_T = (1-\rho), \end{aligned} \quad (2.6)$$

where  $\lambda_1 \geq \lambda_2 = \lambda_3 = \dots = \lambda_T$ . In addition, it can be easily shown that

$$\lambda_{\max}[\sigma_v^2(I_N \otimes V)] = \sigma_v^2[1 + (T-1)\rho]. \quad (2.7)$$

Therefore the characteristic roots of  $\sigma_u^2 \Omega_{EQ}$  in (2.1) can be obtained from the lemma (2.1), and the equations (2.5) and (2.7) :

$$\begin{aligned}
\sigma_u^2 \Omega_{EQ} &= \sigma_u^2 (I_N \otimes \iota_T \iota_T') + \sigma_\lambda^2 (\iota_N \iota_N' \otimes I_T) + \sigma_v^2 (I_N \otimes V_1) \\
&\leq \lambda_i [\sigma_u^2 (I_N \otimes \iota_T \iota_T') + \sigma_\lambda^2 (\iota_N \iota_N' \otimes I_T) + \lambda_{\max} [\sigma_v^2 (I_N \otimes V_1)]] \\
&\leq \sigma_u^2 T + \sigma_\lambda^2 N + \sigma_v^2 [1 + (T-1)\rho]
\end{aligned} \tag{2.8}$$

A lower bound for the mean of the  $NT-k$  smallest characteristic roots of  $\Omega_{EQ}$  may be derived from (2.3) as follows :

$$\begin{aligned}
\frac{1}{NT-k} \sum_{i=1}^{NT-k} \lambda_{i+k} &= \frac{1}{NT-k} \left( \sum_{i=1}^{NT} \lambda_i - \sum_{i=1}^k \lambda_i \right) \\
&= \frac{1}{NT-k} (tr(\Omega_{EQ}) - \sum_{i=1}^k \lambda_i) \\
&\leq \frac{NT}{NT-k} - \frac{k}{NT-k} (T + N + 1 + (T-1)\rho)
\end{aligned} \tag{2.9}$$

from (2.8). Obviously, the first term on the right hand side of (2.9) tends to one and the second term tends to zero as  $N$  and  $T \rightarrow \infty$ . Thus  $S^2$  is asymptotically unbiased for  $\sigma_u^2$ , regardless of the regressor matrix  $X$ . In summary, in this note it can be shown that the OLS-based estimator for disturbance variance,  $S^2$  is asymptotically unbiased for the two-way error component model with equicorrelated time effects irrespective of any restrictions on the regressor matrix  $X$ .

## References

- [1] Amemiya, T. (1971). The estimation of variances in a variance components model, *International Economic Review*, Vol. 12, 1-13.
- [2] Baltagi, B. and Raj, B. (1992). A survey of recent theoretical developments in the econometrics of panel data, *Empirical Economics*, Vol. 17, 85-107.
- [3] Baltagi, B. (1993). Useful matrix transformations for panel data analysis: a survey, *Statistical Papers*, Vol. 34, 281-301.
- [4] Baltagi, B. and Kraemer, W. (1994). Consistency, asymptotic unbiasedness and bounds on the bias of  $s^2$  in the linear regression model with error component disturbances, *Statistical Papers*, Vol. 35, 323-328.
- [5] Dufour, J.M. (1986). Bias of  $s^2$  in linear regression with dependent errors, *The American*

- Statistician*, Vol. 40, 284-285.
- [6] Dufour, J.M. (1988). Estimators of the disturbance variance in econometric models, *Journal of Econometrics*, Vol. 37, 277-292.
  - [7] Graybill, F.A. (1983). *Matrices with Applications in Statistics*, 2nd edn. Wadsworth, Belmont.
  - [8] Horn, R.A. and Johnson, C.R. (1985). *Matrix Algebra*, Cambridge University Press, Cambridge.
  - [9] Horn, R.A. and Johnson, C.R. (1990). *Topics in Matrix Analysis*, Cambridge University Press, Cambridge.
  - [10] Hsiao, C. (1986). *Analysis of Panel Data*, Cambridge University Press, Cambridge.
  - [11] Kiviet, J.F. and Kraemer, W. (1992). Bias of  $s^2$  in the linear regression model with correlated errors, *The Review of Economics and Statistics*, Vol. 74, 362-365.
  - [12] Kraemer, W. (1991). The asymptotic unbiasedness of  $s^2$  in the linear regression model with AR(1)-disturbances, *Statistical Papers*, Vol. 32, 71-72.
  - [13] Moulton, B.R. (1986). Random group effects and the precision of regression estimates, *Journal of Econometrics*, Vol. 32, 385-397.
  - [14] Nerlove, M. (1971). A note on error components model. *Econometrica*, Vol. 47, 383-396.
  - [15] Neudecker, H. (1977). Bounds for the bias of the least squares estimator of  $\sigma^2$  in case of a first-order autoregressive process (positive autocorrelation), *Econometrica*, Vol. 45, 1258-1262.
  - [16] Neudecker, H. (1978). Bounds for the bias of the LS estimator in case of a first-order (positive) autoregressive process where regression contains a constant term, *Econometrica*, Vol. 46, 1223-1226.
  - [17] Song, S.H. (1994). Consistency and asymptotic unbiasedness of  $s^2$  in the linear regression model with serially correlated two-way error components, *Technical Reports Nr. 3*, Department of Statistics, Dortmund University, Germany.
  - [18] Song, S.H. (1995). Consistency and asymptotic unbiasedness of  $s^2$  in the linear regression model with serially correlated error components, (*submitted for publication*).
  - [19] Swamy, P.A. and Arora, S.S. (1972). The exact finite sample properties of the estimators of coefficients in the error components regression models, *Econometrica*, Vol. 40, 253-260.
  - [20] Sathe, S.T. and Vinod, H.D. (1974). Bounds on the variance of regression coefficients due to heteroscedastic or autoregressive errors, *Econometrica*, Vol. 42, 333-340.
  - [21] Wallace, T.D. and Hussain, A. (1969). The use of error components models in combining cross-section with time-series data. *Econometrica*, Vol. 37, 55-72.
  - [22] Watson, G.S. (1955). Serial correlation in regression analysis I, *Biometrika*, Vol. 42, 333-340.