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On Certain Patterned Matrices with Statistical Applications

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Abstract

This paper presents the interesting properties of a certain patterned matrix that plays an significant role in the multi-way balanced designs. The necessary and sufficient condition on the existence of the inverse of the patterned matrix and its determinant are described. In special cases of the patterned matrix, explicit formulas for its inverse, determinant and the characteristic equation are obtained.

1. Introduction

A matrix is called a patterned matrix if the matrix has a particular structure or a pattern. For a patterned matrix the task of finding the inverse or the determinant can be significantly reduced. In this paper we consider the following patterned matrix

$$C_{IJ}^{k} \equiv \begin{bmatrix} c_{11}I_1 & c_{12}J_{12} & \cdots & c_{1k}J_{1k} \\ c_{12}J_{12}^T & c_{22}I_2 & \cdots & c_{2k}J_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1k}J_{1k}^T & c_{2k}J_{2k}^T & \cdots & c_{kk}I_k \end{bmatrix}, \tag{1.1}$$

where I_i $(i=1,\ldots,k)$ is the identity matrix of size $s_i(>0)$, J_{ij} $(i\neq j)$ is an $s_i \times s_j$ matrix of ones and T denotes the transpose. Now, we assume that $c_{ii} \neq 0$ for all i in case $s_i = 1$. To get the inverse and the determinant of C_{IJ}^{k}, we shall deduce N_k from C_{IJ}^{k}, which consists of constants and size for blocks:

$$N_k \equiv \begin{bmatrix} c_{11} & s_2c_{12} & \cdots & s_kc_{1k} \\ s_1c_{12} & c_{22} & \cdots & s_kc_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ s_1c_{1k} & s_2c_{2k} & \cdots & c_{kk} \end{bmatrix}. \tag{1.2}$$

In this paper, we propose the exact formulas for the inverse and the determinant of C_{IJ}^{k} by manipulating N_k, which has smaller size than C_{IJ}^{k}. When the patterned matrix C_{IJ}^{k} is large,

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the proposed method reduces much of the computational effort. Throughout this paper C^k_{IJ} will be called an IJ-patterned matrix of order k (≥ 2) and N_k called the reduced form of C^k_{IJ}.

When $k = 2$, C^2_{IJ}, which is partitioned into 2^2 blocks, plays an important role in the two-way classification model with equal numbers in the subcells in experimental design theory, and the necessary and sufficient condition on the existence of the inverse of C^2_{IJ} is known. If $(C^2_{IJ})^{-1}$ exists, the explicit formulas of the elements of the inverse matrix and the determinant of C^2_{IJ} are given in Graybill(1983).

In section 2, some results on the IJ-patterned matrix of order k (≥ 2) are described; the necessary and sufficient condition on the existence of the inverse is given, and especially explicit formulas for its inverse and determinant are represented by handling the reduced form N_k having smaller size. These results are useful in finding the explicit form of a generalized inverse of X^TX where X is the design matrix for the balanced multi-way factorial design. And the explicit form can be used in computing the sums of squares for testing the estimable hypotheses.

2. Some results on a certain patterned matrix

We shall state and prove some results on C^k_{IJ} based on the N_k. The following lemma is useful in deriving the necessary and sufficient condition on the existence of the inverse of C^k_{IJ}.

Lemma 2.1. If $|N_k| = 0$ or $c_{ii} = 0$ for some i, then $|C^k_{IJ}| = 0$, where $|\cdot|$ denotes the determinant.

Proof. First, suppose that $|N_k| = 0$. Let $C^k_{IJ}(i)$ be the i-th column block of C^k_{IJ}, that is, $C^k_{IJ}(i) = [\ c_{1i}J^T_{1i}\ ;\ \cdots\ ;\ c_{ii}I_i\ ;\ \cdots\ ;\ c_{ik}J_{ik}]^T$ and $c_s(i)$ be the column vector that is obtained by adding all the columns of $C^k_{IJ}(i)$. Then

$$c_s(i) = [\ s_i c_{1i} 1^T_{s_1}\ ;\ \cdots\ ;\ c_{ii} 1^T_{s_i}\ ;\ \cdots\ ;\ s_i c_{ik} 1^T_{s_k}]^T,$$

where 1_{s_i} is an $s_i \times 1$ vector of ones. Hence for some i, the linear dependence of i-th column of N_k implies the linear dependence of $c_s(i)$. Secondly if $c_{ii}=0$ for some i, then the columns of $C^k_{IJ}(i)$ are all same. This completes the proof.

The following theorem gives the necessary and sufficient condition on the existence, and the form of the inverse of C^k_{IJ} in (1.1).

Theorem 2.1. For C^k_{IJ} in (1.1) and N_k in (1.2), the inverse of C^k_{IJ} exists if and only if $|N_k| \neq 0$ and $c_{ii} \neq 0$ for all $i=1,\ldots,k$. If $(C^k_{IJ})^{-1}$ exists, it is given by

$$(C^k_{IJ})^{-1} = \begin{bmatrix} \frac{1}{c_{11}} I_1 + b_{11}J_1 & b_{12}J_{12} & \cdots & b_{1k}J_{1k} \\ b_{12}J^T_{12} & \frac{1}{c_{22}} I_2 + b_{22}J_2 & \cdots & b_{2k}J_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1k}J^T_{1k} & b_{2k}J^T_{2k} & \cdots & \frac{1}{c_{kk}} I_k + b_{kk}J_k \end{bmatrix}, \tag{2.1}$$

where J_i is an $s_i \times s_j$ matrix of ones and

$$b_{ij} = \begin{cases} [\,c_{ii}N_{ii} - |N_k|\,] \;/\; [\,s_i c_{ii} |N_k|\,] & \text{if } i=j, \\ N_{ij} \;/\; [\,s_i |N_k|\,] & \text{if } i<j, \end{cases} \tag{2.2}$$

and where N_{ij} is the cofactor of the (i,j)-th element of N_k.

Proof. The only if part follows from lemma 2.1. So only the if part will be proved. Suppose that $|N_k| \neq 0$ and $c_{ii} \neq 0$ for all i. Then we can construct $(C^k_{IJ})^{-1}$ in (2.1).

Claim: Let n_{ij} be the (i,j)-th element of N_k in (1.2) and N_{ij} be the cofactor of n_{ij}. Then the following relation holds for all i, j: $s_j N_{ij} = s_i N_{ji}$.

It suffices to prove the relation for the case $i<j$. Let $N_k(i,j)$ and $N^T_k(i,j)$ be the matrices that obtained by deleting i-th row and j-th column in N_k and N^T_k, respectively. Then by the property of the cofactor, we get $N_{ij} = (-1)^{i+j}|N_k(i,j)|$ and

$N_{ji}=(-1)^{j+i}|N_k(j,i)|=(-1)^{i+j}|N_k^T(i,j)|$. Because of $s_j\,|N_k(i,j)|=s_i\,|N_k^T(i,j)|$, we obtain $s_jN_{ij}=s_iN_{ji}$. This completes the proof of the claim.

To show that $(C^k_{IJ})^{-1}$ in (2.1) is actually the inverse of C^k_{IJ} in (1.1), we can show that $C^k_{IJ}(C^k_{IJ})^{-1}=(C^k_{IJ})^{-1}C^k_{IJ}=I$ by the above claim.

The following lemma shows that we can find the inverse of N_k when $(C^k_{IJ})^{-1}$ is given. The fact is also useful in deriving the relationship between the determinants of C^k_{IJ} and N_k.

Lemma 2.2. Let the patterned matrix D^k_{IJ} be defined by

$$D^k_{IJ}=\begin{bmatrix} a_{11}I_1+b_{11}J_1 & b_{12}J_{12} & \cdots & b_{1k}J_{1k} \\ b_{12}J^T_{12} & a_{22}I_2+b_{22}J_2 & \cdots & b_{2k}J_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1k}J^T_{1k} & b_{2k}J^T_{2k} & \cdots & a_{kk}I_k+b_{kk}J_k \end{bmatrix}$$

and l_{ij} be the (i,j)-th element of the inverse of the reduced form of $(D^k_{IJ})^{-1}$ when $(D^k_{IJ})^{-1}$ exists. Then l_{ij} is given by

$$l_{ij}=\begin{cases} a_{ii}+s_ib_{ii} & \text{if } i=j, \\ s_jb_{ij} & \text{if } i<j, \\ s_jb_{ji} & \text{if } i>j. \end{cases} \tag{2.3}$$

Proof. The proof follows immediately from the expression of b_{ij} in (2.2).

Remark 1. Let L_k be the $k\times k$ matrix whose (i,j)-th element is l_{ij} given in lemma 2.2. Then L_k is defined by the reduced form of D^k_{IJ} as opposed to C^k_{IJ}.

Theorem 2.2. For C^k_{IJ} in (1.1) and N_k in (1.2),

$$|C^k_{IJ}| \;=\; |N_k|\prod_{i=1}^{k}c_{ii}^{s_i-1}. \tag{2.4}$$

Proof. To prove the validity of (2.4), we first note that it is true for the case $k=2$ since $|C^2_{IJ}|=c_{11}^{s_1-1}c_{22}^{s_2-1}(c_{11}c_{22}-s_1s_2c^2_{12})=c_{11}^{s_1-1}c_{22}^{s_2-1}|N_2|$. Now we assume it is true for $k=n$, that is, $|C^n_{IJ}|=|N_n|\prod_{i=1}^{n}c_{ii}^{s_i-1}$. Consider the case $k=n+1$. If $|C^{n+1}_{IJ}|=0$, by theorem 2.1, the equation (2.4) holds for $k=n+1$. Next, if $|C^{n+1}_{IJ}|\neq 0$, then we have

$$|C^{n+1}_{IJ}|=\begin{vmatrix} C^n_{IJ} & C_{n+1} \\ C^T_{n+1} & c_{(n+1)(n+1)}I_{n+1}\end{vmatrix},$$

where $C^T_{n+1}=[\ c_{1(n+1)}J_{(n+1)1}\ ;\ c_{2(n+1)}J_{(n+1)2}\ ;\ \cdots\ ;\ c_{n(n+1)}J_{(n+1)n}]$. By the property of the determinant of a partitioned matrix, $|C^{n+1}_{IJ}|$ is also given by

$$|C^{n+1}_{IJ}|=|c_{(n+1)(n+1)}I_{n+1}|\ |C^n_{IJ}-C_{n+1}C^T_{n+1}/c_{(n+1)(n+1)}|.$$

Similarly,

$$|N_{n+1}|=\begin{vmatrix} N_n & w \\ z^T & c_{(n+1)(n+1)}\end{vmatrix}=c_{(n+1)(n+1)}\ |N_n-wz^T/c_{(n+1)(n+1)}|,$$

where $w^T=s_{n+1}[\ c_{1(n+1)}\ ;\ c_{2(n+1)}\ ;\ \cdots\ ;\ c_{n(n+1)}]$ and $z^T=[\ s_1c_{1(n+1)}\ ;\ s_2c_{2(n+1)}\ ;\ \cdots\ ;\ s_nc_{n(n+1)}]$. To complete the proof, we shall show that

$$|C^n_{IJ}-C_{n+1}C^T_{n+1}/c_{(n+1)(n+1)}|=|N_n-wz^T/c_{(n+1)(n+1)}|\prod_{i=1}^{n}c_{ii}^{s_i-1}. \tag{2.5}$$

Let $\hat{D}_{ij}$ be the (i,j)-th block of $C^n_{IJ}-C_{n+1}C^T_{n+1}/c_{(n+1)(n+1)}$ and $\hat{n}_{ij}$ be the (i,j)-th element of $N_n-wz^T/c_{(n+1)(n+1)}$. Then $\hat{D}_{ij}$ and $\hat{n}_{ij}$ are given by

$$\hat{D}_{ij}=\begin{cases} c_{ii}I_i-[\ s_{n+1}c^2_{i(n+1)}/c_{(n+1)(n+1)}]\ J_i & \text{if } i=j, \\ [\ c_{ij}-s_{n+1}c_{i(n+1)}c_{j(n+1)}/c_{(n+1)(n+1)}]\ J_{ij} & \text{if } i<j, \\ [\ c_{ji}-s_{n+1}c_{i(n+1)}c_{j(n+1)}/c_{(n+1)(n+1)}]\ J_{ji} & \text{if } i>j, \end{cases}$$

and

$$\hat{n}_{ij}=\begin{cases} c_{ii}-[\ s_i s_{n+1} c^2_{i(n+1)}/c_{(n+1)(n+1)}] & \text{if } i=j, \\ s_j[\ c_{ij}-s_{n+1} c_{i(n+1)} c_{j(n+1)}/c_{(n+1)(n+1)}] & \text{if } i<j, \\ s_j[\ c_{ji}-s_{n+1} c_{i(n+1)} c_{j(n+1)}/c_{(n+1)(n+1)}] & \text{if } i>j. \end{cases}$$

Since $C^n_{IJ}-C_{n+1}C^T_{n+1}/c_{(n+1)(n+1)}$ is a form of D^n_{IJ} in lemma 2.2 and $N_n-wz^T/c_{(n+1)(n+1)}$ is the reduced form of $C^n_{IJ}-C_{n+1}C^T_{n+1}/c_{(n+1)(n+1)}$ from the expressions of $\hat{D}_{ij}$ and $\hat{n}_{ij}$, (2.5) is clearly verified by the assumption for $k=n$.

Theorem 2.1 and 2.2 tell us that the explicit formulas for the inverse and determinant of C^k_{IJ} depend upon those of the reduced form N_k. For simplicity we shall restrict our consideration to the special cases of the reduced form N_k. If we take

(i) $c_{ii}=\prod_{\substack{h=1\\h\neq i}}^{k}(s_h+1),\ i=1,\ldots,k$ and $c_{ij}=\prod_{\substack{h=1\\h\neq i,\ j}}^{k}(s_h+1),\ i<j,$

(ii) For all $i<j$, c_{ij}'s are same,

we obtain the explicit formulas for the determinant, the elements of the inverse and the characteristic equation. The condition (i) is always satisfied if the given matrix is the leading principal submatrix of X^TX when permutated to have full rank, where X is the design matrix for the balanced k-way factorial design with no interactions. On the other hand, the second condition is satisfied if the given matrix is the above mentioned leading principal submatrix for the balanced k-way factorial design whose the numbers of levels of main factors are all same. We can now state:

Corollary 2.1. Let $c_{ii}=\prod_{\substack{h=1\\h\neq i}}^{k}(s_h+1),\ i=1,\ldots,k$ and $c_{ij}=\prod_{\substack{h=1\\h\neq i,\ j}}^{k}(s_h+1),\ i<j,$ where s_h are as given in theorem 2.1. Then

(1) The determinant of N_k is given by

$$|N_k|=[\ 1+\sum_{i=1}^{k}s_i]\ \ [\ \prod_{i=1}^{k}(s_i+1)]^{\ k-2}.$$

(2) The cofactors of N_k are given by

$$N_{ij}=\begin{cases}(s_i+1)^2(1-s_i+\sum_{h=1}^{k}s_j)\ [\ \prod_{h=1}^{k}(s_h+1)]^{k-3} & \text{if } i=j,\\ -s_i(s_i+1)(s_j+1)\ [\ \prod_{h=1}^{k}(s_h+1)]^{k-3} & \text{if } i\neq j.\end{cases}$$

(3) The characteristic equation of N_k is given by

$$|N_k-\lambda I|=\left[\ \prod_{i=1}^{k}\left(\frac{\prod_{h=1}^{k}(s_h+1)}{(s_i+1)^2}-\lambda\right)\right]\left[\ 1+\sum_{i=1}^{k}\frac{s_i\prod_{h=1}^{k}(s_h+1)}{\prod_{h=1}^{k}(s_h+1)-(s_i+1)^2\lambda}\right]=0.$$

Proof. (1) Let $s=\prod_{h=1}^{k}(s_h+1)$, then we have $c_{ii}=s/(s_i+1)$ and $c_{ij}=s/(s_i+1)(s_j+1)$ by assumption. Since $N_k=D+xy^T$ where D is a diagonal matrix whose diagonal element is $s/(s_i+1)^2$ and vectors x, y are

$$x^T=[\ \frac{s}{s_1+1}\ ;\ \frac{s}{s_2+1}\ ;\ \cdots\ ;\ \frac{s}{s_k+1}]\ ,$$

$$y^T=[\ \frac{s_1}{s_1+1}\ ;\ \frac{s_2}{s_2+1}\ ;\ \cdots\ ;\ \frac{s_k}{s_k+1}]\ ,$$

the determinant of N_k becomes

$$|N_k|=s^{k-2}\ [\ 1+\sum_{i=1}^{k}s_i]\ .$$

(2) Similarly, the cofactors of N_k can be easily derived.

(3) The proof is similar to that of (1).

Remark 2. The coefficient b_{ij} given in (2.3) becomes

$$b_{ij}=\begin{cases}(s_i+1)(2-s_i+\sum_{h=1}^{k}s_h)\ /\ s(1+\sum_{h=1}^{k}s_h) & \text{if } i=j,\\ -(s_i+1)(s_j+1)\ /\ s(1+\sum_{h=1}^{k}s_h) & \text{if } i<j.\end{cases}$$

Corollary 2.2. Let $c_{ij}=a$ for all $i\neq j$, where a is a scalar such that $a\neq -1/(\sum_{i=1}^{k}\frac{s_i}{c_{ii}-as_i})$ and $a\neq c_{ii}/s_i$ for all $i=1,\ldots,k$. Then

(1) The determinant of N_k is given by

$$|N_k|=[\ 1+a\sum_{i=1}^{k}\frac{s_i}{c_{ii}-as_i}]\ \prod_{i=1}^{k}(c_{ii}-as_i).$$

(2) The cofactors of N_k are given by

$$N_{ij}=\begin{cases}[\ 1+a\sum\limits_{\substack{h=1\\h\neq i}}^{k}\frac{s_h}{c_{hh}-as_h}]\ \prod\limits_{\substack{h=1\\h\neq i}}^{k}(c_{hh}-as_h) & \text{if } i=j,\\ (-1)^{i+j}as_i\prod\limits_{\substack{h=1\\h\neq i,\ j}}^{k}(c_{hh}-as_h) & \text{if } i\neq j.\end{cases}$$

(3) The characteristic equation of N_k is given by

$$|N_k-\lambda I|=[\ 1+a\sum_{i=1}^{k}\frac{s_i}{c_{ii}-as_i-\lambda}]\ \ [\ \prod_{i=1}^{k}(c_{ii}-as_i-\lambda)]\ =0.$$

Proof. The proof of corollary 2.2 is not difficult and is omitted.

Remark 3. The coefficient b_{ij} becomes

$$b_{ij}=\begin{cases}\frac{1}{s_i(c_{ii}-as_i)}(1+a\sum\limits_{\substack{h=1\\h\neq i}}^{k}\frac{s_h}{c_{hh}-as_h})(1+a\sum\limits_{h=1}^{k}\frac{s_h}{c_{hh}-as_h})^{-1}-\frac{1}{s_ic_{ii}} & \text{if } i=j,\\ (-1)^{i+j}\frac{a}{(c_{ii}-as_i)(c_{jj}-as_j)}(1+a\sum\limits_{h=1}^{k}\frac{s_h}{c_{hh}-as_h})^{-1} & \text{if } i<j.\end{cases}$$

In corollary 2.2, if we take $c_{ii}=(1+a)s_i$ the determinant of N_k is given by

$$|N_k| = (1+\alpha k) \prod_{i=1}^{k} s_i.$$

Also the cofactors of N_k become

$$N_{ij} = \begin{cases} [\ 1+\alpha(k-1)] \prod_{\substack{h=1 \\ h \neq i}}^{k} s_h & \text{if } i=j, \\ (-1)^{i+j} \alpha s_i \prod_{\substack{h=1 \\ h \neq i,\ j}}^{k} s_h & \text{if } i \neq j. \end{cases}$$

If we denoted by $\alpha \neq 1/k$, we can calculate the inverses of N_k and C^k_{IJ}. The coefficient b_{ij} and (i, j)-th element of N_k^{-1} are given by

$$b_{ij} = \begin{cases} \alpha^2(k-1)\ /\ s_i^2(1+\alpha)(1+\alpha k) & \text{if } i=j, \\ (-1)^{i+j}\alpha\ /\ s_i s_j(1+\alpha k) & \text{if } i < j, \end{cases}$$

and

$$\frac{N_{ji}}{|N_k|} = \begin{cases} [\ 1+\alpha(k-1)]\ /\ s_i(1+\alpha k) & \text{if } i=j, \\ (-1)^{i+j}\alpha\ /\ s_i(1+\alpha k) & \text{if } i \neq j. \end{cases}$$

Further, the characteristic equations of N_k and C^k_{IJ} are given by

$$|N_k - \lambda I| = (1+\alpha \sum_{i=1}^{k} \frac{s_i}{s_i - \lambda}) \prod_{i=1}^{k} (s_i - \lambda) = 0,$$

$$|C^k_{IJ} - \lambda I| = (1+\alpha \sum_{i=1}^{k} \frac{s_i}{s_i - \lambda}) \prod_{i=1}^{k} (s_i - \lambda)[\ (1+\alpha)s_i - \lambda]^{s_i - 1} = 0.$$

3. Concluding Remarks

We considered the important properties on a patterned matrix with statistical applications, which is partitioned into blocks so that the diagonal blocks are I matrices premultiplied by constant and the off-diagonal blocks are rectangular J matrices premultiplied by constant. Based on the reduced form of the patterned matrix, we obtained the necessary and sufficient condition on the existence of the inverse of the patterned matrix, and then we found an formula for the elements of the inverse matrix. The formula is based on the cofactor and the determinant of the reduced form having smaller size. Finally, we could see that the determinant of the patterned matrix depends on the coefficients in the diagonal blocks and the determinant of the reduced form.

As applications, the patterned matrix is useful in computing the Moore-Penrose inverse of the design matrix, a generalized inverse of the cross-products matrix and the sum of squares for testing of estimable hypotheses.

References

[1] Graybill, F.A. (1983). *Matrices with applications in statistics*, 2nd ed. Wadsworth.