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Lagrange Multiplier Test for both Regular and Seasonal Unit Roots[1]

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Abstract

In this paper we consider the multiple unit root tests both for the regular and seasonal unit roots based on the Lagrange Multiplier(LM) principle. Unlike Li(1991)'s method, by plugging the restricted maximum likelihood estimates of the nuisance parameters in the model, we propose a Lagrange multiplier test which does not depend on the existence of the nuisance parameters. The asymptotic distribution of the proposed statistic is derived and empirical percentiles of the test statistic for selected seasonal periods are provided. The power and size of the test statistic are examined for finite samples through a Monte Carlo simulation.

1. Introduction

The analysis of seasonal time series often requires to take both regular and seasonal differences. Hasza and Fuller(1979, 1982) and Li(1991) considered the following time series models to test for the multiple unit roots.

$$(1-\phi B)(1-\phi_d B^d)Y_t=\varepsilon_t, \tag{1.1}$$

$$(1-\phi B)(1-\phi_d B^d)Y_t=\alpha+\varepsilon_t, \tag{1.2}$$

$$(1-\phi B)(1-\phi_d B^d)Y_t=\alpha+\beta t+\varepsilon_t, \tag{1.3}$$

where the ε_t are independent and identically nomally distributed with mean 0 and variance σ^2. The test statistics considered by Hasza and Fuller(1982) are based on the F-type statistics of the regression model and the test statistics suggested by Li(1991) are based on the Lagrangian multiplier method. The null hypotheses considered by them, however, assume that the nuisance parameters in models (1.2) and (1.3), α and β, are zeros. Furthermore, depending on the assumptions about the nuisance parameters in the models, the asymptotic distributions of the test statistics considered by Hasza and Fuller(1982) and Li(1991) are different. Therefore, for a proper choice of the test statistic, *apriori* knowledge about the nuisance parameters is required. However the knowledge may not be readily available.

In this paper we propose a multiple unit roots test which is not affected by the existence of

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the nuisance parameters following the developement by Schmidt and Phillips(1992).

2. Test for Multiple Unit Roots

Consider the following seasonal nonstationary model with possible seasonal deterministic trends,

$$(1-\phi B)(1-\phi_d B^d)Y_t = \sum_{j=1}^{d}(\alpha_j+\beta_j\tau)\delta_{jt}+\varepsilon_t, \tag{2.1}$$

where $\tau=[(t-1)/d+1]$ with [x] denoting the largest integer no larger than x and $Y_{-d},\cdots,Y_0$ are fixed, and the seasonal dummy variables

$$\delta_{jt}=\begin{cases}1 & \text{if } j\equiv t \ (\bmod d)\\ 0 & \text{otherwise.}\end{cases}$$

We are mainly interested in testing for multiple unit roots in model (2.1), that is, the null hypothesis is $\phi=1$ and $\phi_d=1$. We have the following approximate log-likelihood function for a sample of $n=md$ observatons from model (2.1), conditional on $Y_{-d},\cdots,Y_0$, is

$$\ln L^*(\Lambda) = \text{constant} - \frac{n}{2}\ln\sigma^2 - \frac{1}{2}\sigma^2 SSE, \tag{2.2}$$

where $\Lambda' = (\alpha_1,\cdots,\alpha_d,\beta_1,\cdots,\beta_d,\phi,\phi_d)$, $\quad SSE=\sum_{j=1}^{d}SSE_j$, and

$$SSE_j=\sum_{k=1}^{m}[Y_{(k-1)d+j}-\phi Y_{(k-1)d+j-1}-\phi_d Y_{(k-2)d+j}+\phi\phi_d Y_{(k-2)d+j-1}-\alpha_j-k\beta_j]^2.$$

To derive the LM test statistic for the null hypothesis in model (2.1), which does not depend on the nuisance parameters α_j's and β_j's, we need the restricted maximum likelihood estimators(RMLE) of α_j's and β_j's under H_0: $(\phi,\phi_d)=(1,1)$. Maximizing $\ln L^*(\Lambda)$ with respect to α_j's and β_j's under H_0, we obtain the following RMLE of α_j's and β_j's,

$$\tilde\beta_j=-\frac{6}{m(m+1)}(Y_{(m-1)d+j}-Y_{(m-1)d+j-1})+\frac{12}{m(m+1)(m-1)}\sum_{k=1}^{m-1}(Y_{(k-1)d+j}-Y_{(k-1)d+j-1}),$$

$$\tilde\alpha_j=\frac{(Y_{(m-1)d+j}-Y_{(m-1)d+j-1})}{m}-\frac{m+1}{2}\tilde\beta_j.$$

The detail of the derivation is given in Appendix 1.

The LM test using RMLE was first introduced by Schmidt and Phillips(1992) to test for the regular unit root and by Ahn and Cho(1993a) to test for the seasonal unit root. Unlike Li(1991)'s LM test, the effects of nuisance parameters are removed by using RMLE. Let

$\tilde{p}=p(\tilde{\lambda})$ and $\tilde{Q}=Q(\tilde{\lambda})$ denote the score vector and the Hessian matrix evaluated at the RMLE's under H_0, respectively, i.e., $\tilde{\lambda}'=(\tilde{\alpha}_1,\dots,\tilde{\alpha}_d,\tilde{\beta}_1,\dots,\tilde{\beta}_d,\phi,\phi_d)$. For the derivation of the LM test let $M_t=Y_t-Y_{t-d}$ and $N_t=Y_t-Y_{t-1}$. Then for model (2.1) with $\alpha_j=0$ and $\beta_j=0$ under H_0, we have $M_t=M_{t-1}+\varepsilon_t$ and $N_t=N_{t-d}+\varepsilon_t$.

The LM test statistic using RMLE is as follows.

$$\mathrm{LM}=\tilde{\mathbf{p}}'\tilde{\mathrm{Q}}^{-1}\tilde{\mathbf{p}} \tag{2.3}$$

$$\tilde{\mathbf{p}}'=\frac{1}{\sigma^2}(0,\dots,0,0,\dots,0,\tilde{p}_1,\tilde{p}_2), \tag{2.4}$$

and

$$\tilde{Q}=\frac{1}{\sigma^2}\begin{pmatrix} O_p(n)I & O_p(n^2)I & O_p(n^{3/2})\underline{1} & O_p(n^{3/2})\underline{1} \\ & O_p(n^3)I & O_p(n^{5/2})\underline{1} & O_p(n^{5/2})\underline{1} \\ & & \tilde{q}_1 & \tilde{q}_2 \\ & & \tilde{q}_3 & \tilde{q}_4 \end{pmatrix}, \tag{2.5}$$

where

$$\tilde{p}_1=\sum_{j=1}^{d}\sum_{k=1}^{m}(\varepsilon_k-\bar{\varepsilon}^j)M_{(k-1)d+j}+\frac{6}{m(m-1)}\sum_{j=1}^{d}\sum_{k=1}^{m}k(\varepsilon_k-\bar{\varepsilon}^j)\sum_{j=1}^{d}\sum_{k=1}^{m}M_{(k-1)d+j}$$
$$-\frac{12}{m(m+1)(m-1)}\sum_{j=1}^{d}\sum_{k=1}^{m}k(\varepsilon_k-\bar{\varepsilon}^j)\sum_{j=1}^{d}\sum_{k=1}^{m}kM_{(k-1)d+j},$$

$$\tilde{p}_2=\sum_{j=1}^{d}\sum_{k=1}^{m}(\varepsilon_k-\bar{\varepsilon}^j)N_{(k-2)d+j}+\frac{6}{m(m-1)}\sum_{j=1}^{d}\sum_{k=1}^{m}k(\varepsilon_k-\bar{\varepsilon}^j)\sum_{j=1}^{d}\sum_{k=1}^{m}N_{(k-2)d+j}$$
$$-\frac{12}{m(m+1)(m-1)}\sum_{j=1}^{d}\sum_{k=1}^{m}k(\varepsilon_k-\bar{\varepsilon}^j)\sum_{j=1}^{d}\sum_{k=1}^{m}kN_{(k-2)d+j},$$

$$\tilde{q}_1=\sum_{j=1}^{d}\sum_{k=1}^{m}M^2_{(k-1)d+j},\quad \tilde{q}_4=\sum_{j=1}^{d}\sum_{k=1}^{m}N^2_{(k-2)d+j},$$

$$\tilde{q}_2=\tilde{q}_3=\sum_{j=1}^{d}\sum_{k=1}^{m}M_{(k-1)d+j}N_{(k-2)d+j}+\sum_{j=1}^{d}\sum_{k=1}^{m}(\varepsilon_k-\bar{\varepsilon}^j)Y_{(k-2)d+j-1}$$
$$+\frac{6}{m(m-1)}\sum_{j=1}^{d}\sum_{k=1}^{m}k(\varepsilon_k-\bar{\varepsilon}^j)\sum_{j=1}^{d}\sum_{k=1}^{m}Y_{(k-2)d+j-1}$$
$$-\frac{12}{m(m+1)(m-1)}\sum_{j=1}^{d}\sum_{k=1}^{m}k(\varepsilon_k-\bar{\varepsilon}^j)\sum_{j=1}^{d}\sum_{k=1}^{m}kY_{(k-2)d+j-1},$$

and I is an identity matrix, $\underline{1}$ is a vector of 1's and $\bar{\varepsilon}^j=\sum_{k=1}^{m}\varepsilon_{(k-1)d+j}/m$.

The derivation is also given in Appendix 1.

3. Limiting Distribution of the Test Statistic

Let

$$D_n \tilde{\mathrm{p}}' = (\underline{0}, \tilde{\underline{p}}')$$

and

$$D_n \tilde{Q} D_n' = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

where D_n is a suitably chosen diagonal matrix to derive the limiting distribution of the LM test. Using the formula for a partitioned inverse we have

$$\mathrm{LM} = \tilde{\underline{p}}' \, [K_{22} - K_{21}K_{11}^{-1}K_{12}]^{-1} \tilde{\underline{p}} + o_p(1). \tag{3.1}$$

If we choose $D_n = (n^{-1/2+\delta}, \ldots, n^{-1/2+\delta}, n^{-3/2+\delta}, \ldots, n^{-3/2+\delta}, n^{-1}, n^{-1})$ with $\delta > 0$, for (2.1), (3.1) is reduced to

$$\mathrm{LM} = \tilde{\underline{p}}' \, K_{22}^{-1} \tilde{\underline{p}} + o_p(1), \tag{3.2}$$

with

$$K_{22} = \frac{1}{\sigma^2} \begin{pmatrix} \tilde{q}_1/n^2 & \tilde{q}_2/n^2 \\ \tilde{q}_3/n^2 & \tilde{q}_4/n^2 \end{pmatrix}.$$

The asymptotic distribution of the LM test statistic for model (2.1) is obtained in Theorem 1.

Theorem 1 Let Y_t be a seasonal time series given by (2.1). Under the null hypothesis $(\phi, \phi_d) = (1,1)$, the LM test statistic in (3.2) is

$$\mathrm{LM} \xrightarrow{L} (\ell_1, \ell_2) \begin{pmatrix} \mathrm{k}_1 & \mathrm{k}_2 \\ \mathrm{k}_3 & \mathrm{k}_4 \end{pmatrix}^{-1} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix},$$

where

$$\ell_1 = d^{-1}\Big[d\sum_{j=1}^{d}\Big(\int_0^1 W_j(r)dV_j(r)\Big) + 6\sum_{j=1}^{d}\Big(\int_0^1 W_j(r)dr\Big)\sum_{j=1}^{d}\Big(\int_0^1 rdV_j(r)\Big) - 12\sum_{j=1}^{d}\Big(\int_0^1 rW_j(r)dr\Big)\sum_{j=1}^{d}\Big(\int_0^1 rdV_j(r)\Big)\Big],$$

$$\ell_2 = d^{-1}\Big[\sum_{j=1}^{d}\Big(\int_0^1 W_j(r)dV_j(r)\Big) + 6\sum_{j=1}^{d}\Big(\int_0^1 W_j(r)dr\int_0^1 rdV_j(r)\Big) - 12\sum_{j=1}^{d}\Big(\int_0^1 rW_j(r)dr\int_0^1 rdV_j(r)\Big)\Big],$$

$$\mathrm{k}_1 = d^{-1}\sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr, \qquad \mathrm{k}_4 = d^{-2}\sum_{j=1}^{d}\int_0^1 W_j(r)^2 dr,$$

$$\mathrm{k}_2 = \mathrm{k}_3 = d^{-2}\Big[\sum_{i,j=1}^{d}\Big(\int_0^1 W_i(r)W_j(r)dr + \sum_{i,j=1}^{d}\int_0^1 r_2\int_0^{r_2} W_i(r_1)dr_1 dV_j(r_2)\Big) + 6\Big(\sum_{j=1}^{d}\int_0^1 rdV_j(r)\Big)\Big(\sum_{j=1}^{d}\int_0^1\int_0^{r_2} W_i(r_1)dr_1 dr_2\Big) - 12\Big(\sum_{j=1}^{d}\int_0^1 rdV_j(r)\Big)\Big(\sum_{j=1}^{d}\int_0^1 r_2\int_0^{r_2} W_i(r_1)dr_1 dr_2\Big)\Big],$$

and W_j are standard Brownian motions and V_j are Brownian bridges.

Proof of Theorem 1 can be easily established by applying the results of Appendix 2 and Chan and Wei(1988).

4. Simulation Results

Empirical percentiles of the test statistic in (3.2) are obtained for d=2, 4 and 12 and m=10, 15, 20, 25, 50, 100, 200 and are given in Table 1. These results are obtained from simulation, where the ε_t are generated by the RNNOA subroutine of IMSL and are based on 30,000 replications.

Table 1 Percentiles of the LM Test Statistic for the Seasonal Trends Model

		Probability of a Smaller Value								
d	$n=md$	0.01	0.025	0.05	0.10	0.50	0.90	0.95	0.975	0.99
2	20	0.036	0.066	0.113	0.197	1.070	3.536	4.724	6.006	8.248
2	30	0.057	0.100	0.162	0.263	1.232	3.654	4.823	6.020	7.885
2	40	0.069	0.125	0.193	0.306	1.311	3.744	4.897	6.172	8.015
2	50	0.080	0.138	0.208	0.327	1.342	3.786	4.952	6.188	8.050
2	100	0.112	0.183	0.266	0.402	1.439	3.730	4.814	5.884	7.460
2	200	0.121	0.218	0.314	0.469	1.472	3.631	4.527	5.871	7.260
2	400	0.157	0.250	0.343	0.494	1.513	3.775	4.826	6.046	7.400
4	40	0.045	0.077	0.119	0.192	0.859	2.572	3.355	4.195	5.404
4	60	0.062	0.101	0.146	0.235	0.986	2.895	3.737	4.639	5.976
4	80	0.070	0.116	0.170	0.259	1.062	3.113	4.013	4.984	6.241
4	100	0.066	0.114	0.174	0.269	1.099	3.170	4.078	5.023	6.356
4	200	0.082	0.136	0.195	0.297	1.190	3.377	4.322	5.332	6.716
4	400	0.085	0.142	0.209	0.318	1.224	3.504	4.492	5.566	7.061
4	800	0.089	0.149	0.219	0.325	1.264	3.568	4.586	5.655	7.132
12	120	0.052	0.083	0.120	0.183	0.769	2.299	2.938	3.683	4.583
12	180	0.062	0.103	0.144	0.218	0.883	2.628	3.402	4.211	5.357
12	240	0.066	0.106	0.155	0.234	0.917	2.789	3.615	4.501	5.693
12	300	0.069	0.111	0.161	0.242	0.955	2.903	3.720	4.716	5.982
12	600	0.073	0.119	0.172	0.260	1.027	3.181	4.120	5.131	6.419
12	1200	0.069	0.119	0.177	0.263	1.067	3.287	4.286	5.322	6.760
12	2400	0.075	0.125	0.183	0.275	1.081	3.342	4.313	5.368	6.813

To examine the power of the LM test statistic discussed in (2.3) under the various alternatives we perform the simulation study. For the simulation we consider the following model which is similar to the one used in Ahn and Cho(1993b).

$$Y_t = \sum_{j=1}^{4} b_{j} t \delta_{jt} + N_t, \qquad (4.1)$$

$$(1-\phi B)(1-\phi_4 B^4)N_t = \varepsilon_t.$$

It is noted that if we let $\beta_j=(1-\phi)(1-\phi_4)b_j$ and $\alpha_j=\phi_4(1-\phi)b_j$, model (1.3) and (4.1) are equivalent, see Schmidt and Phillips(1992) for details.

Using the above relationship, the empirical powers of the proposed test are obtained at the significance level .05 based on 10,000 replications, Tables 2-4. The powers are computed for the seasonal period $d=4$, various values of ϕ and ϕ_d, and $b_j=1,3,4,2$ for Tables 2 and 3 and $b_j=10,30,40,20$ for Table 4 as in Ahn and Cho(1993b). It is observed that the size of the test for $n=100$, Tables 2 and 4, is slightly lower than for $n=200$, Table 3. It is interesting to note that the power of the test when $\phi_d=1$ and $\phi<1$ is higher than when $\phi=1$ and $\phi_d<1$. The power of the test improves as the value of ϕ gets smaller with ϕ_d

Table 2 Empirical Power of Size 0.05 Test for $n=100$
$b_j=(1,3,4,2)$

ϕ	ϕ_d 0.8	0.9	0.95	0.98	0.99	0.995	1.0
0.8	.4641	.5691	.6834	.7838	.8267	.8397	.8666
0.9	.3013	.3618	.4272	.4961	.5401	.5574	.5841
0.95	.2203	.2175	.2374	.2605	.2725	.2795	.2945
0.98	.1755	.1515	.1224	.1016	.0928	.0892	.0891
0.99	.1676	.1158	.0782	.0619	.0566	.0545	.0575
0.995	.1614	.1005	.0679	.0518	.0479	.0473	.0452
1.0	.1671	.0875	.0523	.0506	.0505	.0425	.0448

Table 3 Empirical Power of Size 0.05 Test for $n=200$
$b_j=(1,3,4,2)$

ϕ	ϕ_d 0.8	0.9	0.95	0.98	0.99	0.995	1.0
0.8	.5087	.5778	.6950	.8325	.8848	.9153	.9422
0.9	.3735	.3980	.4808	.6008	.6703	.7122	.7625
0.95	.2558	.2580	.2810	.3203	.3505	.3812	.3944
0.98	.1932	.1661	.1483	.1359	.1372	.1351	.1290
0.99	.1860	.1405	.1037	.0853	.0737	.0689	.0671
0.995	.1860	.1333	.0847	.0595	.0537	.0574	.0551
1.0	.2048	.1203	.0733	.0547	.0528	.0518	.0543

Table 4 Empirical Power of Size 0.05 Test for $n=100$
$b_j=(1,3,4,2)*10$

ϕ	ϕ_d						
	0.8	0.9	0.95	0.98	0.99	0.995	1.0
0.8	.7754	.8471	.9090	.9536	.9653	.9697	.9802
0.9	.7420	.7922	.8503	.9123	.9365	.9465	.9605
0.95	.6910	.7636	.8480	.9223	.9506	.9622	.9729
0.98	.6985	.8143	.9183	.9632	.9751	.9804	.9785
0.99	.7280	.8520	.9115	.8804	.8424	.8191	.7795
0.995	.7482	.8515	.8544	.7251	.6169	.5603	.4776
1.0	.7682	.8342	.7253	.4236	.1652	.0707	.0473

fixed. Although the LM test statistic is derived such that it does not depend on the nuisance parameters, the power gets large as the values of nuisance parameters get large. However, in summary, the test proposed in this paper can be used when there is no *apriori* knowledge about deterministic trends available unlike the tests considered in Hasza and Fuller(1982) and Li(1991).

5. Numerical Examples

The proposed test for multiple unit roots test is applied two real data. The first one is the quarterly United Kingdom data $\{Y_{1t}\}$ used in Ahn and Cho(1993b). To examine whether this data contains multiple unit roots, we take the following procedure. We first obtain a new seires $Z_{1t}=(1-B)(1-B^4)Y_{1t}$ by applying the filter $(1-B)(1-B^4)$ and model the differenced series as follows,

$$(1+0.1869B+0.4677B^4)Z_{1t}=\sum_{j=1}^{4}(\alpha_{1j}+\beta_{1j}\tau)\delta_{jt}+\varepsilon_t,$$

using PROC ARIMA of SAS. By letting $W_{1t}=(1+0.1869B+0.4677B^4)Y_{1t}$ and applying the proposed test to W_{1t}, we obtain the RMLE for nuisance parameters α_{1j}'s and β_{1j}'s

$\tilde{\alpha}_{1j}$	-2.7797	-11.5396	-7.4555	-3.0646
$\tilde{\beta}_{1j}$	-47.6808	167.1559	124.1050	133.5203

and the test statistic LM1=11.6447. Compared with percentiles in Table 1, we reject the null hypothesis of multiple unit roots at the significance level .05. The result implies that the data contains either a regular unit root or a seasonal unit root but not both, and the nonstationarity of the data is not due to a stochastic trend but to a deterministic trend.

The second example is the quarterly Korean G.N.P. series, $\{Y_{2t}\}$, over the period 1970 through 1991. The time series plot of the Korean G.N.P. series is given in Figure 1. The plot shows a regular and a seasonal variation with trend. Following the same modeling procedure, we obtain a new seires $Z_{2t}=(1-B)(1-B^4)Y_{2t}$ and

$$(1+0.3177B+0.2506B^4)Z_{2t}=\sum_{j=1}^{4}(\alpha_{2j}+\beta_{2j}\tau)\delta_{jt}+\varepsilon_t$$

Let $W_{2t}=(1+0.3177B+0.2506B^4)Y_{2t}$. If we apply the proposed LM test to W_2, we obtain the RMLE's of α_{2j}'s and β_{2j}'s

$\tilde{\alpha}_{2j}$	-646.983	44.122	133.429	850.1007
$\tilde{\beta}_{2j}$	33.503	0.780	-2.586	-40.302

and LM2=1.9944. When compared to the percentiles in Table 1, we do not reject the null hypothesis of multiple unit roots at the significance level .05. We conclude that the appropriate model for the Korean G.N.P. series contains $(1-B)(1-B^4)$ terms.

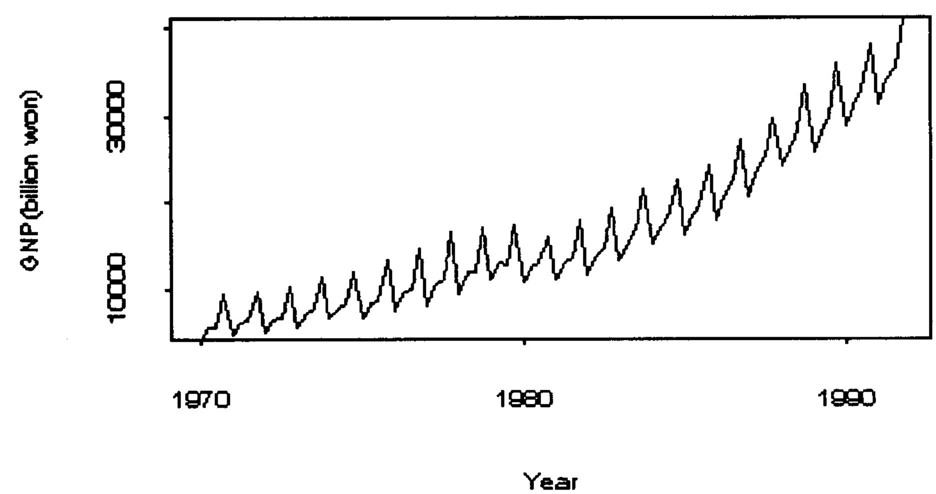

Figure 1 Time Series Plot of the Korean G.N.P. Series

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Appendix 1

If we rewrite (2.1) as follows,

$$\varepsilon_t = Y_t - \phi Y_{t-1} - \phi_d Y_t - d + \phi\phi_d Y_{t-d-1} - \sum_{j=1}^{d}(\alpha_j + \beta_j t)\delta_{jt}.$$

we have

$$\varepsilon_{(k-1)d+j} = Y_{(k-1)d+j} - \phi Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j} + \phi\phi_d Y_{(k-2)d+j-1} - \alpha_j - k\beta_j \qquad \text{(A.1)}$$

and

$$SSE = \sum_{j=1}^{d} SSE_j = \sum_{j=1}^{d}\sum_{k=1}^{m} \varepsilon^2_{(k-1)d+j}. \qquad \text{(A.2)}$$

To obtain the RMLE of the α's and β's under H_0, the partial derivative of (A.2) with respect to α's and β's, $j=1,\ldots,d$, are obtained and set to zero. Since

$$\frac{\partial SSE}{\partial \alpha_j} = \frac{\partial SSE_j}{\partial \alpha_j} = -2\sum_{k=1}^{m} \varepsilon_{(k-1)d+j}$$
$$= -2\sum_{k=1}^{m}(Y_{(k-1)d+j} - Y_{(k-1)d+j-1} - Y_{(k-2)d+j} + Y_{(k-2)d+j-1} - \alpha_j - k\beta_j),$$

and

$$\frac{\partial SSE}{\partial \beta_j} = \frac{\partial SSE_j}{\partial \beta_j} = -2\sum_{k=1}^{m} \varepsilon_{(k-1)d+j} \cdot k$$
$$= -2\sum_{k=1}^{m}(Y_{(k-1)d+j} - Y_{(k-1)d+j-1} - Y_{(k-2)d+j} + Y_{(k-2)d+j-1} - \alpha_j - k\beta_j) \cdot k,$$

by setting $\partial SSE/\partial \alpha_j = 0$ and $\partial SSE/\partial \beta_j = 0$, $j = 1,\ldots,d$, we have

$$m\alpha_j + \frac{m(m+1)}{2}\beta_j = (Y_{(m-1)d+j} - Y_{(m-1)d+j-1})$$
$$\frac{m(m+1)}{2}\alpha_j + \frac{m(m+1)(2m+1)}{6}\beta_j$$
$$= \sum_{k=1}^{m}(Y_{(k-1)d+j} - Y_{(k-1)d+j-1} - Y_{(k-2)d+j} + Y_{(k-2)d+j-1}) \cdot k$$

After some algebra, we have

$$\tilde{\beta}_j = -\frac{6}{m(m+1)}(Y_{(m-1)d+j} - Y_{(m-1)d+j-1})$$
$$+ \frac{12}{m(m+1)(m-1)}\sum_{k=1}^{m-1}(Y_{(k-1)d+j} - Y_{(k-1)d+j-1}) \quad \text{(A.3)}$$

$$\tilde{\alpha}_j = \frac{Y_{(m-1)d+j} - Y_{(m-1)d+j-1}}{m} - \frac{m+1}{2}\tilde{\beta}_j \quad \text{(A.4)}$$

For the evaluation of SSE, we express $\varepsilon_{(k-1)d+j}$ in terms of the α's and β's as follows. From (A.1) under H_0,

$$(m+1)\sum_{k=1}^{m}\varepsilon_{(k-1)d+j} = (m+1)(Y_{(m-1)d+j} - Y_{(m-1)d+j-1}) - m(m+1)\alpha_j - \frac{m(m+1)^2}{2}\beta_j \quad \text{(A.5)}$$

$$\sum_{k=1}^{m}k \cdot \varepsilon_{(k-1)d+j} = \sum_{k=1}^{m}(Y_{(k-1)d+j} - Y_{(k-1)d+j-1} - Y_{(k-2)d+j} + Y_{(k-2)d+j-1}) \cdot k - \sum_{k=1}^{m}k\alpha_j - \sum_{k=1}^{m}k^2\beta_j$$
(A.6)

From (A.5) and (A.6) with $\bar{\varepsilon}^j = \sum_{k=1}^{m}\varepsilon_{(k-1)d+j}/m$

$$\sum_{k=1}^{m}k \cdot (\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j) = \frac{m-1}{2}(Y_{(m-1)d+j} - Y_{(m-1)d+j-1})$$
$$-\sum_{k=1}^{m-1}(Y_{(k-1)d+j} - Y_{(k-1)d+j-1}) - \frac{m(m+1)(m-1)}{12}\beta_j \quad \text{(A.7)}$$

Since the right hand side of (A.7) becomes $\{m(m+1)(m-1)/12\}(\tilde{\beta}_j - \beta_j)$ by (A.3), we have

$$\tilde{\beta}_j = \beta_j + \frac{12}{m(m+1)(m-1)} \sum_{k=1}^{m} k \cdot (\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j). \tag{A.8}$$

Also by (A.4)

$$\sum_{k=1}^{m} \varepsilon_{(k-1)d+j} = m\tilde{\alpha}_j + \frac{m(m+1)}{2} \tilde{\beta}_j - m\alpha_j - \frac{m(m+1)}{2} \beta_j. \tag{A.9}$$

Dividing both sides of (A.9) by m, we have

$$\bar{\varepsilon}^j = m(\tilde{\alpha}_j - \alpha_j) + \frac{m+1}{2} (\tilde{\beta}_j - \beta_j).$$

Thus

$$\tilde{\alpha}_j = \alpha_j + \bar{\varepsilon}^j - \frac{6}{m(m-1)} \sum_{k=1}^{m} k(\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j). \tag{A.10}$$

To obtain the score vector $\boldsymbol{p}$ and the Hessian matrix Q we need the following partial derivatives. Since $\partial \ln L(\Lambda)/\partial \Lambda = -(1/2\sigma^2)\partial SSE/\partial \Lambda$ and $-\partial^2 \ln L(\Lambda)/\partial \Lambda \partial \Lambda' = (1/2\sigma^2)\partial^2 SSE/\partial \Lambda \partial \Lambda'$, we provide partial derivatives of SSE instead of those of the log-likelihood.

$$\begin{aligned}\frac{\partial SSE}{\partial \phi} &= \sum_{j=1}^{d} \frac{\partial SSE_j}{\partial \phi} = -2\sum_{j=1}^{d}\sum_{k=1}^{m} \varepsilon_{(k-1)d+j}(Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j-1}) \\ &= -2\sum_{k=1}^{m}(Y_{(k-1)d+j} - \phi Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j} + \phi\phi_d Y_{(k-2)d+j-1} - \alpha_j - k\beta_j) \\ &\qquad (Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j-1}),\end{aligned}$$

$$\begin{aligned}\frac{\partial SSE}{\partial \phi_d} &= \sum_{j=1}^{d} \frac{\partial SSE_j}{\partial \phi_d} = -2\sum_{j=1}^{d}\sum_{k=1}^{m} \varepsilon_{(k-1)d+j}(Y_{(k-2)d+j} - \phi Y_{(k-2)d+j-1}) \\ &= -2\sum_{k=1}^{m}(Y_{(k-1)d+j} - \phi Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j} + \phi\phi_d Y_{(k-2)d+j-1} - \alpha_j - k\beta_j) \\ &\qquad (Y_{(k-2)d+j} - \phi Y_{(k-2)d+j-1}),\end{aligned}$$

$$\frac{\partial^2 SSE}{\partial \alpha_j^2} = \frac{\partial^2 SSE_j}{\partial \alpha_j^2} = 2m, \qquad \frac{\partial^2 SSE}{\partial \beta_j^2} = \frac{\partial^2 SSE_j}{\partial \beta_j^2} = 2\{m(m+1)(2m+1)/6\},$$

$$\frac{\partial^2 SSE}{\partial \phi^2} = \sum_{j=1}^{d} \frac{\partial^2 SSE_j}{\partial \phi^2} = 2\sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j-1})^2,$$

$$\frac{\partial^2 SSE}{\partial \phi_d^2} = \sum_{j=1}^{d} \frac{\partial^2 SSE_j}{\partial \phi_d^2} = 2\sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-2)d+j} - \phi Y_{(k-2)d+j-1})^2,$$

$$\frac{\partial^2 SSE}{\partial \alpha_j \partial \beta_j} = 2m(m+1)/2,$$

$$\frac{\partial^2 SSE}{\partial \alpha_j \partial \phi} = 2\sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j-1}),$$

$$\frac{\partial^2 SSE}{\partial \beta_j \partial \phi} = 2\sum_{j=1}^{d}\sum_{k=1}^{m} k(Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j-1}),$$

$$\frac{\partial^2 SSE}{\partial \alpha_j \partial \phi_d} = 2\sum_{j=1}^{d}\sum_{k=1}^{m} (Y_{(k-2)d+j} - \phi Y_{(k-2)d+j-1}),$$

$$\frac{\partial^2 SSE}{\partial \beta_j \partial \phi_d} = 2\sum_{j=1}^{d}\sum_{k=1}^{m} k(Y_{(k-2)d+j} - \phi Y_{(k-2)d+j-1}),$$

$$\begin{aligned}\frac{\partial^2 SSE}{\partial \phi \partial \phi_d} &= 2\sum_{j=1}^{d}\sum_{k=1}^{m} (Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j-1})(Y_{(k-2)d+j} - \phi Y_{(k-2)d+j-1}) \\ &\quad + 2\sum_{j=1}^{d}\sum_{k=1}^{m} \varepsilon_{(k-1)d+j} Y_{(k-2)d+j-1} \\ &= 2\sum_{j=1}^{d}\sum_{k=1}^{m} (Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j-1})(Y_{(k-2)d+j} - \phi Y_{(k-2)d+j-1}) \\ &\quad + 2\sum_{j=1}^{d}\sum_{k=1}^{m} (Y_{(k-1)d+j} - \phi Y_{(k-1)d+j-1} - \phi_d Y_{(k-2)d+j} + \phi\phi_d Y_{(k-2)d+j-1} - \alpha_j - k\beta_j) \\ &\qquad Y_{(k-2)d+j-1},\end{aligned}$$

$$\frac{\partial^2 SSE}{\partial \alpha_j \partial \alpha_k} = 0, \quad \frac{\partial^2 SSE}{\partial \beta_j \partial \beta_k} = 0, \quad \frac{\partial^2 SSE}{\partial \alpha_j \partial \beta_k} = 0, \ \text{for } j \neq k.$$

Now, we can obtain the score vector $\tilde{p}$ and the Hessian matrix $\tilde{Q}$ under the RMLE, by plugging $\tilde{\alpha}_j$ and $\tilde{\beta}_j$ of (A.10) and (A.8) to the above partial derivatives.

$$\begin{aligned}\frac{\partial SSE}{\partial \phi} &= -2\sum_{j=1}^{d}\sum_{k=1}^{m} (Y_{(k-1)d+j} - Y_{(k-1)d+j-1} - Y_{(k-2)d+j} + Y_{(k-2)d+j-1} - \tilde{\alpha}_j - k\tilde{\beta}_j) \\ &\qquad (Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1}) \\ &= -2\Big[\sum_{j=1}^{d}\sum_{k=1}^{m} (\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j)(Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1}) \\ &\quad + \frac{6}{m(m-1)} \sum_{j=1}^{d}\sum_{k=1}^{m} k(\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j) \sum_{k=1}^{m} (Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1}) \\ &\quad - \frac{12}{m(m+1)(m-1)} \sum_{j=1}^{d}\sum_{k=1}^{m} k(\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j) \sum_{k=1}^{m} k(Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1})\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial SSE}{\partial \phi_d} &= -2\sum_{j=1}^{d}\sum_{k=1}^{m} (Y_{(k-1)d+j} - Y_{(k-1)d+j-1} - Y_{(k-2)d+j} + Y_{(k-2)d+j-1} - \tilde{\alpha}_j - k\tilde{\beta}_j) \\ &\qquad (Y_{(k-2)d+j} - Y_{(k-2)d+j-1}) \\ &= -2\Big[\sum_{j=1}^{d}\sum_{k=1}^{m} (\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j)(Y_{(k-2)d+j} - Y_{(k-2)d+j-1}) \\ &\quad + \frac{6}{m(m-1)} \sum_{j=1}^{d}\sum_{k=1}^{m} k(\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j) \sum_{k=1}^{m} (Y_{(k-2)d+j} - Y_{(k-2)d+j-1}) \\ &\quad - \frac{12}{m(m+1)(m-1)} \sum_{j=1}^{d}\sum_{k=1}^{m} k(\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^j) \sum_{k=1}^{m} k(Y_{(k-2)d+j} - Y_{(k-2)d+j-1})\Big],\end{aligned}$$

$$\frac{\partial^2 SSE}{\partial \phi^2} = 2\sum_{j=1}^{d}\sum_{k=1}^{m} (Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1})^2,$$

$$\frac{\partial^2 SSE}{\partial \phi_d^2} = 2\sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-2)d+j} - Y_{(k-2)d+j-1})^2,$$

$$\frac{\partial^2 SSE}{\partial \alpha_j \partial \beta_j} = m(m+1) = O_p(n^2),$$

$$\frac{\partial^2 SSE}{\partial \alpha_j \partial \phi} = 2\sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1}) = O_p(n^{3/2}),$$

$$\frac{\partial^2 SSE}{\partial \beta_j \partial \phi} = 2\sum_{j=1}^{d}\sum_{k=1}^{m}k(Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1}) = O_p(n^{5/2}),$$

$$\frac{\partial^2 SSE}{\partial \alpha_j \partial \phi_d} = 2\sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-2)d+j} - Y_{(k-2)d+j-1}) = O_p(n^{3/2}),$$

$$\frac{\partial^2 SSE}{\partial \beta_j \partial \phi_d} = 2\sum_{j=1}^{d}\sum_{k=1}^{m}k(Y_{(k-2)d+j} - Y_{(k-2)d+j-1}) = O_p(n^{5/2}),$$

$$\begin{aligned}\frac{\partial^2 SSE}{\partial \phi \partial \phi_d} = & \ 2\Big[\sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1})(Y_{(k-2)d+j} - Y_{(k-2)d+j-1}) \\ & + \sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-1)d+j} - Y_{(k-1)d+j-1} - Y_{(k-2)d+j} + Y_{(k-2)d+j-1} - \tilde{\alpha}_j - k\tilde{\beta}_j)Y_{(k-2)d+j-1}\Big] \\ = & \ 2\Big[\sum_{j=1}^{d}\sum_{k=1}^{m}(Y_{(k-1)d+j-1} - Y_{(k-2)d+j-1})(Y_{(k-2)d+j} - Y_{(k-2)d+j-1}) \\ & + \frac{6}{m(m-1)}\sum_{j=1}^{d}\sum_{k=1}^{m}k(\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^{j})\sum_{k=1}^{m}Y_{(k-2)d+j-1} \\ & - \frac{12}{m(m+1)(m-1)}\sum_{j=1}^{d}\sum_{k=1}^{m}k(\varepsilon_{(k-1)d+j} - \bar{\varepsilon}^{j})\sum_{k=1}^{m}kY_{(k-2)d+j-1}\Big].\end{aligned}$$

Appendix 2

Let

$$M_t = Y_t - Y_{t-d} \quad \text{and} \quad N_t = Y_t - Y_{t-1},$$

then for model (2.1) with $\alpha_j = 0$ and $\beta_j = 0$,

$$M_t = M_{t-1} + \varepsilon_t \quad \text{and} \quad N_t = N_{t-d} + \varepsilon_t.$$

Using the results of Chan and Wei(1988), we have

$$\begin{aligned} n^{-3/2}\sum_{k=1}^{m}N_{(k-1)d+j} &= n^{-3/2}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(k-1-\ell)d+j}\Big) \\ &\xrightarrow{L} d^{-3/2}\sigma\int_0^1 W_j(r)dr \end{aligned}$$

$$\begin{aligned} n^{-1}\sum_{t=1}^{n}N_{t-d}\varepsilon_t &= n^{-1}\sum_{t=1}^{n}\Big(\sum_{\ell=1}^{[(t-d)/d]}\varepsilon_{t-d\ell}\Big)\varepsilon_t = n^{-1}\sum_{j=1}^{d}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+j}\Big)\varepsilon_{(k-1)d+j} \\ &\xrightarrow{L} d^{-1}\sigma^2\sum_{j=1}^{d}(W_j(1)^2-1)/2 \end{aligned}$$

$$n^{-2}\sum_{t=1}^{n}N_{t-d}^{2}=n^{-2}\sum_{t=1}^{n}\Big(\sum_{\ell=1}^{[(t-d)/d]}\varepsilon_{t-d\ell}\Big)^{2}=n^{-2}\sum_{j=1}^{d}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+j}\Big)^{2}$$
$$\xrightarrow{L} d^{-2}\sigma^{2}\sum_{j=1}^{d}\int_{0}^{1}W_{j}(r)^{2}dr$$

$$n^{-5/2}\sum_{k=1}^{m}kN_{(k-1)d+j}=n^{-5/2}\sum_{k=1}^{m}k\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(k-1-\ell)d+j}\Big)$$
$$\xrightarrow{L} d^{-5/2}\sigma\int_{0}^{1}rW_{j}(r)dr$$

$$n^{-3/2}\sum_{k=1}^{m}M_{(k-1)d+j}=n^{-3/2}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{(k-1)d+j}\varepsilon_{\ell}\Big)=n^{-3/2}\sum_{k=1}^{m}\sum_{j=1}^{d}\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+j}\Big)$$
$$\xrightarrow{L} d^{-3/2}\sigma\sum_{j=1}^{d}\int_{0}^{1}W_{j}(r)dr$$

$$n^{-5/2}\sum_{k=1}^{m}kM_{(k-1)d+j}=n^{-5/2}\sum_{k=1}^{m}k\Big(\sum_{\ell=1}^{(k-1)d+j}\varepsilon_{\ell}\Big)=n^{-5/2}\sum_{k=1}^{m}k\sum_{j=1}^{d}\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+j}\Big)$$
$$\xrightarrow{L} d^{-5/2}\sigma\sum_{j=1}^{d}\int_{0}^{1}rW_{j}(r)dr$$

$$n^{-1}\sum_{t=1}^{n}M_{t-1}\varepsilon_{t}\xrightarrow{L}\sigma^{2}(W(1)^{2}-1)/2 \qquad n^{-2}\sum_{t=1}^{n}M_{t-1}^{2}\xrightarrow{L}\sigma^{2}\int_{0}^{1}W(r)^{2}dr$$

$$n^{-2}\sum_{t=1}^{n}M_{t-1}N_{t-d}=n^{-2}\sum_{t=1}^{n}\Big(\sum_{\ell=1}^{t-1}\varepsilon_{t}\Big)\Big(\sum_{\ell=1}^{[(t-d)/d]}\varepsilon_{t-d\ell}\Big)$$
$$=n^{-2}\sum_{j=1}^{d}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+j}\Big)\Big(\sum_{\ell=1}^{(k-1)d+j-1}\varepsilon_{\ell}\Big)$$
$$=n^{-2}\sum_{j=1}^{d}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+j}\Big)\Big(\sum_{i=1}^{d}\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+i}\Big)$$
$$=n^{-2}\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+j}\Big)\Big(\sum_{\ell=1}^{k-1}\varepsilon_{(\ell-1)d+i}\Big)$$
$$\xrightarrow{L} d^{-2}\sigma^{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_{0}^{1}W_{i}(r)W_{j}(r)dr$$

$$n^{-5/2}\sum_{k=1}^{m}Y_{(k-1)d+j}=n^{-5/2}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{(k-1)d+j}N_{\ell}\Big)=n^{-5/2}\sum_{k=1}^{m}\sum_{j=1}^{d}\Big(\sum_{\ell=1}^{k}N_{(\ell-1)d+j}\Big)$$
$$\xrightarrow{L} d^{-5/2}\sigma\sum_{j=1}^{d}\int_{0}^{1}\int_{0}^{r_2}W_{j}(r_1)dr_1dr_2$$

$$n^{-7/2}\sum_{k=1}^{m}kY_{(k-1)d+j}=n^{-7/2}\sum_{k=1}^{m}k\Big(\sum_{\ell=1}^{(k-1)d+j}N_{\ell}\Big)=n^{-7/2}\sum_{k=1}^{m}k\sum_{j=1}^{d}\Big(\sum_{\ell=1}^{k}N_{(\ell-1)d+j}\Big)$$
$$\xrightarrow{L} d^{-7/2}\sigma\sum_{j=1}^{d}\int_{0}^{1}r_2\int_{0}^{r_2}W_{j}(r_1)dr_1dr_2$$

$$n^{-2}\sum_{t=1}^{n}Y_{t-1}\varepsilon_{t}=n^{-2}\sum_{j=1}^{d}\sum_{k=1}^{m}Y_{(k-1)d+j}\varepsilon_{(k-1)d+j}$$
$$=n^{-2}\sum_{j=1}^{d}\sum_{k=1}^{m}\Big(\sum_{i=1}^{d}\sum_{\ell=1}^{k-1}N_{(\ell-1)d+i}\Big)\varepsilon_{(k-1)d+j}$$
$$=n^{-2}\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{m}\Big(\sum_{\ell=1}^{k-1}N_{(\ell-1)d+i}\Big)\varepsilon_{(k-1)d+j}$$
$$\xrightarrow{L} d^{-2}\sigma^{2}\sum_{j=1}^{d}\sum_{i=1}^{d}\int_{0}^{1}\int_{0}^{r_2}W_{i}(r_1)dr_1dW_{j}(r_2).$$