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Interior and Exterior Trimmed Means in an Exponential Model

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Abstract

In an exponential distribution, the properties of the interior and exterior trimmed means will be introduced, and reliability estimators using the two trimmed means will be compared with the UMVUE of reliability function through simulations.

1. The Interior and Exterior Trimmed Means

The trimmed mean proposed by Tukey had been defined as the average of observations remaining after a fixed number of outlying observation have been removed. Fan(1991) has considered the properties of the interior and exterior trimmed means in the Uniform distribution.

Here we shall consider properties of the interior and exterior means in an exponential distribution with mean σ, and also compare the UMVUE and estimators using the trimmed means for the reliability in the exponential distribution.

Let $X_1,\dots,X_n$ be a simple random sample(SRS) from $X \sim EXP(\sigma)$, an exponential distribution with mean σ, and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the corresponding order statistics.

The interior and exterior trimmed means are defined as followings;

For k=0,1,...,[n/2]-1,

$$T_k = (X_{(k+1)} + X_{(k+2)} + \dots + X_{(n-k)})/(n-2k) \text{ : (n-2k)-exterior trimmed mean,}$$

$$H_k = (X_{(1)} + \dots + X_{(k)} + X_{(n-k+1)} + \dots + X_{(n)})/2k \text{ : 2k-interior trimmed mean,}$$

where, [x] is the greatest integer not exceeding x. (1.1)

The statistic T_k means that we take an average of samples with size n-2k in which the first k and the last k order statistics are eliminated, and the statistic H_k takes an average of the first k and the last k order statistics.

From the results of Balakrishnan and Cohen(1991), it is known that:

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$$E(X_k) = \sigma \sum_{i=1}^{k} \frac{1}{n-i+1} = \alpha_{k:n}, \qquad VAR(X_k) = \sigma^2 \sum_{i=1}^{k} \frac{1}{(n-i+1)^2} = \beta_{k:n},$$

and

$$COV(X_{(i)}, X_{(j)}) = VAR(X_{(i)}), \qquad \text{if } i \le j, \tag{1.2}$$

where, i,j, and k=1,2,...,n.

From the expressions (1.1) and (1.2),the expectations and variances of the interior and exterior trimmed means can be obtained as follows;

$$E(T_k) = \sum_{i=k+1}^{n-k} \frac{1}{n-2k} \alpha_{i:n}, \qquad VAR(T_k) = \sum_{i=k+1}^{n-k} \frac{2n-2i+3}{(n-2k)^2} \beta_{i:n}, \tag{1.3}$$

and $E(H_k) = \frac{1}{2k}\left(\sum_{i=1}^{k} \alpha_{i:n} + \sum_{i=n-k+1}^{n} \alpha_{i:n}\right)$,

$$VAR(H_k) = \frac{1}{4k^2}\left[\sum_{i=1}^{k}(4k-2i+1)\beta_{i:n} + \sum_{i=n-k+1}^{n}(2n-2i+1)\beta_{i:n}\right]. \tag{1.4}$$

From the results (1.3) and (1.4), Tables 1.1 through 1.5 show the exact numerical values for biases and variances of the interior and exterior trimmed means in an exponential distribution, only when n=10,16,20,24,30.

Through the exact numerical evaluations, we get the following numerical results.

FACT 1. Let s=[n/2].

A.If s is even, then

a.The absolute bias of T_k is greater than that of H_{s-k}, for every $k < s/2$.

b.The absolute bias of T_k is equal to that of H_{s-k}, for every $k = s/2$.

c.The absolute bias of T_k is less than that of H_{s-k}, for every $k > s/2$.

B.If s is odd, then

a.The absolute bias of T_k is greater than that of H_{s-k}, for every $k \le (s-1)/2$.

b.The absolute bias of T_k is less than that of H_{s-k}, for every $k > (s-1)/2$.

FACT 2. a. $VAR(T_k) \le VAR(\overline{X}) \le VAR(H_{n-k})$, for each k=1,2,...,s-1.

b. The variance of H_k is decreasing as k is increasing.

From the Fact 2, the ordering relations of variances for the two trimmed means in an exponential distribution are slightly different from those of Fan(1991) in the Uniform distribution.

The following result is applied to consider the reliability in an exponential distribution later. The tables 1.1 through 1.5 show the following numerical result.

FACT 3. $\min_k VAR(T_k) = VAR(T_{k(0)})$, $k(0)=[n/5]$ only when n=10,16,20,24,30.

Next, in an exponential distribution with mean σ we shall consider the reliability estimators, which are proposed as follows;

$$\hat{R}_{1k}(t)=\exp(-\frac{t}{T_k}), \quad \hat{R}_{2k}(t)=\exp(-\frac{t}{H_k}), \quad k=1,2,\ldots,s-1,$$

and $\hat{R}_U(t)=(1-\frac{t}{S_n})^{n-1}$, for $t < S_n$, where, $S_n=\sum_{i=1}^{n} X_i$. (Zacks and Even(1966))

Tables 2.1 through 2.4 show the simulated values for the biases and mean square errors(MSE) of the proposed three reliability estimators in the exponential distribution, when n=10(10)30. Through the simulations, we can get the following simulated results.

FACT 4. a. The bias of $\hat{R}_{1k}(t)$ is smaller than that of $\hat{R}_{2k}(t)$, $1 \le k \le s-1$.

b. The variance of $\hat{R}_{2k}(t)$ is smaller than that of $\hat{R}_{1k}(t)$, $1 \le k \le s-1$.

c. The estimator $\hat{R}_{1k(0)}(t)$ is more efficient than the UMVUE of reliability, where, k(0)=n/5 only when n=10(10)40.

From the Fact 4.c, the reliability estimator $\hat{R}_{1k(0)}(t)$ would be more useful than the UMVUE in reliability estimation for an exponential distribution with mean σ.

Table 1.1 The biases, variances and MSE's of T_k and H_k (n=10, σ=2)

Trimmed mean	BIAS	VAR	MSE
T_1 H_4	0.25724 0.12718	0.34511 0.46263	0.41128 0.47780
T_2 H_3	0.38968 0.31138	0.33088 0.58609	0.48993 0.68305
T_3 H_2	0.46706 0.58452	0.35164 0.84520	0.56979 1.18686
T_4 H_1	0.50873 1.02897	0.38463 1.57977	0.64344 2.63855

Table 1.2 The biases, variances and MSE's of T_k and H_k (n=16, σ=2)

Trimmed mean	BIAS	VAR	MSE
T_1	0.20618	0.21856	0.26107
H_7	0.07847	0.27228	0.27844
T_2	0.32552	0.20834	0.31430
H_6	0.17779	0.30669	0.33829
T_3	0.40689	0.20610	0.37166
H_5	0.30352	0.35910	0.45122
T_4	0.46484	0.20891	0.42499
H_4	0.46484	0.44133	0.65741
T_5	0.50587	0.21582	0.47172
H_3	0.67815	0.58016	1.04005
T_6	0.53338	0.22682	0.51131
H_2	0.97656	0.85162	1.80529
T_7	0.54926	0.24332	0.54501
H_1	1.44323	1.59607	3.67898

Table 1.3 The biases, variances and MSE's of T_k and H_k (n=20, σ=2)

Trimmed mean	BIAS	VAR	MSE
T_1	0.18308	0.17664	0.21016
H_9	0.06250	0.21383	0.21774
T_2	0.29351	0.16769	0.25384
H_8	0.13809	0.23354	0.27168
T_3	0.37201	0.16410	0.30249
H_7	0.22933	0.26111	0.31370
T_4	0.43091	0.16373	0.34941
H_6	0.33999	0.29994	0.41553
T_5	0.47587	0.16576	0.39221
H_5	0.47587	0.35611	0.58256
T_6	0.50999	0.16986	0.42995
H_4	0.64636	0.44157	0.85935
T_7	0.53510	0.17595	0.46228
H_3	0.86801	0.58321	1.33665
T_8	0.55236	0.18421	0.48931
H_2	1.17406	0.85713	2.23555
T_9	0.56246	0.19558	0.51194
H_1	1.64774	1.60366	4.31871

Table 1.4 The biases, variances and MSE's of T_k and H_k (n=24, σ=2)

Trimmed mean	BIAS	VAR	MSE
T_1	0.16524	0.14855	0.17585
H_{11}	0.05192	0.17608	0.17878
T_2	0.26787	0.14089	0.21264
H_{10}	0.11284	0.18878	0.20151
T_3	0.34281	0.13714	0.25466
H_9	0.18411	0.20561	0.23951
T_4	0.40077	0.13568	0.29629
H_8	0.26764	0.22789	0.29952
T_5	0.46671	0.13587	0.33542
H_7	0.36612	0.25771	0.39175
T_6	0.48341	0.13739	0.37107
H_6	0.48341	0.29853	0.53222
T_7	0.51256	0.14010	0.40282
H_5	0.62539	0.35646	0.74757
T_8	0.53529	0.14393	0.43046
H_4	0.80153	0.44348	1.08593
T_9	0.55232	0.14893	0.45399
H_3	1.02843	0.58652	1.64419
T_{10}	0.56418	0.15524	0.47354
H_2	1.33936	0.86169	2.65557
T_{11}	0.57117	0.16353	0.48976
H_1	1.81762	1.60933	4.91307

Table 1.5 The biases, variances and MSE's of T_k and H_k (n=30, σ=2)

Trimmed mean	BIAS	VAR	MSE
T_1	0.14488	0.12020	0.14120
H_{14}	0.04142	0.13924	0.14095
T_2	0.23778	0.11412	0.17066
H_{13}	0.08852	0.14677	0.15461
T_3	0.30747	0.11068	0.20522
H_{12}	0.14196	0.15624	0.17639
T_4	0.36290	0.10879	0.24049
H_{11}	0.20259	0.16809	0.20913
T_5	0.40825	0.10797	0.27464
H_{10}	0.27152	0.18293	0.25665
T_6	0.44591	0.10800	0.30684
H_9	0.35022	0.20168	0.32433
T_7	0.47736	0.10872	0.33659
H_8	0.44065	0.22569	0.41986
T_8	0.50360	0.11007	0.36368
H_7	0.54556	0.25706	0.55470
T_9	0.52533	0.11201	0.38798
H_6	0.66887	0.29928	0.74667
T_{10}	0.54305	0.11451	0.40941
H_5	0.81651	0.35849	1.02518
T_{11}	0.55713	0.11760	0.42799
H_4	0.99797	0.44669	1.44263

Continue

Trimmed mean	BIAS	VAR	MSE
T_{12} H_3	0.56783 1.22988	0.12130 0.59081	0.44373 2.10341
T_{13} H_2	0.57353 1.54556	0.12572 0.86697	0.45466 3.25573
T_{14} H_1	0.57982 2.02832	0.13128 1.61548	0.46747 5.72956

Table 2.1 Simulated biases and variances for the reliability estimators $\hat{R}_{1k}(t)$ and $\hat{R}_{2k}(t)$. (n=10,σ=1)

R(t) \ t		0.2	0.4	0.6	0.8	1.0
$\hat{R}_{11}(t)$	BIAS VAR	0.036758 0.001623	0.062413 0.004135	0.070241 0.006074	0.072665 0.000796	0.097784 0.008994
$\hat{R}_{24}(t)$	BIAS VAR	0.108809 0.000102	0.190783 0.000781	0.247504 0.000555	0.285103 0.000919	0.320483 0.001134
$\hat{R}_{12}(t)$	BIAS VAR	0.015413 0.003624	0.026078 0.008068	0.023907 0.010798	0.024730 0.011370	0.045106 0.013398
$\hat{R}_{23}(t)$	BIAS VAR	0.120144 0.000063	0.211593 0.000172	0.277881 0.000350	0.323163 0.000592	0.363833 0.000766
$\hat{R}_{13}(t)$	BIAS VAR	0.001142 0.004617	-0.002901 0.012717	-0.000334 0.012945	-0.004489 0.015341	0.011103 0.017688
$\hat{R}_{22}(t)$	BIAS VAR	0.133424 0.000030	0.236549 0.000092	0.314543 0.000181	0.370319 0.000312	0.416905 0.000444
$\hat{R}_{14}(t)$	BIAS VAR	-0.005877 0.005401	-0.032585 0.020680	-0.019630 0.015551	-0.028847 0.019170	-0.004369 0.021056
$\hat{R}_{21}(t)$	BIAS VAR	0.150511 0.000007	0.269147 0.000017	0.362031 0.000037	0.431708 0.000050	0.488193 0.000091

Table 2.2 Simulated biases and variances for the reliability estimators $\hat{R}_{1k}(t)$ and $\hat{R}_{2k}(t)$. (n=20,σ=1)

R(t) \ t		0.2	0.4	0.6	0.8	1.0
$\hat{R}_{11}(t)$	BIAS VAR	0.022040 0.000836	0.034286 0.002255	0.028323 0.004322	0.042055 0.004578	0.038301 0.004943
$\hat{R}_{29}(t)$	BIAS VAR	0.075555 0.000172	0.126988 0.000470	0.157287 0.001049	0.184042 0.001323	0.193414 0.001632

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R(t) \ t		0.2	0.4	0.6	0.8	1.0
$\hat{R}_{12}(t)$	BIAS VAR	0.005574 0.001122	0.010602 0.002948	-0.003725 0.005544	0.010785 0.005258	0.003971 0.005543
$\hat{R}_{28}(t)$	BIAS VAR	0.081816 0.000146	0.138085 0.000401	0.172881 0.000943	0.202022 0.001188	0.214257 0.001477
$\hat{R}_{13}(t)$	BIAS VAR	-0.006690 0.001479	-0.007409 0.003557	-0.026547 0.006672	-0.010937 0.005916	-0.019285 0.006301
$\hat{R}_{27}(t)$	BIAS VAR	0.088758 0.000122	0.150256 0.000335	0.190065 0.000819	0.221974 0.001050	0.237535 0.001302
$\hat{R}_{14}(t)$	BIAS VAR	-0.017148 0.001896	-0.022945 0.004309	-0.044655 0.007416	-0.028756 0.006328	-0.040797 0.007202
$\hat{R}_{26}(t)$	BIAS VAR	0.096204 0.000100	0.163766 0.000279	0.209000 0.000688	0.244269 0.000906	0.263789 0.001127
$\hat{R}_{15}(t)$	BIAS VAR	-0.026371 0.002368	-0.036129 0.005330	-0.060965 0.008239	-0.044822 0.007798	-0.059660 0.008028
$\hat{R}_{25}(t)$	BIAS VAR	0.104248 0.000083	0.178632 0.000233	0.230175 0.000567	0.269265 0.000757	0.293174 0.000937
$\hat{R}_{16}(t)$	BIAS VAR	-0.034595 0.002924	-0.046095 0.006200	-0.073532 0.009025	-0.057864 0.007410	-0.072619 0.008151
$\hat{R}_{24}(t)$	BIAS VAR	0.113150 0.000064	0.295235 0.000183	0.253996 0.000430	0.297703 0.000604	0.326237 0.000735
$\hat{R}_{17}(t)$	BIAS VAR	-0.040658 0.003499	-0.053149 0.006999	-0.084040 0.009639	-0.068235 0.008346	-0.081384 0.008300
$\hat{R}_{23}(t)$	BIAS VAR	0.123270 0.000047	0.214287 0.000128	0.281944 0.000300	0.331757 0.000431	0.365598 0.000534
$\hat{R}_{18}(t)$	BIAS VAR	-0.043514 0.004089	-0.059522 0.007783	-0.092149 0.009861	-0.076552 0.009599	-0.088649 0.008750
$\hat{R}_{22}(t)$	BIAS VAR	0.135310 0.000028	0.237549 0.000067	0.315988 0.000176	0.374448 0.000260	0.416386 0.000330
$\hat{R}_{19}(t)$	BIAS VAR	-0.045939 0.004510	-0.063102 0.008590	-0.097122 0.010568	-0.082097 0.010502	-0.091944 0.009273
$\hat{R}_{21}(t)$	BIAS VAR	0.150688 0.000007	0.268514 0.000013	0.361456 0.000038	0.432743 0.000054	0.486128 0.000078

Table 2.3 Simulated biases and variances for the reliability estimators $\hat{R}_{1k}(t)$ and $\hat{R}_{2k}(t)$. (n=30,σ=1)

R(t) \ t		0.2	0.4	0.6	0.8	1.0
$\hat{R}_{11}(t)$	BIAS VAR	0.017795 0.000711	0.030250 0.001720	0.025078 0.003069	0.022795 0.003370	0.034380 0.003661
$\hat{R}_{214}(t)$	BIAS VAR	0.058954 0.000250	0.100855 0.000611	0.118847 0.001116	0.130875 0.001467	0.145842 0.001805
$\hat{R}_{12}(t)$	BIAS VAR	0.005396 0.000870	0.009902 0.002025	0.001141 0.003900	0.003179 0.003824	0.008310 0.003874
$\hat{R}_{213}(t)$	BIAS VAR	0.063001 0.000207	0.107967 0.000564	0.128751 0.001024	0.141837 0.001382	0.158027 0.001740
$\hat{R}_{13}(t)$	BIAS VAR	-0.003579 0.001033	-0.004942 0.002456	-0.016687 0.004576	-0.022064 0.004183	-0.011193 0.004205
$\hat{R}_{212}(t)$	BIAS VAR	0.067327 0.000190	0.115686 0.000520	0.139063 0.000934	0.152869 0.001308	0.171336 0.001664

Continue ...

R(t) \ t		0.2	0.4	0.6	0.8	1.0
$\hat{R}_{14}(t)$	BIAS VAR	-0.011206 0.001202	-0.016837 0.002713	-0.030873 0.005139	-0.037721 0.004655	-0.026800 0.004486
$\hat{R}_{211}(t)$	BIAS VAR	0.071906 0.000172	0.123983 0.000480	0.150219 0.000853	0.167090 0.001240	0.185761 0.001585
$\hat{R}_{15}(t)$	BIAS VAR	-0.017398 0.001342	-0.026196 0.003010	-0.042897 0.005734	-0.050304 0.005014	-0.039903 0.004759
$\hat{R}_{210}(t)$	BIAS VAR	0.076785 0.000160	0.132936 0.000440	0.162343 0.000769	0.181510 0.001167	0.201622 0.001499
$\hat{R}_{16}(t)$	BIAS VAR	-0.022613 0.001474	-0.036738 0.003377	-0.054310 0.006246	-0.060913 0.005313	-0.050983 0.004983
$\hat{R}_{29}(t)$	BIAS VAR	0.081968 0.000140	0.142377 0.000404	0.175294 0.000681	0.197267 0.001084	0.218998 0.001409
$\hat{R}_{17}(t)$	BIAS VAR	-0.027341 0.001604	-0.044679 0.003733	-0.064272 0.006749	-0.069392 0.005507	-0.061212 0.005211
$\hat{R}_{28}(t)$	BIAS VAR	0.087559 0.000126	0.152456 0.000364	0.189270 0.000599	0.214412 0.000988	0.237882 0.001317
$\hat{R}_{18}(t)$	BIAS VAR	-0.031815 0.001772	-0.051433 0.004036	-0.072729 0.007120	-0.076613 0.005752	-0.069334 0.005433
$\hat{R}_{27}(t)$	BIAS VAR	0.093561 0.000111	0.163470 0.000325	0.204503 0.000527	0.233316 0.000898	0.258768 0.001203
$\hat{R}_{19}(t)$	BAIS VAR	-0.035666 0.001947	-0.058005 0.004376	-0.080371 0.007513	-0.082521 0.005926	-0.076340 0.005747
$\hat{R}_{26}(t)$	BIAS VAR	0.100123 0.000097	0.175428 0.000290	0.221182 0.000454	0.254368 0.000791	0.281743 0.001077
$\hat{R}_{110}(t)$	BIAS VAR	-0.039287 0.107375	-0.063493 0.188529	-0.086918 0.239719	-0.086720 0.277869	-0.081469 0.308041
$\hat{R}_{25}(t)$	BIAS VAR	0.002111 0.000083	0.004825 0.000249	0.007842 0.000381	0.006071 0.000690	0.005934 0.000944
$\hat{R}_{111}(t)$	BIAS VAR	-0.042277 0.115484	-0.066454 0.203448	-0.091723 0.261258	-0.089787 0.304924	-0.085779 0.338451
$\hat{R}_{24}(t)$	BIAS VAR	0.002239 0.000067	0.005086 0.000202	0.008216 0.000311	0.006310 0.000568	0.006214 0.000795
$\hat{R}_{112}(t)$	BIAS VAR	-0.044993 0.002380	-0.068832 0.005439	-0.096511 0.008661	-0.091482 0.006534	-0.089933 0.006410
$\hat{R}_{23}(t)$	BIAS VAR	0.124749 0.000048	0.220787 0.000154	0.286647 0.000225	0.336823 0.000423	0.375016 0.000588
$\hat{R}_{113}(t)$	BIAS VAR	-0.046546 0.002423	-0.070218 0.005852	-0.099147 0.009090	-0.093113 0.006871	-0.093504 0.006661
$\hat{R}_{22}(t)$	BIAS VAR	0.136010 0.000028	0.241935 0.000095	0.317978 0.000120	0.376692 0.000252	0.421420 0.000381
$\hat{R}_{114}(t)$	BIAS VAR	-0.048217 0.002430	-0.072108 0.006363	-0.100597 0.009240	-0.095986 0.006961	-0.096684 0.007028
$\hat{R}_{21}(t)$	BIAS VAR	0.150677 0.000007	0.270064 0.000023	0.360925 0.000030	0.432665 0.000064	0.487191 0.000076

Table 2.4 Simulated MSE's of reliability estimators $\hat{R}_{1k_{(0)}}(t)$ and $\hat{R}_U(t)$ of reliability function. (σ=1,k(0)=n/5)

n	t	MSE	
		$\hat{R}_{1k(0)}(t)$	$\hat{R}_U(t)$
10	0.2	0.00386	0.01143
	0.4	0.00875	0.03441
	0.6	0.01137	0.05825
	0.8	0.01198	0.07828
	1.0	0.01543	0.09487
20	0.2	0.00219	0.00584
	0.4	0.00484	0.01625
	0.6	0.00941	0.02553
	0.8	0.00715	0.03628
	1.0	0.00887	0.03889
30	0.2	0.00198	0.00367
	0.4	0.00471	0.01067
	0.6	0.00919	0.01453
	0.8	0.00902	0.01796
	1.0	0.00785	0.02239
40	0.2	0.00109	0.00213
	0.4	0.00609	0.00618
	0.6	0.00524	0.00998
	0.8	0.00983	0.01117
	1.0	0.00532	0.01293

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