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# An Invariance Principle of Uniform CLT for the Baker's Transformation

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## Abstract

The baker's transformation is an ergodic transformation defined on the half open unit square. This paper considers the limiting behavior of the partial sum process of a martingale sequence constructed from the baker's transformation in the context of an invariance principle of a uniform central limit theorem.

## 1. Introduction

Let $\Omega=[0,1)\times[0,1)$ be the sample space, $\mathfrak{I}$ be the Borel sets and $P$ be the Lebsgue measure. The baker's transformation on the half open unit square is defined by

$$\phi(x,y) \longrightarrow (2x, \frac{y}{2}),\ 0 \le x < \frac{1}{2},$$

and

$$\phi(x,y) \rightarrow (2x-1, \frac{y+1}{2}),\ \frac{1}{2} \le x < 1.$$

The name of the transformation comes from imagining the unit square to be bread dough that is stretched in the $x$-direction until it is twice as long and half as high and then cut along $x=1$ to make two loaves. We can think about $(\ldots, x_{-1}, x_0, x_1, \ldots) \in \{0,1\}^Z$ as a point $(x,y)$ in the half open unit square $[0,1)\times[0,1)$by putting $x=\sum_{i=0}^{\infty}\frac{x_i}{2^{i+1}}$ and $y=\sum_{i=1}^{\infty}\frac{x_{-i}}{2^i}$. It is known that the transformation is ergodic (See Durrett, 1991).

For $t\in[0,1]$, we define

$$\begin{aligned} f_t(x,y) &= 1_{[0,t]}(y),\ 0 \le x < \frac{1}{2} \\ &= -1_{[0,t]}(y),\ \frac{1}{2} \le x < 1 \end{aligned}$$

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Consider the class of functions $\Pi = \{f_t(x, y)\}_{0\le t\le 1}$. We denote $\phi^i(x, y) = (x_i, y_i)$ for $(x, y) \in [0, 1)\times[0, 1)$, $i = 0, 1, \cdots$. Define

$$Z_n(t) := Z_n(f_t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_t(x_i, y_i),\ f_t \in \Pi.$$

We are interested in the limiting behavior of the process $S_n$. The following Theorem 1 is proved in Levental (1989) using a result of a Uniform CLT for uniformly bounded families of martingale differences.

**Theorem 1.** The distribution of $\{Z_n(t) : 0 \le t \le 1\}$ converges to that of $\{B(t) : 0 \le t \le 1\}$, where $B(t)$ is standard Brownian motion.

We generalize Theorem 1 to the context of an invariance principle of a uniform central limit theorem.

## 2. An Invariance Principle of Uniform CLT

In the proof of an invariance principle of a uniform central limit theorem for the baker's transformation we use a result of martingale difference. So we introduce the following setup : $(\Omega = S^Z, \mathfrak{I} = B^Z, P)$ will be our basic probability space. We denote by $T$ the left shift on $\Omega$. We assume that $P$ is invariant under $T$, i.e., $PT^{-1} = P$, and that $T$ is ergodic. We denote by $X = (\ldots, X_{-1}, X_0, X_1, \ldots)$ the coordinate maps on $\Omega$. From our assumptions it follows that $\{X_i\}_{i\in Z}$ is a stationary and ergodic process. Next we define for each $i \in Z$ a $\sigma$-fields

$$M_i := \sigma(X_j : j \le i)$$

and

$$H_i := \{f \in \Omega \to R : f \in M_i \ and\ f \in L^2(\Omega)\}.$$

We also denote for each $f \in L^2(\Omega)$,

$$E_{i-1}(f) := E(f \mid M_{i-1}),$$

and

$$H_0 \ominus H_{-1} := \{f \in H_0 : E(f \cdot g) = 0 \text{ for each } g \in H_{-1}\}.$$

Finally for every $f, g \in L^2(\Omega)$ we put $d(f, g) := [E(f-g)^2]^{\frac{1}{2}}$.

Let $F \subseteq H_0 \ominus H_{-1}$. From our setup it follows that for every $f \in F$, $\{f(T^i(X)), M_i\}$ is a stationary martingale difference sequence. We write $V_i := T^i(X)$, and $V := T^0(X)$ $(= X)$. Let

$$S_n(f, t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f(V_i),$$

where $[a]$ denotes the integer part of $a$.

Let $\delta > 0$. For $F \subseteq L^2(\Omega)$ we define the covering number with bracketing $\nu^B(\delta, F, d)$, as the smallest $n$ for which there exists $\{f^l_{0,\delta}, f^u_{0,\delta}, \cdots, f^l_{n,\delta}, f^u_{n,\delta}\} \subseteq L^2(\Omega)$ so that for every $f \in F$ there exist some $0 \le i \le n$ satisfying $f^l_{i,\delta} \le f \le f^u_{i,\delta}$ and $\|f^u_{i,\delta} - f^l_{i,\delta}\| < \delta$ (See Dudley, 1984).

For a function $\varphi: F \to R$, let $\|\varphi\|_F := \sup_{f \in F} |\varphi(f)|$ and let $\|\varphi\|_\delta := \sup_{(\delta)} |\varphi(f) - \varphi(g)|$ where $(\delta) := \{(f, g) \in F \times F : d(f, g) < \delta\}$. The following proposition and corollary appear in Bae (1993).

**Proposition 1.** Assume that (a) $\int_0^1 [\ln \nu^B(u, F, d)]^{1/2} du < \infty$,

and (b) there exists a constant $D > 0$ such that

$$P^*\{\sup_{f, g \in H_0} \sum_{i=1}^{n} \frac{E_{i-1}[f(V_i) - g(V_i)]^2}{n\, d^2(f, g)} > D\} \to 0.$$

Then for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\limsup_n P^*\{\sup_{0 \le t \le 1} \|S_n(\cdot, t)\|_\delta > \varepsilon\} < \varepsilon,$$

where $P^*$ denotes outer probability measure.

**Corollary 1.** Under the assumptions of Proposition 1,

$$S_n \Rightarrow G,$$

where $G(f,t)$ ia a Gaussian process with $EG(f,t)=0$ and $EG(f_1,t_1)G(f_2,t_2) = (t_1 \wedge t_2) Ef_1(X) f_2(X)$ which is uniformly continuous in $(f,t)$ a.s. with respect to the metric $e$ defined by $e((f_1,t_1),(f_2,t_2)) = \max\{d(f_1,f_2),\ |t_1-t_2|\}$.

Now we generalize Theorem 1 to the context of an invariance principle of a uniform central limit theorem for the baker's transformation.

Define

$$S_n(f_t, s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} f_t(x_i, y_i), \quad f_t \in G, \quad s \in [0,1],$$

where $[a]$ is the integer part of $a$ as before.

**Theorem 2.** The distribution of $\{S_n(f_t, s) : f_t \in \Pi,\ s \in [0,1]\}$ converges to that of $\{G(f_t, s) : f_t \in \Pi,\ s \in [0,1]\}$ where $G$ is a Gaussian process with $EG(f_t, s) = 0$ and $EG(f_{t_1}, s_1) G(f_{t_2}, s_2) = (t_1 \wedge t_2)(s_1 \wedge s_2)$ where $a \wedge b$ denotes the minimum of $a$ and $b$.

The following lemma appears in Pollard (1984).

**Lemma 1.** (Bernstein's Inequality) Let $Y_1, \ldots, Y_n$ be independent random variables with zero means and bounded ranges : $|Y_i| \le M$. Write $\sigma_i^2$ for the variance of $Y_i$ . Suppose $V \ge \sigma_1^2 + \cdots + \sigma_n^2$. Then for each $\eta > 0$,

$$P\{|\sum_{i=1}^{n} Y_i| > \eta\} \le 2\exp\{-\frac{\frac{\eta^2}{2}}{V + \frac{M\eta}{3}}\}.$$

**Proof of Theorem 2.** We observe that $E(f_{t(x,y)} | y) = 0$ for each $0 \le t \le 1$, which means that $\Pi = \{f_t(x,y) : 0 \le t \le 1\} \subseteq H_0 \ominus H_{-1}$. For $0 \le t_1 < t_2 \le 1$ and $0 \le s_1 < s_2 \le 1$ we have

$$E(\frac{1}{\sqrt{n}} \sum_{i=1}^{[ns_1]} f_{t_1}(x_i, y_i) \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns_2]} f_{t_2}(x_i, y_i)) = \frac{[ns_1]}{n} t_1$$

whose limit is the covariance structure of a Gaussian process. We apply Proposition 1 to the class of functions $\Pi = \{f_t(x,y) : 0 \le t \le 1\}$. Now we verify the conditions of Proposition 1. Since there is a one to one correspondence between $\Pi = \{f_t(x,y) : 0 \le t \le 1\}$ and $[0,1]$ the integralability condition of covering number with bracketing is obvious. Next note that the metric $d$ in this case is given by $d^2(f_{t_1}, f_{t_2}) = |t_1 - t_2|$.

Note also that $E_{i-1}(f_{t_1} - f_{t_2})^2(x_i, y_i) = 1_{[t_1,t_2]}(y_i)$. Condition (b) will follow from: There exist $D > 0$ such that

$$P\{\sup_{t\in[0,1]} \sum_{i=1}^{n} \frac{1_{[0,t]}(y_i)}{nt} > D\} \to 0, \quad as\ n \to \infty.$$

It will be enough to show that

$$P\{\sup_{0\le k\le[\sqrt{n}]-1} \sum_{i=1}^{n} 1_{[\frac{k}{\sqrt{n}},\frac{k+1}{\sqrt{n}}]}(y_i) > 8\sqrt{n}\} \to 0, \quad as\ n \to \infty. \tag{2.1}$$

**Claim** $$2^m P\{\sum_{i=1}^{2^{2m}} (1_{[0,\frac{1}{2^m}]}(y_i) - \frac{1}{2^m}) > 2^m\} \to 0, \ as\ m \to \infty.$$

To prove the claim we observe that

$$Y_k = \sum_{2km < i \le (2k+1)m} (1_{[0,\frac{1}{2^m}]}(y_i) - \frac{1}{2^m}), \quad k = 0, 1, 2, \cdots.$$

is an i.i.d. sequence since the $Y_k$ depend on disjoint subsets of the sequence of $0's$ and $1's$. Note that $EY_k = 0$, and

$$\begin{aligned} Var(Y_k) &= Var(\sum_{2km < i \le (2k+1)m} 1_{[0,\frac{1}{2^m}]}(y_i)) \\ &\le E(\sum_{2km < i \le (2k+1)m} 1_{[0,\frac{1}{2^m}]}(y_i)) \\ &\le E(m \sum_{2km < i \le (2k+1)m} 1_{[0,\frac{1}{2^m}]}(y_i)) \\ &\le \frac{m^2}{2^m}. \end{aligned}$$

Apply Lemma 1 with $V=m^2 2^m$, an upper bound of $\sum_{k=0}^{[\frac{2^{2m}}{2m}]} Y_k$, $M=m$, and $\eta=2^{m-1}$ to have

$$2^m P\{\sum_{k=0}^{[\frac{2^{2m}}{2m}]} Y_k > 2^{m-1}\} \to 0, \text{ as } m\to\infty, \tag{2.2}$$

as follows from

$$2^m P\{\sum_{k=0}^{[\frac{2^{2m}}{2m}]} Y_k > 2^{m-1}\} \le 2^m \cdot 2 \cdot \exp\{-\frac{\frac{2^{2m-2}}{2}}{m^2 2^m + \frac{m2^{m-1}}{3}}\}$$

$$= 2^{m+1} \cdot \exp\{-\frac{2^m}{8m^2+\frac{4m}{3}}\}$$

$$\to 0, \text{ as } m\to\infty.$$

Similarly for the $Z_k$ defined by

$$Z_k = \sum_{(2k+1)m < i \le (2k+2)m} (1_{[0,\frac{1}{2^m}]}(y_i) - \frac{1}{2^m}), \ k=0,1,2,\cdots,$$

we have

$$2^m P\{\sum_{k=0}^{[\frac{2^{2m}}{2m}]} Z_k > 2^{m-1}\} \to 0, \text{ as } m\to\infty. \tag{2.3}$$

Claim follows from (2.2) and (2.3). Using the representation of $0's$ and $1's$ we see that the distribution of $\{1_{[\frac{k}{\sqrt{n}},\frac{k+1}{\sqrt{n}}]}(y_i)\}_{1\le i\le 2^{2m}}$ does not depend on $k$, so that of $\sum_{i=1}^{2^{2m}} 1_{[\frac{k}{\sqrt{n}},\frac{k+1}{\sqrt{n}}]}(y_i)$ does not depend on $k$ either. Therefore from the claim we get

$$P\{\sup_{0\le k\le 2^{m-1}} \sum_{i=1}^{2^{2m}} 1_{[\frac{k}{\sqrt{n}},\frac{k+1}{\sqrt{n}}]}(y_i) > 2\cdot 2^m\} \to 0, \text{ as } m\to\infty.$$

We have proved (2.1) for $n=2^{2m}$, $m=1,2,\cdots$. For $2^{2m} < n < 2^{2(m+1)}$ we have

$$P\{\sup_{0\le k\le [\sqrt{n}]-1} \sum_{i=1}^{n} 1_{[\frac{k}{\sqrt{n}},\frac{k+1}{\sqrt{n}}]}(y_i) > 8\sqrt{n}\}$$

$$\le \sum_{k=0}^{[\sqrt{n}]-1} P\{\sum_{i=1}^{n} 1_{[\frac{k}{\sqrt{n}},\frac{k+1}{\sqrt{n}}]}(y_i) > 8\sqrt{n}\}$$

$$= [\sqrt{n}]\ P\{\sum_{i=1}^{n} 1_{[0,\frac{1}{\sqrt{n}}]}(y_i) > 8\sqrt{n}\}$$

$$\le 2^{m+1} P\{\sum_{i=1}^{2^{2(m+1)}} 1_{[0,\frac{1}{2^m}]}(y_i) > 8\cdot 2^m\}$$

$$\le 4\cdot 2^{m+1} P\{\sum_{i=1}^{2^{2m}} 1_{[0,\frac{1}{2^m}]}(y_i) > 2\cdot 2^m\}$$

$$\to 0, \text{ as } m\to\infty,$$

as follows from the claim. The proof of (2.1) is completed. This complete the proof of Theorem 2.

**Remark.** Observe that $Z_n(t) = S_n(f_t, 1)$ and $B(t) = G(f_t, 1)$. Apply Theorem 2 to get Theorem 1

# References


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