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Minimum Mean Squared Error Accelerated Life Test Plans for Exponential Lifetime Distribution

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Abstract

This paper considers model robust accelerated life test plans for estimating the log mean or percentile of product life which is exponentially distributed. A linear relationship between the log mean life and the stress is assumed as usual, while the true relationship is quadratic. Optimum plans are then obtained by minimizing asymptotic mean squared error of maximum likelihood estimator of the log mean life.

1. Introduction

Accelerated life tests (ALTs) provide timely information on the lifetime distribution of products or materials. Units are tested at higher-than-usual levels of stress to induce early failures. The level of stress at which units are tested is called the test stress. The life data from the test stresses are then used in conjunction with a statistical model relating lifetime to stress in order to estimate the lifetime distribution characteristics at a design stress. Such testing saves time and cost over testing at the design stress. Further savings result from a good test plan. Common plans use equally spaced test stresses and the same number of test units allocated to each test stress. Such common plans are usually inefficient for estimating the lifetime distribution characteristics at the design stress. In order to obtain more precise estimate for the same number of test units and test time, optimum ALT plans have been studied. Optimum ALT plans are determined by choosing the test stresses and the number of test units at each test stress to satisfy specified optimality criterion. The usual optimality criterion is to minimize asymptotic variance (AsVar) of maximum likelihood estimator (MLE) of the mean or percentile of (log) lifetime at the design stress.

The literature on optimum ALT plans deals with a variety of lifetime distributions, life-stress relationships, estimation methods and censoring types. Chernoff (1962) considers ML estimation of the failure rate of an exponential distribution at the design stress. Assuming that the failure rate is a quadratic function of the stress, he suggests optimum plans for both Type I censored and complete data. Optimum plans for estimating the percentile of a normal (or lognormal) or extreme value (or Weibull) distribution at the design stress have been

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studied by Barton (1991), Kielpinski and Nelson (1975), Meeker and Nelson (1975), Nelson and Kielpinski (1976) and Nelson and Meeker (1978). Most of the previous works assume that the life-stress relationship is linear. For given design stress and upper limit of the test stress, they consider ALT plans with two test stresses and the high test stress set to the upper limit. The low test stress and proportions of test units allocated to the high and low test stresses are then determined by minimizing AsVar of MLE of the percentile of interest.

The life-stress relationship is often either very complicated or unknown. Even though the linear relationship seems appropriate, it is a safe design strategy to take account of the possibility that a quadratic or higher order model could provide better approximation. If the sample ALT data indicate a curvilinear relationship, it might be tempting to fit a quadratic model. In this context Nelson and Kielpinski (1976) suggest compromise plans with three test stresses. Meeker (1984) compares optimum test plans and some compromise test plans with respect to the ability to detect quadratic departure from the assumed linear relationship. However, data analysts are usually reluctant to use such a quadratic model if the desired inferences require much extrapolation. It is desirable to make ALT experiments and corresponding data analysis as simple as possible. Therefore, if the life-stress relationship is well approximated by the linear model and quadratic departure of small magnitude seems to be involved in the linear model, data analysis may be performed under the linear model. However, in order to protect ourselves from the model departure, we need ALT plans for the linear model which are robust to the quadratic model departure. Such robust test plans may be obtained by assuming that the true model is quadratic. This approach has been used in the field of response surface designs. For example, see Box and Draper (1959) and Draper and Guttman (1986). This paper considers optimum ALT plans for estimating the log mean life of an exponential lifetime distribution at the design stress when there exists the quadratic model departure. In this case an appropriate optimality criterion is to minimize asymptotic mean squared error (AsMSE) of MLE obtained under the linear relationship. The low test stress and the proportions of test units allocated to each test stress are then chosen so that AsMSE is minimized. The test procedure, the model assumptions and the estimation method are described in Section 2. Section 3 derives AsMSE of MLE of the log mean life and optimal ALT plans.

2. The model

This section first describes the test procedure and the model for the lifetime as a funtion of the stress x. We assume that the lifetime is exponentially distributed with mean life μ and the log mean life is approximately a linear function of the stress, i.e..

$$\ln \mu(x) = \gamma_0 + \gamma_1 x . \tag{2.1}$$

The popular inverse law and Arrhenius models are special cases of this simple linear relationship. In order to estimate the characteristics of the lifetime distribution at the design stress x_d under the assumptions mentioned above, we consider the usual ALT plans with two test stresses such that the high test stress x_h is set to the specified upper limit of the test stress, n_l units randomly chosen from n units are tested at the low test stress x_l and the remaining n_h units at x_h, and each unit is tested until some prespecified censoning time η if it does not fail sooner. Let $\pi_i = n_i/n$ for $i = h, l$. It is then necessary to determine the values of x_l and π_l optimally in some sense.

It is convenient to express the stress x as $x = x_d - \xi(x_d - x_h)$, where

$$\xi = \xi(x) = \frac{(x_d - x)}{(x_d - x_h)}$$

is the standardized stress so that $\xi_d = \xi(x_d) = 0$ and $\xi_h = \xi(x_h) = 1$. Then model (2.1) is written as

$$\ln \mu(\xi) = \beta_0 + \beta_1 \xi \,, \tag{2.2}$$

where $\beta_0 = \gamma_0 + \gamma_1 x_d$ and $\beta_1 = -\gamma_1(x_d - x_h)$. Suppose our design objective is the efficient estimation of β_0, the log mean life at ξ_d. By differentiating the log likelihood with respect to β_0 and β_1 and equating to zero, MLE of β_0 is obtained as

$$\widehat{\beta_0} = \frac{\xi_l}{(1-\xi_l)} \ln\left(\frac{\widetilde{T_l}}{r_l}\right) - \frac{1}{(1-\xi_l)} \ln\left(\frac{\widetilde{T_h}}{r_h}\right), \tag{2.3}$$

where r_i is the number of units failed before η at ξ_i, $\widetilde{T_i} = \sum_{j=1}^{r_i} T_{ij} + (n_i - r_i)\eta$ and T_{ij} is j-th failure time at ξ_i for $i = h, l$. This estimator is meaningful only when $r_i > 0$. Most of the previous works determine x_l and π_l by minimizing AsVar of MLE of the parameter of interest. However, this approach is applicable when model (2.2) is the true model. We now consider the situation where model (2.2) seems to be appropriate but quadratic model departure is concerned. It is then a reasonable design strategy to develop an ALT plan for model (2.2) which protects us against the quadratic model departure. Such an ALT plan is the plan that minimizes AsMSE of $\widehat{\beta_0}$ obtained under the assumption that the true life-stress relationship is

$$\ln \mu(\xi) = \beta_0 + \beta_1 \xi + \beta_2 \xi^2 . \tag{2.4}$$

As MSE of $\widehat{\beta}_0$ and optimum ALT plans will be derived in the following section.

3. Optimum ALT plans

Mendenhall and Lehman (1960) showed that

$$E\left(\frac{\widetilde{T}_i}{\eta r_i}\right)=\frac{-1}{\ln q_i}-\frac{q_i}{p_i}+n\pi_i E\left(\frac{1}{r_i}\right)-1 \tag{3.1}$$

and

$$Var\left(\frac{\widetilde{T}_i}{\eta r_i}\right)=n^2\pi_i^2 Var\left(\frac{1}{r_i}\right)+\left\{\frac{1}{(\ln q_i)^2}-\frac{q_i}{p_i^2}\right\}E\left(\frac{1}{r_i}\right), \tag{3.2}$$

where p_i is the probability that a unit will fail by η at ξ_i and $q_i=1-p_i$. They also suggested simple approximation formula for the moments of $1/r_i$, which are

$$E\left(\frac{1}{r_i}\right)\approx\frac{(n_i-2)}{n_i\{(n_i-1)p_i-1\}}$$

and

$$Var\left(\frac{1}{r_i}\right)\approx\frac{(n_i-2)\{n_i-(n_i-1)p_i-1\}}{n_i^2\{(n_i-1)p_i-1\}^2\{(n_i-1)p_i-2\}}.$$

Inserting these approximate moments into (3.1) and (3.2) and assuming that n is sufficiently large, asymptotic mean and variance of $\widetilde{T}_i/(\eta r_i)$ are given by $(-\ln q_i)^{-1}$ and $\{n\pi_i p_i(-\ln q_i)^2\}^{-1}$, respectively. Utilizing the asymptotic normality property of MLEs, asymptotic mean and variance of $\ln\{\widetilde{T}_i/(\eta r_i)\}$ are obtained as $-\ln(-\ln q_i)$ and $(n\pi_i p_i)^{-1}$. Since $p_i=1-\exp\{-\eta/\exp(\beta_0+\beta_1\xi_i+\beta_2\xi_i^2)\}$ under quadratic model (2.4), bias of $\widehat{\beta}_0$ is $-\beta_2\xi_l$ and AsMSE of $\widehat{\beta}_0$ is

$$\text{AsMSE}\ (\widehat{\beta}_0)=\beta_2^2\xi_l^2+\frac{1}{(1-\xi_l)^2}\ \frac{1}{n\pi_l p_l}+\frac{\xi_l^2}{(1-\xi_l)^2}\ \frac{1}{n(1-\pi_l)p_h}. \tag{3.3}$$

We illustrate by an example the effect of quadratic model departure on AsMSE$(\widehat{\beta}_0)$. Consider the classical optimal ALT designs minimizing AsVar$(\widehat{\beta}_0)$ for $p_d=0.01$ and $p_h=0.99$. AsMSE$(\widehat{\beta}_0)$ of the designs are depicted against β_2 in Figure 1 for n=30, 50, 100. Plots for other values of p_d and p_h show similar features. The effect of model departure on AsMSE$(\widehat{\beta}_0)$ is significant. It is therefore sensible to consider the designs minimizing

AsMSE($\widehat{\beta}_0$).

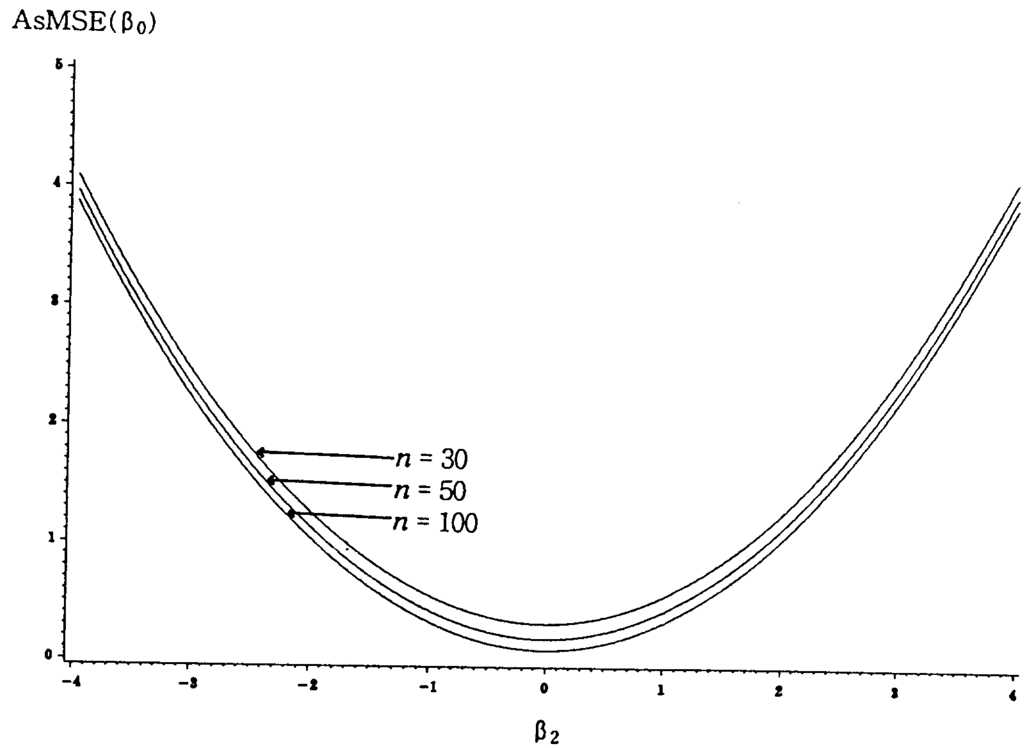

Figure 1. AsMSE($\widehat{\beta}_0$) of the designs minimizing AsVar($\widehat{\beta}_0$) for $p_d=0.01$ and $p_h=0.99$.

ALT plans attaining the minimum of (3.3) will be referred to as minimum mean squared error (MMSE) ALT plans. Unfortunately we could not show the convexity of (3.3) due to the complexity. However, we verified numerically via a two-dimensional search method that ξ_l^* and π_l^*, the optimal values of ξ_l and π_l, are given by the solution of equation

$$0=2\beta_2^2\xi_l+\frac{1}{n(1-\xi_l)^3}\left(\frac{1}{\sqrt{p_l}}+\frac{\xi_l}{\sqrt{p_h}}\right)\cdot \left[2\left(\frac{1}{\sqrt{p_l}}+\frac{1}{\sqrt{p_h}}\right)-\frac{q_l\ln q_l}{p_l\sqrt{p_l}}(1-\xi_l)\{\ln(-\ln q_d)-\ln(-\ln q_h)-\beta_2(1-2\xi_l)\}\right] \quad (3.4)$$

and

$$\pi_l^*=\frac{\sqrt{p_h}}{\sqrt{p_l^*}\,\xi_l^*+\sqrt{p_h}}\,, \quad (3.5)$$

where p_l^* is the p_l evaluated at ξ_l^*. These equations are obtained by equating the derivatives of (3.3) with respect to ξ_l and π_l to zero. The optimum plans depend on the model parameters p_d, p_h and β_2 which are usually unknown. In practice one must approximate the parameters from experiences, similar data or a preliminary test. If n and preliminary estimates of p_d, p_h and β_2 are available, p_l for arbitrary ξ_l is computed as

$$p_l = 1 - \exp[-(-\ln q_d)^{(1-\xi_l)}(-\ln q_h)^{\xi_l}\exp(\beta_2\xi_l - \beta_2\xi_l^2)]$$

and then equation (3.4) is easily solved by standard numerical methods. The log 100pth percentile, t_p, at the design stress is $\widehat{\beta_0} + \ln\{-\ln(1-p)\}$. Therefore t_p is estimated by $\widehat{\beta_0} + \ln\{-\ln(1-p)\}$. Optimum plans for estimating β_0 are also optimum for estimating t_p at the design stress.

Finally we consider the problem of determining the samle size n. Since the data analysis is performed under the linear model, the $(1-\alpha)100\%$ asymptotic confidence interval for β_0 is $\widehat{\beta_0} \pm Z_{(1-\alpha/2)}\{\text{AsVar}(\widehat{\beta_0})\}^{1/2}$, where $Z_{(1-\alpha/2)}$ is $(1-\alpha/2)100$th percentile of standard normal distribution and $\text{AsVar}(\widehat{\beta_0})$ is the sum of second and third terms of the right hand side of (3.3) evaluated under model (2.2). Therefore the desired accuracy of $\widehat{\beta_0}$ can be expressed by specifying α and the desired value of $Z_{(1-\alpha/2)}\{\text{AsVar}(\widehat{\beta_0})\}^{1/2}$, say w. Once p_d, p_h and β_2 are estimated and α is specified, we compute ξ_l^*, π_l^* and the corresponding $Z_{(1-\alpha/2)}\{\text{AsVar}(\widehat{\beta_0})\}^{1/2}$ for different n. Then by comparing $Z_{(1-\alpha/2)}\{\text{AsVar}(\widehat{\beta_0})\}^{1/2}$ with w, we can determine a sample size satisfying the specified accuracy.

4. Concluding remarks

This paper suggests MMSE ALT plans for the linear life-stress model that provide us with protection against the quadratic model departure. In order to select a specific MMSE ALT plan, preliminary estiamtes of p_d, p_h and β_2 are required. Since $\ln\mu(\xi)$ is generally a decreasing function for $0 \le \xi \le 1$, the specified value of β_2 should be between $\ln(-\ln q_d) - \ln(-\ln q_h)$ and $\ln(-\ln q_h) - \ln(-\ln q_d)$. A numerical study on the MMSE ALT plan showed that the low test stress is closer to the design stress and more units are allocated to the low test stress as compared with the classical ALT plan minimizing AsVar.

Further studies are necessary for different lifetime distributions and other types of model departure.

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