삼차원 판이론의 유한요소해석

Finite Element Analysis and Evaluation of a Three-dimensional Plate Theory

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요 약

종속변수와 기본 탄성방정식의 가중잔차 근사식의 Fourier 급수 전개를 이용한 3차원 판이론을 제시하였다. 판의 가중잔차 평형방정식은 가중된 변위로 표시되며, 그 결과는 다시 위치에너지 Functional을 이용하여 유한요소해석을 수행하였다. 본 해석은 Strip판에 적용되어 2가지 예를 분석하였으며, 예제의 결과는 이론해와 잘 일치하였다.

Abstract

Based on the weighted residual concept[4], a three-dimensional plate theory is derived using a Fourier series expansion of a dependent variable and a weighted residual approximation of the basic elasticity equations. The weighted residual equilibrium equations of the plate are expressed in terms of weighted displaced quantities, and the results are then interpreted by means of a potential energy functional. The potential energy expression is used to develop a finite element implementation. For illustrative purposes, the application of the theory to a strip plate is considered and two numerical examples of a cantilever and a simply-supported strip plate are studied.

1. Introduction

The finite element method is a numerical technique for solving, with acceptable accuracy, continuum mechanics problems in engineering. Due to the generality and richness of the ideas underlying the method and the difficulty of analyzing a continuum exactly, the method is extensively used for a wide range of problems in engineering. Conventional finite element methods for plate problems, in which the vertical displacements and their rotations are the degrees of freedom, are useful ; however, there are some difficulties related to convergence, compatibility, etc.[1]. These problems are largely due to the two-dimensional approximation

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of three-dimensional elasticity theory by plate theory. As a result, many different plate elements have been proposed to date and the best element may depend upon the problem to be solved[1].

The effects of the transverse shear stresses and the transverse normal stress on the deflection of a plate, which are usually negligible in a thin plate, are not considered in a classical plate theory[3]. In order to improve the assumptions of classical plate theory, a sixth order plate theory was developed by E. Reissner[6, 7] ; many other higher order theories have been proposed by others, for example see [8, 9].

In the present paper a three-dimensional analysis of homogeneous, linearly elastic plates experiencing small lateral displacements is developed for several specific conditions. Motivated by the extended plane stress theory given in [4], the weighted residual concept is applied to the bending of plates. Assuming no body forces, anti-symmetric boundary conditions and normal stresses applied to the faces of the plate, the weighted equilibrium equations in terms of weighted displacements are derived. In the process Fourier expansions of the displacements and their weighted norms are introduced. These representations recognize the even or odd nature of the functions in reference to the plate midsurface. From these equilibrium equations a potential energy functional is derived and used for finite element implementation in terms of the displacement quantities. For an illustration of the theory an one-dimensional finite element analysis for a strip plate is performed using a sub-parametric finite element and element condensation of element degrees of freedom. The results for displacements and stresses of two numerical examples are studied. The effect of the number of Fourier terms retained in the solution is observed.

2. Derivation of a Three-dimensional Plate Theory and Finite Element Implementation

In rectangular Cartesian coordinates, small lateral deflection of a homogeneous, linearly elastic, isotropic plate, of which the thickness h is bounded by $z=\pm h/2$ and the reference surface is in the x-y plane, is considered.

2.1 Basic Equations for a Three-dimensional Plate

The basic three-dimensional elasticity equations in rectangular Cartesian coordinates are as follows(for simplicity, body forces are neglected) :

$$\sigma_x=\alpha\, u_{,x}+\lambda(v_{,y}+w_{,z}) \tag{1}$$

$$\sigma_y=\alpha\, v_{,y}+\lambda(u_{,x}+w_{,z}) \tag{2}$$

$$\sigma_z=\alpha\, w_{,z}+\lambda(u_{,x}+v_{,y}) \tag{3}$$

$$\tau_{xy}=\mu(u_{,y}+v_{,x}) \tag{4}$$

$$\tau_{yz}=\mu(v_{,z}+w_{,y}) \tag{5}$$

$$\tau_{xz}=\mu(u_{,z}+w_{,x}) \tag{6}$$

$$\sigma_{x,x}+\tau_{xy,y}+\tau_{xz,z}=0 \tag{7}$$

$$\tau_{xy,x}+\sigma_{y,y}+\tau_{yz,z}=0 \tag{8}$$

$$\tau_{xz,x}+\tau_{yz,y}+\sigma_{z,z}=0 \tag{9}$$

where $\alpha=(1-\nu)E/[(1+\nu)(1-2\nu)]$

$\lambda=\nu E/[(1+\nu)(1-2\nu)]$

$\mu=E/[2(1+\nu)]$

The σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{xz} quantities are the stress components, and the u, v and w are the displacements in the x, y and z directions, respectively. The E and ν are Young's modulus and Poisson's ratio. If a plate is subjected to anti-symmetric loading on its faces, u and v are odd functions while w is an even function in terms of z. Consequently, σ_x, σ_y, σ_z and τ_{xy}

are odd functions, and τ_{xz} and τ_{yz} are even functions, thus :

$$u|_{z=0}=v|_{z=0}=0 \quad (10)$$

$$\sigma_x|_{z=0}=\sigma_y|_{z=0}=\sigma_z|_{z=0}=\tau_{xy}|_{z=0}=0 \quad (11)$$

The analysis is limited to cases where the surface's loading consists only of normal stresses, i.e. :

$$\sigma_z|_{z=\pm h/2}=\pm q/2 \quad (12)$$

$$\tau_{xz}|_{z=\pm h/2}=0 \quad (13)$$

$$\tau_{yz}|_{z=\pm h/2}=0 \quad (14)$$

Since the stress σ_z is to be an odd function for the analysis, the case where $\sigma_z|_{z=+h/2}=q$ and $\sigma_z|_{z=-h/2}=0$ would need to be represented by the summation of the case considered herein and symmetric case of $\sigma_z|_{z=\pm h/2}=q/2$. The symmetric loading case can be handled by a similar analysis for symmetric plate behavior [4].

2.2 Fourier Expansion and Weighted Norms of Basic Equations

The weighted residual concept is applied to develop a three-dimensional plate theory. The residuals(functions of z) are the errors in the satisfaction of the differential equations. To obtain weighted residuals the equations are multiplied by a weighting function and integrated over the thickness of the plate[5]. Thus, to approximate Eqs. (1) to (9), the resulting weighted residuals are then set equal to zero, i.e. :

$$\int_{-h/2}^{h/2} W(z)(i)dz=0 \quad (15)$$

The quantity W(z) is a weight function in terms of z and (i) is the equation with number i, for i=1 to 9. In preparation for this process, the Fourier expansions of even and odd functions are discussed in the following subsections 2.2.1 and 2.2.2.

2.2.1 Representation of an Even Function

Over the depth of the plate, h, an even function can be expanded as follows :

$$f(z)=f_0+\sum_{n=1}^{N} f_n \cos(2n\pi z/h) \quad (16)$$

$$\text{where } f_0=(1/h)\int_{-h/2}^{h/2} f(z)\,dz \quad (17)$$

$$f_n=(2/h)\int_{-h/2}^{h/2} f(z)\cos(2n\pi z/h)\,dz \quad (18)$$

Here, the Fourier coefficients f_0 and f_n can be alternatively thought of as the "constant" and "cosine" weighted norms of the even function f(z). The surface value of the function, f_e, is expressed in terms of f_0 and f_n.

$$f_e=f|_{z=+h/2}=f_0+f_e^* \quad (19)$$

$$\text{where } f_e^*=\sum_{n=1}^{N} f_n(-1)^n$$

2.2.2 Representation of an Odd Function

To form a continuous function so as to insure "uniform" convergence of the Fourier series expansion for all values of z, an odd function f(z) is represented as follows :

$$f(z)=g(z)+f_e(2z/h)$$

The function g(z) is expanded in a Fourier series with coefficients f_n, thus :

$$f(z)=\sum_{n=1}^{N} f_n \sin(2n\pi z/h)+f_e(2z/h) \quad (20)$$

$$\text{where } f_n=(2/h)\int_{-h/2}^{h/2} g(z)\sin(2n\pi z/h)\,dz$$

$$f_e=f|_{z=h/2}$$

It is noted that the derivative of g(z) is continuous at $z=\pm h/2$. At this point the weighted norms of f(z) that will be needed later are introduced :

$$f_n^{\#}=(2/h)\int_{-h/2}^{h/2} f(z)\sin(2n\pi z/h)\,dz$$
$$=f_n-2(-1)^n f_e/(n\pi) \qquad (21)$$

and

$$f_0^{\#}\equiv f_0=(1/h)\int_{-h/2}^{h/2} f(z)\,2z/h\,dz$$
$$=f_e/3-\sum_{n=1}^{N}(-1)^n f_n/(n\pi) \qquad (22)$$

From the above equation, the surface value, f_e, is expressed as

$$f_e=3f_0+f_e^* \qquad (23)$$

where $f_e^*=3\sum_{n=1}^{N}(-1)^n f_n/(n\pi)$

2.2.3 Plate Equations

The first step in the development of the analysis is to apply weighting functions to the elasticity equations and integrate the equations over the thickness of the plate.

(a) n=0 for Even functions

Eqs. (1) to (9) are multiplied by the weighting function $W(z)=1/h$ and the results are integrated over the depth of the plate to give the desired weighted residual equations. In the process it is noted that the norms of the odd functions are zero and hence only Eqs. (5), (6) and (9) give non-trivial results(Recall that f_0 is defined by $f_0=(1/h)\int_{-h/2}^{h/2} f(z)\,dz$ and that $f_e=f|_{z=+h/2}$).

$$\tau_{xz0}=(1/h)\int_{-h/2}^{h/2}\tau_{xz}\,dz$$
$$=(1/h)\int_{-h/2}^{h/2}\mu(u_{,z}+w_{,x})dz$$
$$=2\mu\, u_e/h+\mu\, w_{0,x} \qquad (24)$$
$$\tau_{yz0}=(1/h)\int_{-h/2}^{h/2}\tau_{yz}\,dz$$
$$=(1/h)\int_{-h/2}^{h/2}\mu(v_{,z}+w_{,y})dz$$
$$=2\mu\, v_e/h+\mu\, w_{0,y} \qquad (25)$$

The equilibrium Eq. (9) becomes

$$(1/h)\int_{-h/z}^{h/z}(\tau_{xz,x}+\tau_{yz,y}+\sigma_{z,z})dz=0$$

or

$$\tau_{xz0,x}+\tau_{yz0,y}+2\sigma_{ze}/h=0$$

Using Eq. (12)

$$\tau_{xz0,x}+\tau_{yz0,y}+q/h=0$$

Substituting from Eqs. (17) and (19), and using Eq. (23) gives :

$$\mu(6u_{0,x}/h+2u_{e,x}^*/h+w_{0,xx})$$
$$+\mu(6v_{0,y}/h+2v_{e,y}^*/h+w_{0,yy})+q/h=0 \qquad (26)$$

(b) n=0 for Odd Equations

The weighting function $W(z)=2/h$ is used and the definition of f_0 from Eq. (22) is recalled. Multiplying Eqs. (1) to (9) by the weighting function and integrating over the depth of the plate gives(Eqs. (5), (6), and (9) are identically zeros) :

$$\sigma_{x0}=(2/h^2)\int_{-h/2}^{h/2}\sigma_x\, z dz$$
$$=(2/h^2)\int_{-h/2}^{h/2}(\alpha u_{,x}+\lambda v_{,y}+\lambda w_{,z})zdz$$

or

$$\sigma_{x0}=\alpha u_{0,x}+\lambda v_{0,y}+2\lambda(w_e-w_0)/h \qquad (27)$$

In a similar manner,

$$\sigma_{y0}=\alpha v_{0,y}+\lambda u_{0,x}+2\lambda(w_e-w_0)/h \qquad (28)$$

Eq. (3) gives :

$$\sigma_{z0}=(2/h^2)\int_{-h/2}^{h/2}\sigma_z\, zdz$$
$$=(2/h^2)\int_{-h/2}^{h/2}(\alpha w_{,z}+\lambda u_{,x}+\lambda v_{,y})zdz$$

or

$$\sigma_{z0}=2\alpha(w_e-w_0)/h+\lambda u_{0,x}+\lambda v_{0,y} \tag{29}$$

Eq. (4) gives :

$$\tau_{xy0}=(2/h^2)\int_{-h/2}^{h/2}\tau_{xy}\, zdz$$
$$=(2/h^2)\int_{-h/2}^{h/2}\mu(u_{,y}+v_{,x})zdz$$

or

$$\tau_{xy0}=\mu u_{0,y}+\mu v_{0,x} \tag{30}$$

The equilibrium equation (7) gives :

$$(2/h^2)\int_{-h/2}^{h/2}(\sigma_{x,x}+\tau_{xy,y}+\tau_{xz,z})zdz$$
$$=\sigma_{x0,x}+\tau_{xy0,y}+2(\tau_{xze}-\tau_{xz0})/h=0$$

or using Eq. (13)

$$\sigma_{x0,x}+\tau_{xy0,y}-2\tau_{xz0}/h=0$$

Substituting from Eqs. (24), (27) and (30)

$$(\alpha u_{0,xx}+\lambda v_{0,xy}+2\lambda w_{e,x}^*/h)+(\mu u_{0,yy}+\mu v_{0,xy})$$
$$-2\mu(6u_0/h+2u_e^*/h+w_{0,x})/h=0 \tag{31}$$

Likewise Eq. (8) gives :

$$(\alpha v_{0,yy}+\lambda u_{0,xy}+2\lambda w_{e,y}^*/h)+(\mu u_{0,xy}+\mu v_{0,xx})$$
$$-2\mu(6v_0/h+2v_e^*/h+w_{0,y})/h=0 \tag{32}$$

(c) n=1→N for Even Function

The weighting function $W(z)=(2/h)\cos(2n\pi z/h)$ is used and the definition of f_n Eq. (18) is recalled. Eq. (5) gives :

$$\tau_{xzn}=(2/h)\int_{-h/2}^{h/2}\tau_{xz}\cos(2n\pi z/h)dz$$
$$=(2/h)\int_{-h/2}^{h/2}\mu(u_{n,z}+w_{n,x})\cos(2n\pi z/h)dz$$

Integrating the first term and recalling the defintion of $f_n^{\#}$ for an odd function from Eq. (21) gives :

$$\tau_{xyn}=(2\mu/h)\,[2(-1)^n u_e+n\pi u_n^{\#}]+\mu w_{n,x}$$
$$=(2\mu/h)\,(n\pi u_n)+\mu w_{n,x} \tag{33}$$

Similarly,

$$\tau_{yzn}=(2\mu/h)\,[2(-1)^n v_e+n\pi v_n^{\#}]+\mu w_{n,y}$$
$$=(2\mu/h)\,(n\pi v_n)+\mu w_{n,y} \tag{34}$$

The equilibrium Eq. (9) gives :

$$(2/h)\int_{-h/2}^{h/2}(\tau_{xzn,x}+\tau_{yzn,y}+\sigma_{zn,z})\cos(2n\pi z/h)dz=0$$

or

$$\tau_{xzn,x}+\tau_{yzn,y}+(2/h)[2\sigma_{ze}(-1)^n+n\pi\sigma_{zn}]=0$$

Using Eq. (12)

$$\tau_{xzn,x}+\tau_{yzn,y}+(2/h)[q(-1)^n+n\pi\sigma_{zn}^{\#}]=0 \tag{35}$$

(d) n=1→N for Odd Function

The weighting function $W(z)=(2/h)\sin(2n\pi z/h)$ is used. Eq. (1) gives :

$$\sigma_{xn}^{\#}=(2/h)\int_{-h/2}^{h/2}\sigma_{xn}\sin(2n\pi z/h)dz$$
$$=(2/h)\int_{-h/2}^{h/2}(\alpha u_{n,x}+\lambda v_{n,y}+\lambda w_{n,z})\sin(2n\pi z/h)dz$$
$$=\alpha u_{n,x}^{\#}+\lambda v_{n,y}^{\#}-2\lambda n\pi w_n/h$$
$$=\alpha[u_{n,x}-2(-1)^n u_{e,x}/(n\pi)]$$
$$+\lambda[v_{n,y}-2(-1)^n v_{e,y}/(n\pi)]-2\lambda n\pi w_n/h \tag{36}$$

Similarly

$$\sigma^{\#}_{yn}=\alpha[v_{n,y}-2(-1)^n v_{e,y}/(n\pi)] +\lambda(u_{n,x}-2(-1)^n u_{e,x}/(n\pi)]-2\lambda n\pi w_n/h \tag{37}$$

$$\sigma^{\#}_{zn}=\lambda[v_{n,y}-2(-1)^n v_{e,y}/(n\pi)] +\lambda(u_{n,x}-2(-1)^n u_{e,x}/(n\pi)]-2\alpha n\pi w_n/h \tag{38}$$

$$\tau^{\#}_{xzn}=\mu[u_{n,y}-2(-1)^n u_{e,y}/(n\pi)] +\mu[v_{n,x}-2(-1)^n v_{e,x}/(n\pi)] \tag{39}$$

Substituting from Eqs. (33), (34) and (38) into Eq. (35) gives :

$$(2\mu n\pi u_{n,x}/h-\mu w_{n,xx})+(2\mu n\pi v_{n,y}/h+\mu w_{n,yy}) +2q(-1)^n/h+(2n\pi/h)\,[-2\alpha n\pi w_n/h +\lambda\{u_{n,x}-2(-1)^n u_{e,x}/(n\pi)\} +\lambda\{v_{n,y}-2(-1)^n v_{e,y}/(n\pi)\}]=0 \tag{40}$$

Equilibrium Eqs. (7) and (8) are treated in a similar fashion, i.e. :

$$(2/h)\int_{-h/2}^{h/2}(\sigma_{xn,x}+\tau_{xyn,y}+\tau_{xzn,z})\sin(2n\pi z/h)dz =\alpha[u_{n,xx}-2(-1)^n u_{e,xx}/(n\pi)] +\lambda[v_{n,xy}-2(-1)^n v_{e,xy}/(n\pi)] -2\lambda n\pi w_{n,x}/h+\mu[u_{n,yy}-2(-1)^n u_{e,yy}/(n\pi)] +\mu[v_{n,xy}-2(-1)^n v_{e,xy}/(n\pi)] -(2\mu n\pi/h)\,(2n\pi u_n/h+w_{n,x})=0 \tag{41}$$

and

$$(2/h)\int_{-h/2}^{h/2}(\sigma_{yn,y}+\tau_{xyn,x}+\tau_{yzn,z})\sin(2n\pi z/h)dz =\alpha[v_{n,yy}-2(-1)^n v_{e,yy}/(n\pi)] +\lambda[u_{n,xy}-2(-1)^n u_{e,xy}/(n\pi)] -2\lambda n\pi w_{n,y}/h+\mu[u_{n,xy}-2(-1)^n u_{e,xy}/(n\pi)] +\mu[v_{n,xx}-2(-1)^n v_{e,xx}/(n\pi)] -(2\mu n\pi/h)\,(2n\pi v_n/h+w_{n,y})=0 \tag{42}$$

2.3 Finite Element Implementation

In the present finite element analysis only a strip plate is considered, i.e. all quantities are independent of y, and v_0 and v_n are zero. It is seen that equilibrium Eqs. (32) and (42) are identically satisfied. Using the calculus of variations it can be easily shown that the remaining weighted equilibrium equations (26), (31), (40) and (41) are equivalent to minimizing the following N+1 functionals. The displacement quantities, u and w, must be continuous but the derivatives $u_{,x}$ and $w_{,x}$ need not be.

$$\Pi_0=\int_L [-3\alpha u_{0,x}^2/2-18\mu u_0^2/h^2 -6\mu w_{0,x}u_0/h-6\lambda w_e^* u_{0,x}/h -12\mu u_e^* u_0/h^2-2\mu u_e^* w_{0,x}/h -\mu w_{0,x}^2/2+qw_0/h]dx \tag{43}$$

$$\Pi_n=\int_L [-\alpha u_{n,x}^2/2+2\alpha(-1)^n u_{e,x}u_{n,x}/(n\pi) -2\mu(n\pi/h)^2 u_n^2-2\lambda n\pi w_n u_{n,x}/h -2\mu n\pi w_{n,x}u_n/h-\mu w_{n,x}^2/2 +2q(-1)^n w_n/h-2\alpha(n\pi/h)^2 w_n^2 -4\lambda(-1)^n u_{e,x}w_n/h]dx \tag{44}$$

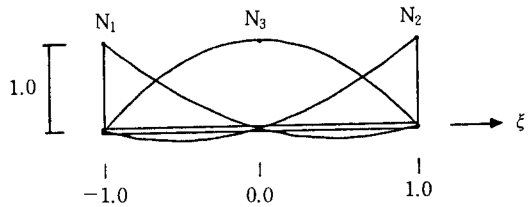
L=element length

It is noted that the above N+1 functionals are all coupled beacause the u_e and w_e terms are functionals of u_0, u_1, ⋯, u_n and w_0, w_1, ⋯, w_n, respectively.

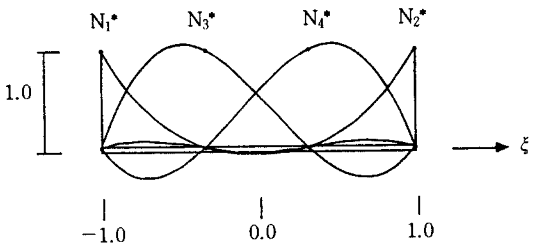
2.3.1 Element Approximation

Because, for thin strip plate theory, u and w are quadratic and cubic functions, respectively, for the case of constant shear, quadratic and cubic shape functions for u and w are used in the analysis, see Fig. 1. The approximate displacements within an element are expressed by means of the shape functions N_i and N_i^* in terms of the local coordinated ξ :

$$u=\sum_{i=1}^{3} N_i\, u_i,\qquad w=\sum_{i=1}^{4} N_i^*\, w_i \tag{45}$$



(a) Shape functions of u



(b) Shape functions of w

Fig. 1. Shape functions

where

$N1 = -\xi(1-\xi)/2$

$N2 = \xi(1+\xi)/2$

$N3 = 1-\xi^2$

$N1^* = -(9\xi^3-9\xi^2-\xi+1)/16$

$N2^* = (9\xi^3+9\xi^2-\xi-1)/16$

$N3^* = (27\xi^3-9\xi^2-27\xi+9)/16$

$N4^* = -(27\xi^3+9\xi^2-27\xi-9)/16$

Here, u_i and w_i represent u_{0i} or u_{ni}, and w_{0i} or w_{ni}, respectively, depending on which functionals are being approximated.

The derivatives of the displacements are

$$u_{i,x} = \sum_{i=1}^{3} N_{i,x}\, u_i = \sum_{i=1}^{3} F_i\, u_i$$

$$w_{i,x} = \sum_{i=1}^{4} N_{i,x}^*\, w_i = \sum_{i=1}^{4} F_i^*\, u_i \qquad (46)$$

where

$F_1 = (2\xi-1)/L$

$F_2 = (2\xi+1)/L$

$F_3 = -4\xi/L$

$F_1^* = -(27\xi^2-18\xi-1)/(8L)$

$F_2^* = (27\xi^2+18\xi-1)/(8L)$

$F_3^* = (81\xi^2-18\xi-27)/(8L)$

$F_4^* = -(81\xi^2+18\xi-27)/(8L)$

Substituting these approximations into Eqs. (43) and (44) gives the following expressions for potential energy in terms of the node point values of the weighted norms of the displacements :

$$\begin{aligned}\Pi_0 = \int_{-1}^{1} [&-3\alpha(F_i u_{0i})^2/2 - 18\mu(N_i u_{0i})^2/h^2 \\ &-6\mu(F_i^* w_{0i})(F_j u_{0j})/h \\ &-6\lambda(N_i^* w_{ei}^*)(F_j u_{0j})/h \\ &-12\mu(N_i u_{ei}^*)(N_j u_{0j})/h^2 \\ &-2\mu(N_i u_{ei}^*)(F_j^* w_{0j})/h \\ &-\mu(F_i^* w_{0i})^2/2 + q(N_i^* w_{0i})/h]\, |J|\, d\xi \qquad (47)\end{aligned}$$

$$\begin{aligned}\Pi_n = \int_{-1}^{1} [&-\alpha(F_i u_{ni})^2/2 - 2\alpha(-1)^n(F_i u_{ei}) \\ &(F_j u_{nj})/(n\pi) - 2\mu(n\pi/h)^2(N_i u_{ni})^2 \\ &-2\lambda n\pi(N_i^* w_{ni})(F_j u_{nj})/h \\ &-2\mu n\pi(F_i^* w_{ni})(N_j u_{nj})/h \\ &-\mu(F_i^* w_{ni})^2/2 + 2q(-1)^n(N_i^* w_{ni})^2/h \\ &+2\alpha(n\pi/h)^2(N_i^* w_{ni})^2 \\ &-4\lambda(-1)^n(F_i u_{ei})(N_j^* w_{nj})/h]\, |J|\, d\xi \qquad (48)\end{aligned}$$

Here, indices i and j take on values 1, 2 and 3 for the quadratic functions and 1, 2, 3 and 4 for the cubic functions, and when repeated in a term, they imply summation. The quantity |J| is the Jacobian from the coordinate transformation.

2.3.2 Element Stiffness Matrix and Load Vector

Element stiffness matrix and load vector can be obtained by differentiating the potential energy expressions (47) and (48) with respect to the node point displacements, i.e. :

$$\partial\Pi_0/\partial u_{0i}=(-3\alpha A_{ij}-36\mu B_{ij}/h^2)u_{0j}-6\mu C_{ij}w_{0j}/h -6\lambda D_{ij}w_{ej}^*/h-12\mu B_{ij}u_{ej}^*/h^2 \quad (49)$$

$$\partial\Pi_0/\partial w_{0i}=-6\mu C_{ji}u_{0j}/h-\mu F_{ij}w_{oj} -2\mu C_{ji}u_{ej}^*/h+qO_i/h \quad (50)$$

$$\partial\Pi_n/\partial u_{ni}=[-\alpha A_{ij}-4\mu(n\pi/h)^2B_{ij}]u_{nj} +2n\pi(\lambda D_{ij}-\mu C_{ij})w_{nj}/h +2\alpha(-1)^nA_{ij}u_{ej}/(n\pi) \quad (51)$$

$$\partial\Pi_n/\partial w_{ni}=2n\pi(\lambda D_{ji}-\mu C_{ji})u_{nj}/h -[\mu F_{ij}+4\alpha(n\pi/h)^2H_{ij}]w_{nj} +q(-1)^nO_i/h \quad (52)$$

where

$$A_{ij}=\int_{-1}^{1}F_iF_j|J|d\xi, \qquad B_{ij}=\int_{-1}^{1}N_iN_j|J|d\xi$$
$$C_{ij}=\int_{-1}^{1}N_iF_j^*|J|d\xi, \qquad D_{ij}=\int_{-1}^{1}F_iN_j^*|J|d\xi$$
$$F_{ij}=\int_{-1}^{1}F_i^*F_j^*|J|d\xi, \qquad H_{ij}=\int_{-1}^{1}N_i^*N_j^*|J|d\xi$$
$$O_i=\int_{-1}^{1}N_i^*|J|d\xi, \qquad |J|=L/2$$

These equations give a 7×7 element stiffness matrix and a 7×1 element load vector for each n, n=0→N. Before assembling the global stiffness and load matrices, the element matrices are condensed to a 4×4 stiffness and a 4×1 load matrices. The condensation eliminates the unknowns at the three interior nodes, see Fig. 1. It is important to note that Eqs. (49) to (52) are coupled due to the existence of the terms u_e and w_e.

2.3.3 Solution

To solve the equations, a finite element analysis program for a one-dimensional problem is developed. To handle the coupling in Eqs. (49) to (52) an iteration method is used. Using succesive substitution(which is the easiest method to apply, however, it tends to be slowly convergent) the Fourier terms of the nodal weighted displacements(u_0, u_1, ⋯, u_N, u_e and w_0, w_1, ⋯, w_N, w_e) are estimated by treating the n=0→N terms as uncoupled. These results are then used to find u_e and w_e frϕ Eqs. (19) and (23). Using these estimates for u_e and w_e, n=0→N uncoupled problems are again solved for new estimates of u_n and w_n which are in turn used to find new estimates of u_e and w_e, etc. These process is continued until convergence occurs. From the converged results the displacements are calculated for specified values of z :

$$w(z)=w_0+\sum_{n=1}^{N}w_n\cos(2n\pi z/h) \quad (53)$$

$$u(z)=\sum_{n=1}^{N}u_n\sin(2n\pi z/h)+u_e\,2z/h \quad (54)$$

The stresses are found from the following equations :

$$\sigma_x=\sum_{n=1}^{N}\sigma_{xn}\sin(2n\pi z/h) \quad (55)$$

$$\sigma_z=\sum_{n=1}^{N}\sigma_{zn}\sin(2n\pi z/h)+qz/h \quad (56)$$

$$\tau_{xz}=\tau_{xz0}+\sum_{n=1}^{N}\tau_{xzn}\cos(2n\pi z/h) \quad (57)$$

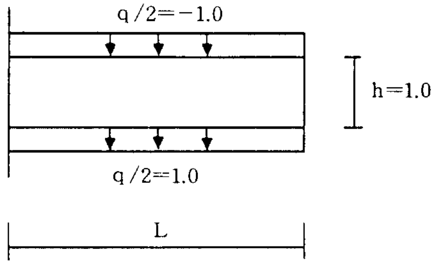
3. Numerical Examples and Discussion

Two examples are studied for the evaluation of the theory. In each case results are given and discussed for several thickness-to-length ratios and for a variable numbers of terms in the solution. The following two examples are selected for the study(Fig. 2) :

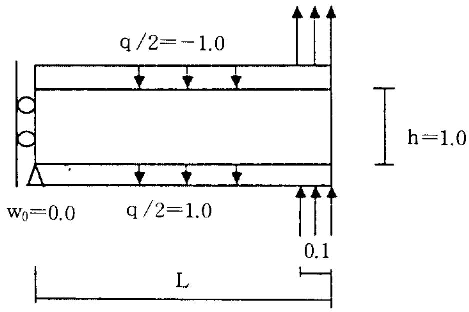
(1) A cantilever plate with a uniformly distributed load

(2) A modified simply-supported plate with a uniformly distributed load.

The loads are considered to be applied to both faces of the plate, see Eq. (12). The convergence criterion for the iteration process is



(a) Example 1 − A cantilever plate



(b) Example 2 − A modified simply-supported plate

Fig. 2. Example problems

$$|[w_0^{(I+1)} - w_0^{(I)}] / w_0^{(I+1)}| < 10^{-4} \qquad (58)$$

The quantity w_0 is the average vertical displacement of the plate at the free end for example (1) and at the center for example (2), and (I) is the iteration number.

3.1 Example(1)

A cantilever strip plate is subjected to a uniform distributed load q=−2. The properties of the plate are E=1000, h=1 and ν=0. The following cases are considered(L is the length of the plate and N is the number of Fourier series terms).

(a) L/h=1, 5 elements are used with N=0, 1 and 2

(b) L/h=5, 10 elements are used with N=0, 1 and 2

(c) L/h=10, 20 elements are used with N=0, 1 and 2

For comparison the deflection at the end of the plate as given by classical plate theory, including shear deformation, is

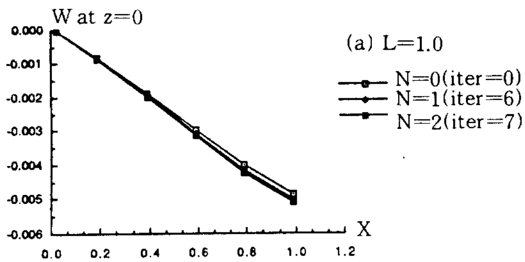
$$\delta = -qL^4(1+4\alpha) / (8D) \qquad (59)$$

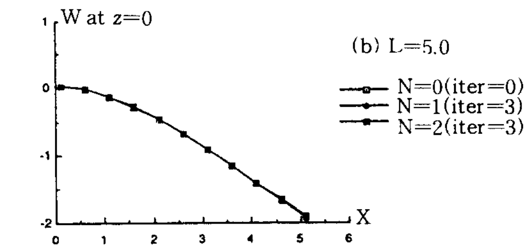
where

$D = Eh^3 / [12(1-\nu^2)]$, $\alpha = D / (\mu A_y L^2)$

$A_y = 5h/6$

Fig. 3 shows the vertical deflections for the three cases. It is seen that the displacement $w|_{z=0}$ converges fast irrespective of depth-to-length ratios. The convergence of $w|_{z=0}$ at the free end of the plate with increasing N is shown in Fig. 4 ; the convergence with increasing number of elements is shown in Fig. 5. Plots of stresses, σ_x, σ_z, and τ_{xz} over the depth of the plate and for three locations along the length are shown in Figs. 6, 7 and 8, re-





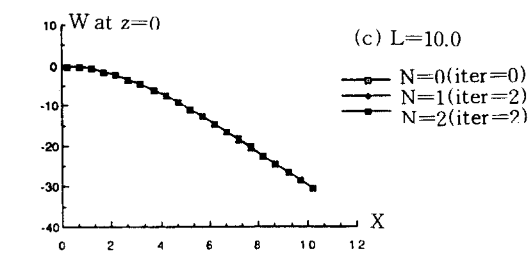


Fig. 3. Deflection for Example 1

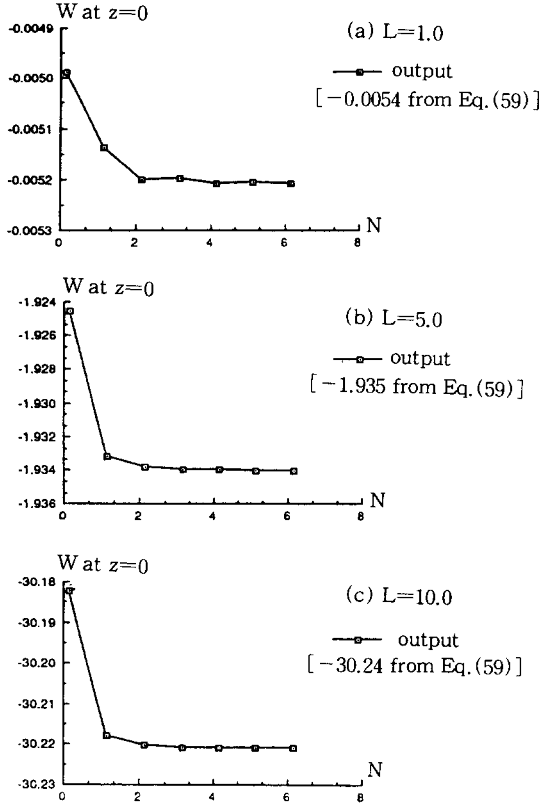


Fig. 4. Convergence of deflection for Example 1

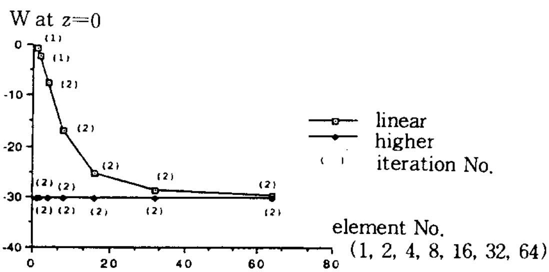


Fig. 5. Convergence of deflection at the free end of the plate for Example 1(L=10.0, N=2) using linear and higher order shape functions

spectively. As expected, the bending stress σ_x has nearly a linear distribution and the distribution of τ_{xz} becomes parabolic as N increases. The stress σ_z has some radical changes near a support, however the magnitude of the stress is negligible.

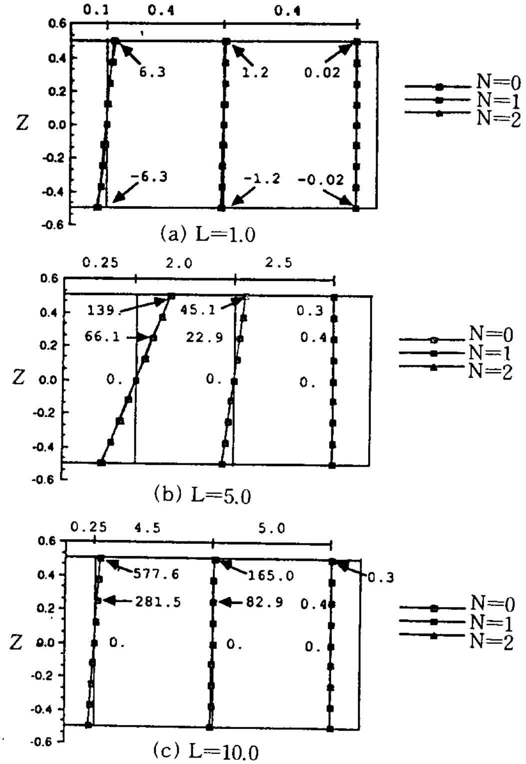


Fig. 6. Stress σ_x for Example 1

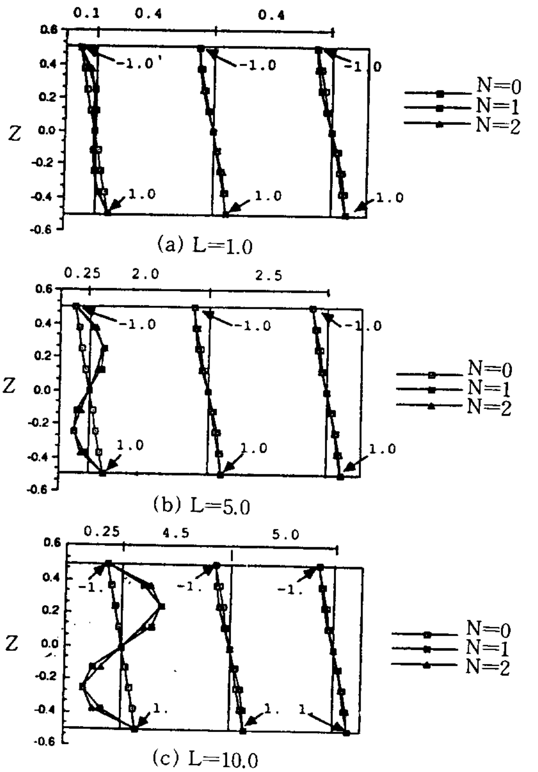


Fig. 7. Stress σ_z for Example 1

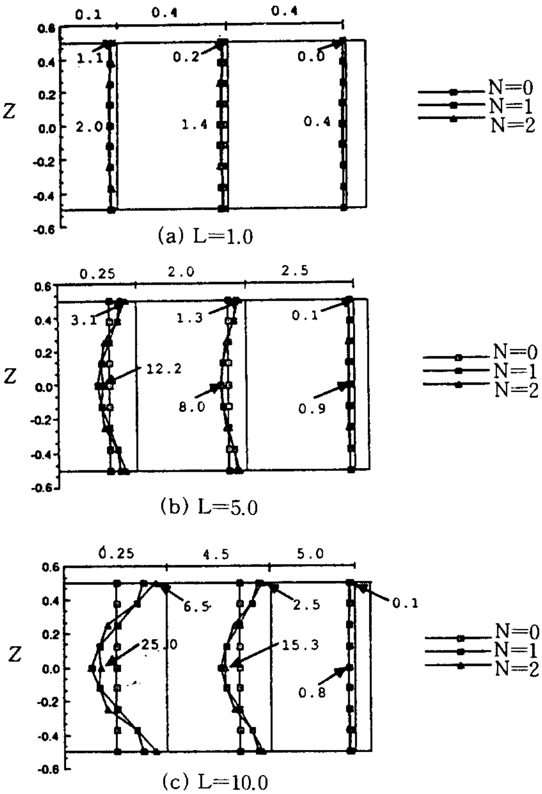


Fig. 8. Stress τ_{xz} for Example 1

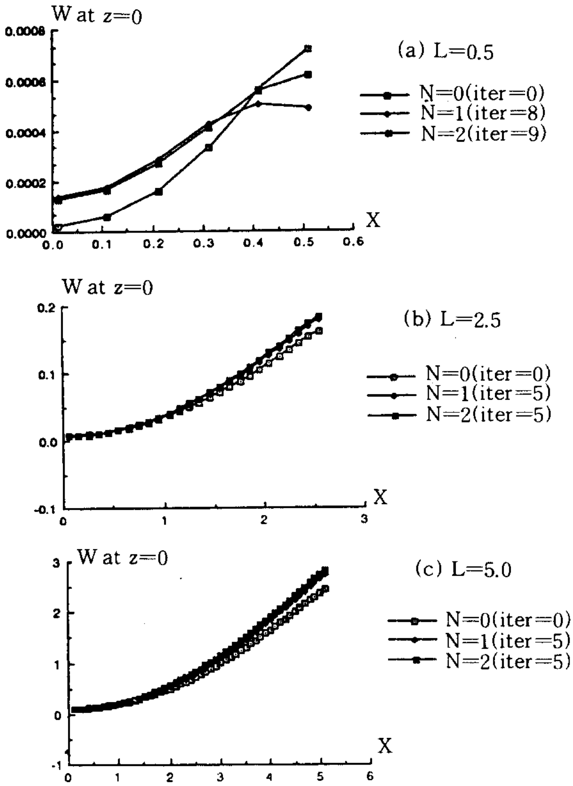


Fig. 9. Deflection for Example 2

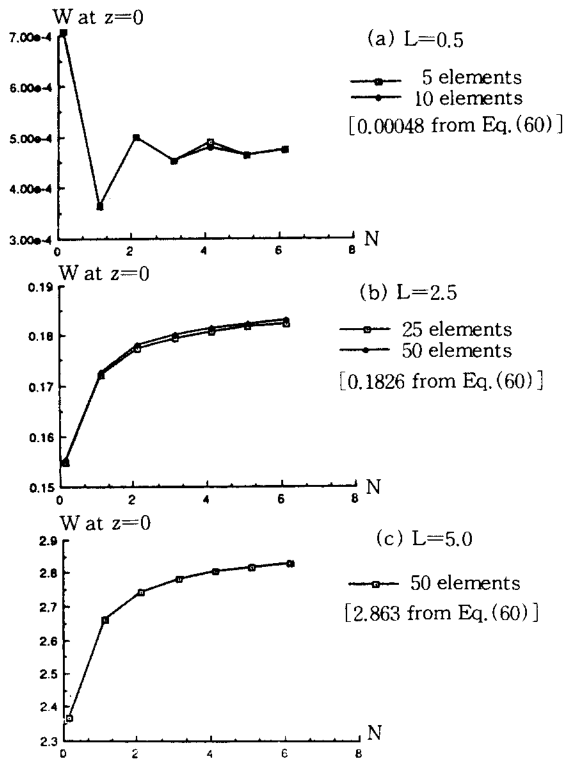


Fig. 10. Convergence of deflection for Example 2

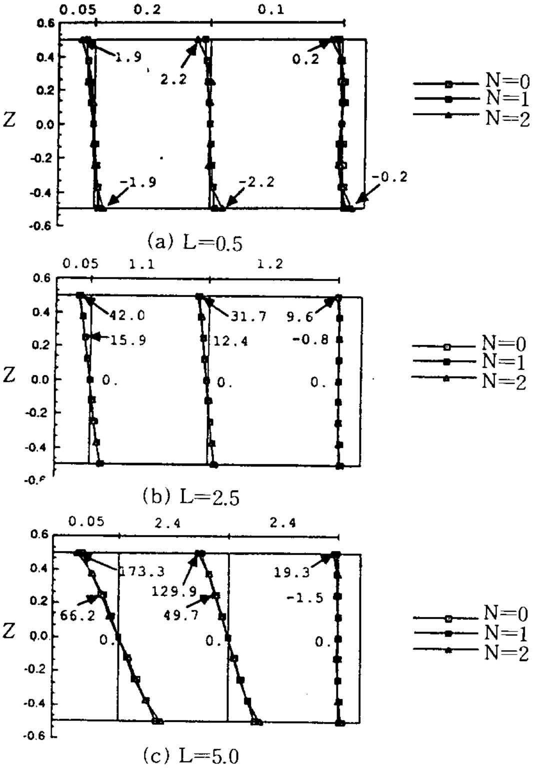
3.2 Example(2)

Using symmetry, only a half a simply-supported plate subjected to a uniformly distributed load q=−2 is considered. The properties of the plate are E=1000, h=1 and ν=0.3. In order to have simple asymmetric boundary conditions as assumed in the development of the analysis, the support reaction is assumed to be uniformly distributed over a finite length of both the top and bottom of the plate. The width of the support is assumed to be constant 0.1. Rigid body motion is tied down by setting w_0=0 at the center line(line of symmetry).

The cases investigated are the following :

(a) L/h=0.5, 5 elements are used with N=0, 1 and 2

(b) L/h=2.5, 25 elements are used with N=0, 1 and 2

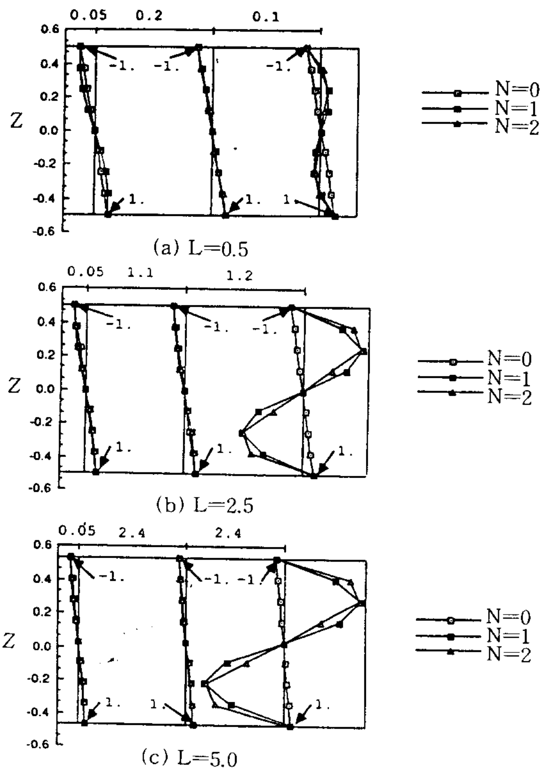


(a) L=0.5

(b) L=2.5

(c) L=5.0

Fig. 11. Stress σ_x for example 2

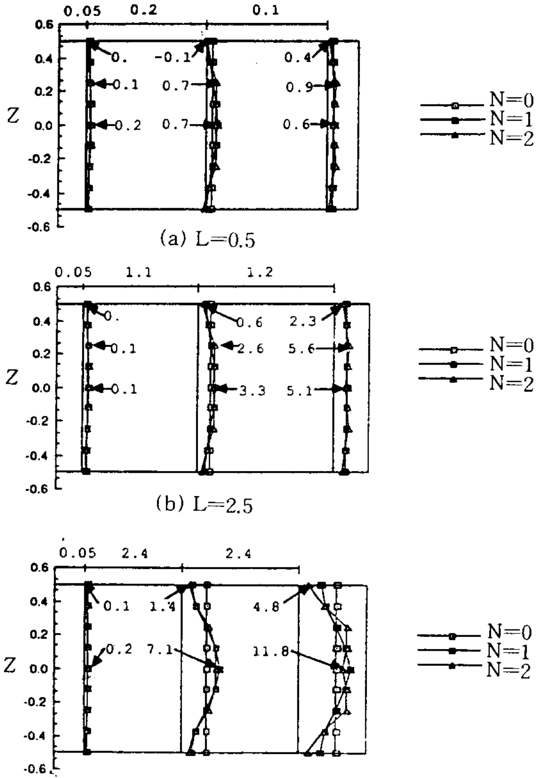


(a) L=0.5

(b) L=2.5

(c) L=5.0

Fig. 12. Stress σ_2 for Example 2



(a) L=0.5

(b) L=2.5

(c) L=5.0

Fig 13. Stress τ_{xz} for Example 2

(c) L/h=5.0, 50 elemenst are used with N=0, 1 and 2

The deflection δ at the center of the plate as predicted by thin plate theory(and ideal simple-supports) is shown for comparison :

$$\delta=-q(2L)^4(5+12\alpha)/(384D) \qquad (60)$$

where $D=Eh^3/[12(1-\nu^2)]$, $\alpha=D/[\mu A_y(2L)^2)$, $A_y=5h/6$

Fig. 9 shows the convergence of the displacement $w|_{z=0}$; the convergence of $w|_{z=0}$ at the center of the plate with increasing number of terms is shown in Fig. 10. The analysis gives a good result for the displacement $w|_{z=0}$, irrespective of depth-to-length ratios of the plate. Almost all the same trends as seen in Example (1) appear for Example (2). It is of interest to note the oscillatory convergence for case (a) as N increases in Fig. 10. Plots of the stresses, σ_x, σ_z, and τ_{xz}, are shown in Figs. 11, 12 and 13. Similar to Example (1) nearly linear stress distributions are found for σ_x, and parabolic distributions for τ_{xz}(with an increasing N). Some radical changes are observed in σ_z near the support.

3.3 Discussion

This analysis gives a good approximation for displacements in both examples. Especially, it is seen in Figs. 4 and 9 that the displacements rapidly converge with a small N. Although the modification used in Example (2) is a rough idealization of a simply-supported plate, the analysis also gives good results for both displacements and stresses. The limitation imposed by requiring an anti-symmetric normal stress σ_z is not overly restrictive since both body forces and loads on a single face can be handled by using a particular solution to reduce the problem to the form considered in this work. Since σ_z is small, radical change of σ_z near supports can be ignored. The distribution of shear stress, τ_{xz}, is expected to be nearly parabolic. As N increases Figs. 8 and 13 display parabolic shear stress distributions for Examples (1) and (2).

In Fig. 5, results obtained using linear shape functions(instead of the quadratic and cubic described previously) are compared with ones found using the higher order shape functions, for the case L/h=10 of Example (1). It shows that the slow convergence given by linear shape functions can be improved by using higher oredr shape functions in finite element analysis.

In summary, the three-dimensional plate theory successively yields an approximate solution for displacements and stresses for thin and thick plates.

4. Summary and Conclusion

The objective of this theory is to obtain good approximations for all displacements and stresses in a plate without using a conventional three-dimensional finite element analysis. For a linear elastic plate experiencing small lateral displacements, the governing equations were weighted by a linear, a constant and N Fourier functions of z and integrated over the thickness of the plate. This process results in the expansion of all "even" equations in a cosine Fourier series and all "odd" ones in a "modified" sine Fourier series. The "modification" involves the inclusion of a linear term in z that results in a uniformly convergent series. Therefore, the three-dimensional elasticity problem can be approximately solved by using a number of two-dimensional finite element analyses. In the present paper, a strip plate, e. a two-dimensional problem, was analyzed by using a one-dimensional finite element program ; two examples are considered. The analyses give results for all the stress components. Although the accuracy of the expansion depends on the number of terms, N, the present examples show rapid convergence for a small N. From the standpoint of computation and accuracy, the proposed theory is very advantageous because the degrees of freedoms are only the Fourier components of the displacements, u, v and w for a two-dimensional grid. Due to the small number of degrees of freedom in comparison with a three-dimensional finite element analysis, this approach requires less computational work and gives better approximations. However, in order for the proposed theory to be complete, further research should be performed with more various loading and boundary conditions in a strip plate as well as for two-dimensional finite element implementation of a three-dimensional plate.

References

1. R. D. Cook, "Concepts and Applications of Finite Element Analysis, 2nd ed.," John Wiley and Sons, New York, 1981.
2. S. Timoshenko and J. Goodier, "Theory of Elasticity, 2nd ed.," McGraw-Hill, New York, 1951.
3. S. Timoshenko and S. Woinowsky-Krieger, "Theory of Plates and Shells, 2nd ed.," McGraw-Hill, New York, 1959.
4. D. D. Lee, "Finite Element Analysis and Evaluation of an Extended Plane Stress Theory," M. S. Thesis, Department of Civil Engineering, University of California, Davis, 1986.
5. O. C. Zienkiewicz, "The Finite Element Method, 3rd ed.," McGraw-Hill, London, 1977.
6. E. Reissner, "The Effect of Transverse Shear Deformation On the Bending of Elastic Plates," *Jour. of Appl., Mech.*, pp.A69-A77, Vol. 12, 1945.
7. E. Reissner, "On the Bending of Elastic Plates," *Quar. Jour. of Appl. Math.*, pp.55-68, Vol. 5, 1947.
8. K. O, Friedricks and R. F. Dressler, "A Boundary Layer Theory for Elastic Plates," *Comm. on Pure and Appl. Math.*, pp.1-33, Feb., 1961.
9. E. L. Reiss and S. Locke, "On the Theory of Plane Stresses," *Quar. Jour. of Mech.*, pp. 195-203, Oct., 1961.

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