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Limit Theorems for Banach Space Valued Random Vectors

Dug Hun Hong[1], Seok Yoon Hwang[2]

Abstract

The purpose of this paper is to generalize Hyede's(1969) results in Banach spaces of R-type 2.

1. Introduction

Let $(B, \| \ \|)$ be a real separable Banach spaces. Let $\{X_n\}$ be a sequence of independent, Banach space valued random vectors, and write $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. Let $\{b_n\}$ be a monotone sequence of positive constants with $b_n \to \infty$ as $n \to \infty$.

Choi and Sung(1988) proved that if $\sum_{n=1}^{\infty} E\|X_n\|^2/b_n^2 < \infty$, then the following are equivalent:

$$\text{(i)} \quad \frac{E\|S_n\|}{b_n} \to 0 \quad ;$$

$$\text{(ii)} \quad \frac{S_n}{b_n} \to 0 \quad a.s. \quad ;$$

$$\text{(iii)} \quad \frac{S_n}{b_n} \to 0 \quad \text{in probabilty}.$$

For a sequence $\{X_n\}$ of i.i.d. random variables, it turned out by Heyde(1969) that the oscillation behavior of normed sums rests essentially on whether or not the summands belong to the domain of partial attraction of normal distribution.

1) Department of Statistics, Hyosung Women's University, Kyungbuk, 713-702, KOREA
2) Department of Mathematics, Taegu University, Kyungbuk, 713-714, KOREA

The purpose of this paper is to generalize Heyde's results in Banach spaces of R-type 2. A sequence $\{X_n\}$ of i.i.d. random vectors in B belongs to the domain of partial attraction of the normal distribution(See Vvedenskaya, 1983) if

$$\liminf_{t\to\infty} \frac{t^2 P(\|X_1\| > t)}{\int_{|x|\le t} x^2 dP(\|X_1\| \le x)} = 0.$$

The Banach space B is of R-type 2 if and only if there exists C such that for any sequence $\{X_n\}$ of independent random vectors in B with $EX_i = 0$, $E\|\sum_{i=1}^{n} X_i\|^2 \le CE(\sum_{i=1}^{n} \|X_i\|^2)$ (See Woyczynski, 1980).

Throughout this paper, $\{X_n\}$ denotes a sequence of i.i.d. random vectors in Banach space B of $R-$type 2.

2. Results

Let $X_j' = X_j I(\|X_j\| \le b_j)$, $X_j'' = X_j - X'$, where I means the indicator function. And let $T_n = \sum_{j=1}^{n} X_j'$.

Theorem 1. Suppose that $\{X_n\}$ are symmetric and they do not belong to the domain of partial attraction of the normal distribution. Let $\{b_n\}$ be a monotone sequence of positive constants with $b_n \to \infty$. Then $S_n/b_n \to 0$ a.s. if and only $\sum_{n=1}^{\infty} P(\|X_1\| > b_n)$ converges.

Proof. Let $\{X_n\}$ not belong to the domain of partial attraction of the normal distribution. Then there exist positive constants a, b such that for all $t > b$,

$$a\int_{|x|\le t} x^2 dP(\|X_1\| \le x) < t^2 P(\|X_1\| > t). \tag{1}$$

If $a > 1$, then for all $t > b$

$$\begin{aligned} t^2\{P(\|X_1\| > t) - P(\|X_1\| > at)\} &\leq \int_{t < |x| \leq at} x^2 dP(\|X_1\| \leq x) \\ &\leq \int_{|x| \leq at} x^2 dP(\|X_1\| \leq x) \\ &\leq \frac{t^2 a^2}{a} P(\|X_1\| > at), \end{aligned}$$

which follows that

$$P(\|X_1\| > t) \leq \frac{a + a^2}{a} P(\|X_1\| > at). \tag{2}$$

If $\{b_n\}$ is a monotone sequence of positive constants with $b_n \to \infty$, then there exists N such that for all $n > N$,

$$\frac{a}{a+\varepsilon^2} P(\|X_1\|\} > b_n) \leq P(\|X_1\| > \varepsilon b_n) \leq P(\|X_1\| > b_n) \quad \text{for} \quad \varepsilon > 1,$$

and

$$P(\|X_1\| > b_n) \leq P(\|X_1\| > \varepsilon b_n) \leq \frac{\varepsilon^2 a + 1}{\varepsilon^2 a} P(\|X_1\| > b_n) \quad \text{for} \quad \varepsilon < 1.$$

Thus if $\sum P(\|X_1\| > \varepsilon b_n)$ converges[resp. diverges] for some $\varepsilon > 0$, then it converges[resp. diverges] for every $\varepsilon > 0$.

The sufficiency : Assume that $\sum P(\|X_1\| > b_n) = \infty$, which implies that $\sum P(\|X_1\| > \varepsilon b_n) = \infty$ for every $\varepsilon > 0$. Since by Borel-Cantelli Lemma, $P(\|X_1\| > \varepsilon b_n \text{ i.o. }) = 1$ and since $\|X_n\| \leq \|S_n\| + \|S_{n-1}\|$, for every $\varepsilon > 0$, $(P(\|S_n\| > \frac{\varepsilon}{2} b_n \text{ i.o. }) = 1$.

The necessity : Assume that $\sum P(\|X_1\| > b_n) < \infty$, which implies that $\sum P(\|X_1\| > \varepsilon b_n) < \infty$ for every $\varepsilon > 0$. Then by Borel-Cantelli Lemma,

$$\frac{S_n - T_n}{b_n} \to 0 \text{ a.s.} \tag{3}$$

And by (1), $E\|X_n'\|^2 \leq Cb_n^2 P(\|X_1\| > b_n)$ for sufficiently large n, which implies that

$$\sum \frac{E\|X_n'\|^2}{b_n^2} < \infty. \tag{4}$$

Given $\varepsilon > 0$,

$$P(\|\frac{T_n}{b_n}\| > \varepsilon) \leq \frac{1}{\varepsilon^2}\frac{1}{b_n^2} E\|T_n\|^2 \leq \frac{C}{\varepsilon^2}\frac{\sum_{i=1}^{n} E\|X_n'\|^2}{b_n^2}.$$

Thus by (4) $T_n/b_n \to 0$ in probability. But from Choi and Sung's(1988) theorem, we have that T_n/b_n converges almost surely, which completes the proof.

Theorem 2. Suppose that $\{X_n\}$ belongs to the domain of attraction of a nonnormal stable law of index $\alpha \neq 1$ which is not one-sided and that $EX_i = 0$ if $E\|X_i\| < \infty$. Let $\{b_n\}$ be a sequence of positive constants with $b_n \to \infty$ as $n \to \infty$. Then

$$\frac{S_n}{b_n} \to \text{ a.s. if and only if } \sum P(\|X_1\| > b_n) < \infty$$

Proof. The proof of sufficiency is the same as in Theorem 1. Now we assume that $\sum P(\|X_1\| > b_n) < \infty$. Under the conditions of theorem, we may write for $x > 0$, by Gnedenko and Kolmogorov's theorem (See, Gnedenko and Kolmogorov, 1954),

$$P(\|X_1\| > x) = M(x)/x^{\alpha},$$

where $M(x)$ is a function which is slowly varying as $x \to \infty$. We consider the two cases $\alpha < 1$ and $1 < \alpha < 2$.

The case $\alpha < 1$:

$$\|EX_n'\| \le E\|X_n'\| = \int_{|x| \le b_n} x dP(\|X_1\| \le x) \le \int_0^{b_n} P(\|X_1\| > x) dx. \quad (5)$$

By intergration by parts and standard propeties of slowly varying function (See Feller, 1966), we have

$$\int_0^{b_n} \frac{P(\|X_1\| > x)dx}{b_n P(\|X_1\| > b_n)} \to \frac{1}{1-\alpha} \text{ as } n \to \infty.$$

We can choose a constant $C_1 > 0$ such that

$$\frac{\|EX_n'\|}{b_n} \le C_1 P(\|X_1\| > b_n) \text{ for each } n. \quad (6)$$

The case $1 < \alpha < 2$:

Since $EX_1 = 0$,

$$\|EX_n'\| = \|EX_n''\| \le E\|X_n''\| = -\int_{b_n}^{\infty} x dP(\|X_1\| > x) = b_n P(\|X_1\| > b_n) + \int_{b_n}^{\infty} P(\|X_1\| > x) dx.$$

But from the integration by parts and standard properties of slowly varying functions, we can choose a constants $C_2 > 0$ satisfying

$$\frac{\|X_n'\|}{b_n} \le C_2 P(\|X_1\| > b_n). \quad (7)$$

If we take $C = \max\{C_1, C_2\}$, from (6) and (7) we obtain

$$\sum_{n=1}^{\infty} \frac{\|EX_n'\|}{b_n} \le C \sum_{n=1}^{\infty} P(\|X_1\| > b_n) < \infty, \quad (8)$$

which implies by the Kronecker Lemma that $ET_n/b_n \to 0$.

It remains to show that $(T_n - ET_n)/b_n \to 0$ a.s. Since $E\|X'_n - EX'_n\|^2 \le E(\|X'_n\| + E\|X'_n\|)^2 \le 4E\|X'_n\|^2$,

$$\sum_{j=1}^{\infty} \frac{E\|X'_j - EX'_j\|^2}{b_j^2} \le 4\sum_{j=1}^{\infty} \frac{E\|X'_j\|^2}{b_j^2} < \infty \quad \text{by (4)},$$

thus

$$\frac{E\|\sum_{j=1}^{n}(X'_j - EX'_j)\|^2}{b_j^2} \le C\frac{\sum_{j=1}^{n} E\|X'_j - EX'_j\|^2}{b_n^2} \to 0.$$

If we apply Choi and Sung's result, we obtain that

$$\frac{T_n - ET_n}{b_n} \to 0 \text{ a.s.},$$

which completes the proof.

Remark. In the case $\alpha < 1$ of Theorem 2, the condition of R-type 2 is not necessary. In fact, by (5) $\sum_{n=1}^{\infty} E\|T_n - ET_n\|/b_n \le 2\sum_{n=1}^{\infty} E\|X'_n\|/b_n < \infty$, which implies by Kronceker's Lemma

$$\frac{\sum_{j=1}^{n} E\|T_n - ET_n\|}{b_n} \to 0.$$

By applying Choi and Sung's result, $(T_n - ET_n)/b_n \to 0$ a.s., from which we can obtain the result.

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