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A Note on the Asymptotic Behavior of Toeplitz Matrices

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Abstract

The eigenvalues of matrix behave asymptotically like the eigenvalues of Toeplitz matrix.

We begin here with a discussion of the asymptotic eigenvalues distribution of the Toeplitz and asymptotically Toeplitz matrices.

Let $\boldsymbol{A}_n$ be an $n \times n$ Toeplitz matrix with bandwidth 3, where $\boldsymbol{A}_n$ is given by

$$\boldsymbol{A}_n = \frac{1}{2}\delta \begin{pmatrix} -\delta & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -\delta & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\delta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\delta & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\delta \end{pmatrix}.$$

Then the eigenvalues $\{\alpha_{nk} ; k = 1,2,\dots,n\}$ of $\boldsymbol{A}_n$ (See, for example, Graybill[2], p. 284) are

$$\alpha_{nk} = -\frac{1}{2}\delta^2 + \delta\cos\left(\frac{k\pi}{n+1}\right) \qquad \text{for} \quad k=1,2,\dots,n$$

And let $\boldsymbol{B}_n$ be an $n \times n$ matrix, where $\boldsymbol{B}_n$ is given by

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$$B_n = \frac{1}{2}\delta \begin{pmatrix} -\delta & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -\delta & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\delta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\delta & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

with eigenvalues $\{\beta_{nk} \,;\, k = 1,2,\dots,n\}$.

Theorem. Let the eigenvalues $\{\alpha_{nk} \,;\, k = 1,2,\dots,n\}$ of Toeplitz matrix $\boldsymbol{A_n}$ behave symptotically like the ordinates of the function $g(U)$ with equidistributed U. Then the eigenvalues $\{\beta_{nk} \,;\, k = 1,2,\dots,n\}$ of a matrix $\boldsymbol{B_n}$ also behave similarly.

Proof. First, we show that

$$tr(\boldsymbol{A_n B_n}) \le \sum_{k=1}^{n} \alpha_{nk}\beta_{nk}$$

Now, let $\alpha_{n1} \le \alpha_{n2} \le \cdots \le \alpha_{nn}$ and it is sufficient to consider that $\boldsymbol{A_n}$ is diagonal matrix; i.e., $\boldsymbol{A_n} = diag(\alpha_{n1}, \dots, \alpha_{nn})$. Then

$$\begin{aligned} tr(\boldsymbol{A_n B_n}) \le \sum_{j=1}^{n} \alpha_{nj}\beta_{jj} &= \sum_{j=1}^{n} \alpha_{nj}(\boldsymbol{e_j}' \boldsymbol{B_j e_j}) \\ &= \sum_{i=1}^{n} (\alpha_{ni} - \alpha_{n(i-1)}) \sum_{j=i}^{n} \boldsymbol{e_j}' \boldsymbol{B_j e_j} \\ &\le \sum_{j=1}^{n} \alpha_{nj}\beta_{nj} \end{aligned}$$

where b_{jj} is the element of matrix $\boldsymbol{B_n}$, $\boldsymbol{B_j}$ is the j^{th} column of $\boldsymbol{B_n}$, $\boldsymbol{e_j}$ is unit vector and $\alpha_{no} = 0$. The latter term is followed by the Ky Fan inequality (See, Beckenbach and Bellman [1], p. 77). So

$$\begin{aligned} \sum_{k=1}^{n} (\alpha_{nk} - \beta_{nk})^2 &= \sum_{k=1}^{n} \alpha_{nk}^2 - 2\sum_{k=1}^{n} \alpha_{nk}\beta_{nk} + \sum_{k=1}^{n} \beta_{nk}^2 \\ &\le tr(\boldsymbol{A}_n^2) - tr(\boldsymbol{A_n B_n}) - tr(\boldsymbol{B_n A_n}) + tr(\boldsymbol{B}_n^2) \\ &= tr(\boldsymbol{A_n} - \boldsymbol{B_n})^2. \end{aligned}$$

Next, let I be uniformly distributed on $\{1, 2, \cdots, n\}$ and let

$$\alpha_{nI} \to \alpha, \qquad \textit{in dist.}$$

Then

$$E\{(\alpha_{nI} - \beta_{nI})^2\} = \frac{1}{n} \sum_{k=1}^{n} (\alpha_{nk} - \beta_{nk})^2$$

$$\to 0, \qquad as \qquad n \to \infty .$$

Therefore

$$\alpha_{nI} - \beta_{nI} \to 0, \qquad \textit{in prob.}$$

Hence .

$$\beta_{nI} \to \alpha, \qquad \textit{in dist.}$$

References

[1] Beckenbach, E. F., Bellman, R. (1965). *Inequalities.* Springer-Verlag, New-York, Heidelberg, Berlin.

[2] Graybill, F. A. (1983). *Matrices with Applications in Statistics,* 2nd ed., Wadsworth, Belmont.