The Korean Journal of Rheology,
Vol. 7, No. 3, December 1995 pp. 192-202

〈연구논문〉

단순전단유동에서 미분 구성방정식의 일차원적 불안정 거동 예

권 영 돈

성균관대학교 공과대학 섬유공학과
(1995년 7월 18일 접수)

Examples of One-Dimensional Dissipative Instabilities in Simple Shear Flow as Predicted by Differential Constitutive Equations

Young-Don Kwon

Department of Textile Engineering, Sung Kyun Kwan University
300, Chunchun-dong, Jangan-ku, Suwon 440-746, KOREA
(Received July 18, 1995)

요 약

이 연구에서는 유변학 구성방정식이 나타내는 일차원 불안정성의 몇가지 예를 보였다. 안정성 해석을 위하여 맥스웰형 미분 구성방정식 Giesekus, Leonov, Larson모델을 선택하였다. 나타난 불안정성은 단순전단유동에서의 정상유동곡선이 무제한적 단순증가성을 위배할 때 발생한다. 단순전단유동에 부과된 섭동하에서 Giesekus와 Larson모델이 일정영역의 모델계수와 전단율속도값에서 불안정 거동을 나타내었다. 또한, 각 모델의 특성치인 임계응력을 넘는 계단파응력하에서 세 구성방정식 모두 "blow-up"형태의 불안정성을 보였다. 이 불안정 거동은 관성력을 고려하지 않은 경우에도 발생함이 증명되었다. 끝으로 이러한 불안정 거동을 개선하는 몇가지 방법을 Leonov와 Giesekus모델에 대하여 제시하였다.

Abstract – More examples of one-dimensional dissipative instabilities exhibited by several rheological constitutive equations (CE's) are illustrated in this paper. Such Maxwell-like differential equations as the Giesekus, the simplest Leonov and the Larson CE's are employed for this stability analysis. The instabilities are associated with the violation of monotonicity and unboundedness of steady flow curves in simple shear. From the consideration of standing wave perturbation imposed on the simple shear flow, it is proven that the Giesekus and the Larson models show a kind of dissipative instability in some values of numerical parameters and the shear rate. Imposing the step stress loading greater than some critical value also incurs severe "blow-up" instability even without inertial terms in the set of equations. At the end, methods of improving such ill-posed behaviors are suggested for the Leonov and the Giesekus CE's.

Keywords: Viscoelastic constitutive equation, one-dimensional stability, dissipative stability, Giesekus model, Leonov model, Larson model, simple shear flow

1. Introduction

In numerical simulation of viscoelastic fluid flows, degradation of the numerical solution or lack of convergence of computational schemes has been frequently observed for large or even modest values of Deborah numbers. It is thought that the main cause of this instability is bad choice of a constitutive equation (CE) for numerical applications (see, e.g. p.314 of Ref. [1]).

The mathematical instability of rheological CE's can be distinguished into two types: (i) Hadamard and (ii) dissipative. Hadamard instability, which shows the unboundedly increasing amplitude of short waves as the wavelength tends to zero, is associated with the nonlinear rapid response of CE's, hence it depends on the type of differential operator in the evolution equation for differential models and the configuration tensor-stress relation, i.e. the elastic potential in the hyper-viscoelastic case (the case where there exists a thermodynamic potential relation). However, the dissipative instability which is inherent only in viscoelastic equations, by definition, results from the dissipative terms of CE's in the case of differential type.

A number of publications on the Hadamard stability of viscoelastic CE's appeared (see e.g. Ref. [2]). However, no result on the global stability (the stability for any flow type and any value of velocity gradient tensor) for the class of CE's has been obtained until a series of works by Leonov et al.[5], and the global Hadamard stability constraints obtained for the class of Maxwell-like differential [3] and for the class of single integral CE's [4] are the condition for the strong monotonicity of stress, i.e. the GCN+ condition (for definition, see the section 52 in Ref. [6]), which is less restrictive. In the paper [5], the necessary and sufficient condition for global Hadamard stability is formulated in the form of algebraic inequality, and thus it seems that the stability analysis of Hadamard type is completely resolved for such two broad classes of CE's as Maxwell-like quasilinear (linear in derivatives) differential and factorable single integral models, which are the only ones in practical applications.

Results regarding the dissipative instability of CE's have been very rare. One-dimensional (1-D) instability of the Giesekus model within a certain range of a parameter was reported for shear flows [7, 8]. Those papers showed that the instability occurs when the numerical parameter is greater than 1/2 and shear rate exceeds a certain critical value. This type of dissipative instability is related to the decreasing branch of the steady shear flow curve.

Recently, in the study of dissipative stability for viscoelastic CE's several general results have been reported. In the paper [3], one can find two theorems proven for the Maxwell-type CE's, one of which illustrates positive definiteness of the configuration tensor in some restricted situation and was first proven by Hulsen [9] in a different way. Another theorem gives a useful sufficient condition for the boundedness of variables in the Maxwell-type CE's, the satisfaction of which guarantees the dissipative stability also for a limited flow history. In the case of single integral CE's, a necessary and sufficient condition for the boundedness of variables was also established in Ref.[10]. However, both theorems assume a predefined strain history, hence whenever mixed stress-strain history is given, they cannot be applied to the dissipative stability analysis. The dissipative instability which occurs when stress history is prescribed, has been demonstrated in the literature [11].

So far there have been a lot of attempts in the literature to apply unstable CE's to real flow instabilities like melt fracture. For example, the hypothesis of short memory was employed for the explanation of these physical phenomena [12], but it turned out that this instability was related to the change of type and furthermore some inconsistency appeared due to the use of different equations for the basic flow [13]. Another approach can be found in the work by Dunwoody and Joseph [14], where they obtained stability criteria of shear flows by applying long wave perturbations to CE's.

Nevertheless, Hadamard and dissipative instabilities of viscoelastic CE's are the genetic flaws incurred by a bad formulation of various terms in CE's. It is generally agreed that the melt fracture is a phenomenon related to real rubber-like fracture at the die entrance corner or stick-slip process of polymeric fluids along the wall. Therefore, this problem has nothing to do with the unstable behavior of CE's, and it should be treated as an adhesion (or fracture) problem of liquid on the wall (or at the corner) under intensive flow. It can be concluded that any CE with any type of instability described above, should be discarded from any further application, however well it describes viscometric data and however deep physical meaning it may contain. An attempt to apply the unstable behavior of equations to real flow instability would cause inconsistency with other sets of experimental data, and this kind of contradiction has been already revealed in [15].

In the recent work [5], a compilation of stability analyses is presented. Surprisingly, there exist only three stable CE's, which are the upper convected Phan Thien-Tanner CE, the FENE model and the Leonov class of CE's under specified stability constraints.

In this paper, more detailed examples of 1-D dissipative instabilities are illustrated. The instability of the Larson as well as the Giesekus CE caused by the decreasing branch of the steady shear flow solution is proved, even though the analysis of the Giesekus model was given in the work [7,8]. In addition, the severe instability exhibited by the Giesekus, the simplest Leonov and the Larson CE's in the case of step shear loading is also demonstrated, while the result only for the Leonov model was obtained in the paper [11].

2. Stability Analyses of Constitutive Equations

As mentioned in the introduction, various kinds of instability have been revealed in the numerical modeling of high Deborah number flows as well as in mathematical analysis of CE's. The dissipative instability may happen even for the Hadamard stable models for which the common constraint of positive definiteness of dissipation is satisfied. It is due to a bad formulation of dissipative terms for differential CE's. The global criteria for dissipative stability of viscoelastic CE's are far away from being completely established. However, in this section one criterion against distinct dissipative instabilities is suggested, which is the monotonicity and the unboundedness of steady flow curves in simple shear. The main objective of this study is to demonstrate some unphysical dissipative instabilities which can occur in CE's when the mixed stress-strain history is considered.

Two general classes of CE's are employed at present for analyzing viscoelastic flows of polymeric liquids. They are of differential and single integral types. In this paper, three Maxwell-like differential CE's such as the Giesekus, the simplest Leonov and the Larson models are studied to expose 1-D dissipative instabilities.

Employing the following canonical form:

$$\overset{\nabla}{\underline{\underline{c}}} + \underline{\underline{\psi}}(\underline{\underline{c}}) = 0, \quad \underline{\underline{\psi}}(\underline{\underline{c}}) = A_0\underline{\underline{\delta}} + A_1\underline{\underline{c}} + A_2\underline{\underline{c}}^2,$$
$$\underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + 2\rho\underline{\underline{c}}\cdot\partial F/\partial\underline{\underline{c}}, \quad \overset{\nabla}{\underline{\underline{c}}} \equiv d\underline{\underline{c}}/dt - \underline{\underline{c}}\cdot\underline{\underline{\nabla v}} - \underline{\underline{\nabla v}}^T\cdot\underline{\underline{c}}$$
$$D = \rho\cdot \mathrm{tr}(\underline{\underline{\psi}}\cdot\partial F/\partial\underline{\underline{c}}) \qquad (1)$$

we can represent all three differential CE's with a proper specification of F and A_i's in $\underline{\underline{\psi}}$. Here $\underline{\underline{c}}$ is a "configuration tensor", $\overset{\nabla}{\underline{\underline{c}}}$ is the upper convected time derivative of $\underline{\underline{c}}$, $\underline{\nabla v}$ and $\underline{\nabla v}^T$ are the velocity gradient tensor and its transposition, $\underline{\underline{\psi}}(\underline{\underline{c}})$ is the dissipative term, an isotropic tensor function of $\underline{\underline{c}}$, $\underline{\underline{\sigma}}$ is the stress tensor, p is an isotropic pressure, $\underline{\underline{\delta}}$ is the unit tensor, ρ is the density, F is the elastic potential, and D is the dissipation.

Details of the specific models are as follows:

(i) the Giesekus model [16, 17]

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+G\underline{\underline{c}},\ A_0=-(1-\alpha)/\theta,\ A_1=(1-2\alpha)/\theta,$$
$$A_2=\alpha/\theta,\ F=(G/2\rho)(I_1-3),$$
$$D=(G/2\theta)\{I_1-3+\alpha(\mathrm{tr}\underline{\underline{c}}^2-2I_1+3)\},\ (0\leq\alpha\leq1),\quad(2)$$

(ii) the simplest Leonov model [18]

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+G\underline{\underline{c}},\ A_0=-(2\theta)^{-1},\ A_1=(I_2-I_1)/6\theta$$
$$A_2=-A_0,\ F=(G/2\rho)(I_1-3),$$
$$D=(G/12\theta)\{2I_1^2+I_1I_2-6I_2-9\},\quad(3)$$

(iii) the Larson model [19]

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+G\underline{\underline{c}}/B(I_1),\ A_0=-B(I_1)/\theta,\ A_1=-A_0,$$
$$A_2=0,\ F=(3G/2\rho\xi)\ln B(I_1),\ D=(G/2\theta)(I_1-3),$$
$$B(I_1)=1+(\xi/3)(I_1-3),\ (0\leq\xi\leq1).\quad(4)$$

Here G is the shear modulus, θ is the relaxation time, $I_1=\mathrm{tr}\underline{\underline{c}}$, $I_2=\mathrm{tr}\underline{\underline{c}}^{-1}$, and α and ξ are numerical nonlinear parameters.

2.1 Simple shear flow

In this type of flow, the matrices of the velocity gradient and configuration tensors for all three differential CE's are of the form:

$$\underline{\underline{\nabla v}}=\dot{\gamma}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},\ \underline{\underline{c}}=\begin{bmatrix}c_{11}&c_{12}&0\\c_{12}&c_{22}&0\\0&0&1\end{bmatrix}\quad(5)$$

Here $\dot{\gamma}$ is the shear rate, and the common viscometric coordinate system is employed.

the Giesekus and the Leonov models

In the Giesekus model

$$\frac{\partial c_{11}}{\partial\hat{t}}+\alpha(c_{11}^2+c_{12}^2)+(1-2\alpha)c_{11}+\alpha-1=2\Gamma c_{12},$$
$$\frac{\partial c_{22}}{\partial\hat{t}}+\alpha(c_{12}^2+c_{22}^2)+(1-2\alpha)c_{22}+\alpha-1=0,$$
$$\frac{\partial c_{12}}{\partial\hat{t}}+ac_{12}(c_{12}+c_{22}-2)+c_{12}=\Gamma c_{22},\quad(6)$$
$$\hat{\sigma}=\frac{\sigma_{12}}{G}=c_{12},\ \hat{N}_1=\frac{\sigma_{11}-\sigma_{22}}{G}=c_{11}-c_{22},$$
$$\hat{N}_2=\frac{\sigma_{22}-\sigma_{33}}{G}=c_{22}-1.\quad(7)$$

When α = 1/2, eqs.(6) with (7) become identical to the relations for the simplest Leonov CE. In this case, the first integral of the set (6) exists, which follows from the condition of incompressibility $\det\underline{\underline{c}}=1$. Then one of the equations in the set, say the first one of eqs.(6), can be replaced by the finite equation,

$$c_{11}=(1+c_{12}^2)/c_{22}.$$

the Larson model

It is represented by

$$\frac{\partial c_{11}}{\partial\hat{t}}+B(c_{11}-1)=2\Gamma c_{12},\ c_{22}=1,$$
$$\frac{\partial c_{12}}{\partial\hat{t}}+Bc_{12}=\Gamma,\ B=1+\frac{\xi}{3}(c_{11}-1),\quad(8)$$
$$\hat{\sigma}=c_{12}/B,\ \hat{N}_1=(c_{11}-1)/B,\ \hat{N}_2=0.\quad(9)$$

In eqs.(6)~(9), the dimensionless variables $\hat{t}=t/\theta$ and $\Gamma=\theta\dot{\gamma}$ are introduced, and stress components are scaled by the elastic modulus G.

In the steady shear flow, all rheological variables can be expressed through the quantity c_{12}.

the Giesekus and the Leonov models

$$c_{11}=\frac{1}{2\alpha}(2\alpha-1+\sqrt{1-4\alpha^2c_{12}^2+8\alpha\Gamma c_{12}}),$$
$$c_{22}=\frac{1}{2\alpha}(2\alpha-1\pm\sqrt{1-4\alpha^2c_{12}^2}),$$
$$\Gamma=\frac{\alpha c_{12}\left[1\pm(2\alpha-1)\sqrt{1-4\alpha^2c_{12}^2}\right]}{(1-2\alpha)(1\mp\sqrt{1-4\alpha^2c_{12}^2})-2\alpha^2(c_{12}^2-1)}\quad(10)$$

The upper branch solution is valid for the condition $\alpha\leq1/2$ or $\Gamma\leq(2\alpha-1)^{-2}$. When $\alpha>1/2$ and also $\Gamma>(2\alpha-1)^{-2}$, the Giesekus CE possesses the lower branch solution. The dimensionless flow curves $\hat{\sigma}$

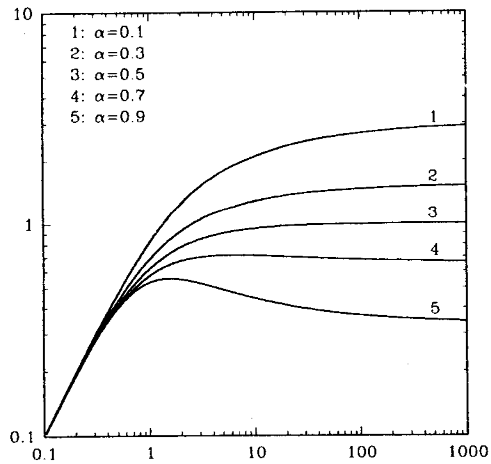

Fig. 1. Dimensionless shear stress of the Giesekus model or the Leonov model (a=1/2) plotted versus dimensionless shear rate in steady simple shear flow.

(Γ) are shown in Fig.1, and have the following features. When $\alpha \leq 1/2$, the shear stress is monotonically increasing and bounded by the value $(\alpha^{-1}-1)^{1/2}$. When $\alpha > 1/2$, the shear stress exhibits a maximum equal to $(2\alpha)^{-1}$ at shear rate $\Gamma = (2\alpha-1)^{-2}$ and then decreases to a value of $(1-\alpha)/\alpha$ with Γ increasing. When $\alpha = 1/2$, the upper branch solution of eq.(10) is related to the simplest Leonov model, of which the flow curve is depicted in Fig. 1 for the case of $\alpha = 1/2$, and it shows the monotonically increasing but bounded (by 1) shear stress with shear rate growing. These results are consistent with those presented in the cited papers by Giessekus [16,17], Yoo and Choi [20], Schleiniger and Weinacht [7. 8], and Leonov et al. [21].

<u>the Larson model</u>

$$c_{11} = 1 + 2c_{12}^2,\ \Gamma = (2\xi c_{12}^2/3+1)c_{12},$$
$$\hat{\sigma} = \frac{c_{12}}{2\xi c_{12}^2/3+1},\ \hat{N}_1 = \frac{2c_{12}^2}{2\xi c_{12}^2/3+1},\ \hat{N}_2 = 0. \qquad (11)$$

The dimensionless flow curves in Fig. 2 calculated for various values of the parameter ξ, are quite similar to those for the Giesekus model

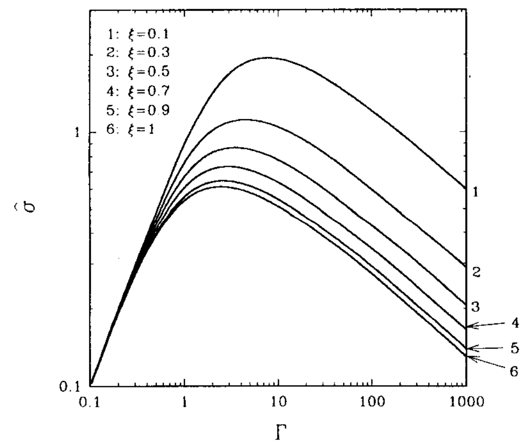

Fig. 2. Dimensionless shear stress of the Larson model plotted versus dimensionless shear rate in steady simple shear flow.

with $\alpha > 1/2$. Here the decreasing branches of flow curves exist for any $\xi > 0$ and the shear stress achieves a maximum value equal to $\sqrt{3/(8\xi)}$ at $\Gamma = \sqrt{6/\xi}$.

2.2. Instability of decreasing flow curves in plane Couette flow

It is evident that the decreasing branches of flow curves for the Giesekus CE with $\alpha > 1/2$ and the Larson CE with any $\xi > 0$ are unstable. Nevertheless, neither linear stability analyses nor straightforward numerical calculations of start-up inertialess flows for all models, with constant values of Γ corresponding to the decreasing branches, showed any instability. It is mathematically illustrated in this section for the Giesekus and the Larson CE's that the inertia term prominently influences those 1-D instabilities.

The equation of motion $\rho(d\underline{v}/dt) = \underline{\nabla} \cdot \underline{\underline{\sigma}}$ combined with CE's (2) and (4) yields in the plane Couette flow

$$R\Gamma^0 \frac{\partial u}{\partial \hat{t}} = \begin{cases} \dfrac{\partial c_{12}}{\partial y} & \text{for the Giesekus model} \\ B^{-1}\dfrac{\partial c_{12}}{\partial y} - B^{-2}\dfrac{\xi}{3}c_{12}\dfrac{\partial c_{11}}{\partial y} & \text{for the Larson model} \end{cases} \qquad (12)$$

Here u is the dimensionless velocity scaled by the velocity V of a moving plate, y is the dimensionless coordinate variable across the gap scaled by the gap thickness h, $B = 1+\xi(c_{11}-1)/3$, the dimensionless steady state shear rate Γ^0 and R are given by $\Gamma^0 = \theta V/h$ and $R=\rho h^2/(G\theta^2)$.

Now consider infinitesimal 1-D disturbances of standing waves imposed on steady state solutions as

$$u = u^0+\delta u = u^0+\sum_n \bar{u}_n \exp(\hat{w}_n \hat{t})\sin(n\pi y),$$
$$\Gamma=\Gamma^0+\delta\Gamma=\Gamma^0(1+\partial\delta u/\partial y),$$
$$\{c_{11}, c_{22}, c_{12}\} = \{c^0_{11}, c^0_{22}, c^0_{12}\}+\{\delta c_{11}, \delta c_{22}, \delta c_{12}\},$$
$$\{\delta c_{11}, \delta c_{22}, \delta c_{12}\} = \sum_n \{\bar{c}_{11}, \bar{c}_{22}, \bar{c}_{12}\}_n \cdot \exp(\hat{w}_n \hat{t})\cdot\cos(n\pi y). \quad (13)$$

Here u^0 is the dimensionless steady velocity without disturbances, c^0_{ij} is a steady state solution of c_{ij} obtained in eqs.(10) and (11), $\bar{u}_n$ and $\hat{w}_n$ are the infinitesimal amplitude of δu and the dimensionless frequency of the disturbing waves scaled by θ in the n-th Fourier mode, and $\bar{c}_{ij}$ is the infinitesimal amplitude of disturbances δc_{ij}, hence for the Larson model $\delta c_{22} = 0$ and $c_{22} = 1$. Then, taking only one mode of disturbing waves, substituting eqs.(13) into eqs.(12) and considering only the first order terms with respect to infinitesimal disturbances, we arrive at

$$\bar{u} = -\frac{n\pi}{R\Gamma^0\hat{w}}\bar{c}_{12} \quad \text{for the Giesekus model,}$$

$$\bar{u} = -\frac{n\pi}{R\Gamma^0\hat{w}B^0}\left[\bar{c}_{12}-\frac{\xi c^0_{12}}{3B^0}\bar{c}_{11}\right] \quad \text{for the Larson model,} \quad (14)$$

where $B^0 = 1+\xi(c^0_{11}-1)/3$, and for the simplicity of notation, the subscript n for the Fourier mode is reserved. Substituting eqs.(13) into eqs.(6) or (8) with the aid of eqs.(14) and again considering only the remaining highest order terms produce in a matrix form the following relations between the amplitudes:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\cdot\begin{bmatrix} \bar{c}_{11} \\ \bar{c}_{22} \\ \bar{c}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for the Giesekus model,}$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\cdot\begin{bmatrix} \bar{c}_{11} \\ \bar{c}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{for the Larson model.} \quad (15)$$

Here $a_{11} = \hat{w}^2 + (2\alpha c^0_{11}-2\alpha+1)\hat{w}$, $a_{12}=0$, $a_{13} = 2(\alpha c^0_{12}-\Gamma^0)\hat{w}+2(n\pi)^2 c^0_{12}/R$, $a_{21}=0$, $a_{22} = \hat{w}+2\alpha c^0_{22}-2\alpha+1$, $a_{23}=2\alpha c^0_{12}$, $a_{31}=\alpha c^0_{12}\hat{w}$, $a_{32}=(\alpha c^0_{12}-\Gamma^0)\hat{w}$, $a_{33}=\hat{w}^2 + (\alpha c^0_{11}+\alpha c^0_{22}-2\alpha+1)\hat{w}+(n\pi)^2 c^0_{22}/R$, and $b_{11} = \hat{w}^2 + (2B^0-1)\hat{w}-\frac{2(n\pi)^2\xi(c^0_{12})^2}{3R(B^0)^2}$, $b_{12} = -2\Gamma^0\hat{w}+\frac{2(n\pi)^2 c^0_{12}}{RB^0}$

$b_{21} = \frac{\alpha}{3}c^0_{12}\hat{w}-\frac{(n\pi)^2\xi c^0_{12}}{3R(B^0)^2}$, $b_{22} = \hat{w}^2+B^0\hat{w}+\frac{(n\pi)^2}{RB^0}$

Nontrivial solutions of $\bar{c}_{ij}$ do exist, if the determinants of two matrices $[a_{ij}]$ and $[b_{ij}]$ vanish. Therefore, the vanishing determinants result in the following fourth order polynomial with respect to nonzero $\hat{w}$ for both the CE's under study:

$$A_0+A_1\hat{w}+A_2\hat{w}^2+A_3\hat{w}^3+\hat{w}^4 = 0. \quad (16)$$

Here the coefficients A_is have awkward forms composed of steady state solutions and model parameters. The stability of basic steady state solutions requires the real part for all roots of $\hat{w}$ to be negative. In this case, we employ the Hurwitz theorem, which states that the following inequalities are necessary and sufficient for real parts of all roots to be negative:

$$A_0>0,\ A_1>0,\ A_1A_2-A_0A_3>0,$$
$$A_1A_2A_3-A_0A_3^2-A_1^2>0. \quad (17)$$

The use of only the first two inequalities in (17) as necessary conditions of stability is enough for our further analyses.

the Giesekus model

The first of inequalities in (17) has a form of

$$A_0 = \pm \frac{(n\pi)^2 \sqrt{Z}}{2\alpha R}\left[\sqrt{Z + 8\alpha\Gamma^0 c_{12}^0}(2\alpha - 1 \pm \sqrt{Z}) - (2\alpha c_{12}^0)^2\right] > 0 \tag{18}$$

where $Z = 1-4(\alpha c_{12}^0)^2$, R is a positive parameter shown in eqs.(12), and Γ^0 is related to c^0_{12} through eqs.(10). The upper branch of (18) is valid for the conditions $\alpha \leq 1/2$ or $\Gamma \leq (2\alpha-1)^{-2}$ and the lower branch is for the case of $\alpha > 1/2$ and also $\Gamma > (2\alpha-1)^{-2}$. It can be readily seen that the square-bracketed term in (18) is always positive. Therefore, the lower branch case of inequality (18) violates the necessary condition for stability.

the Larson model

The first two inequalities of (17) are represented as

$$A_0 = \frac{2\xi(n\pi c_{12}^0)^2}{RB^0}\left[1 - \frac{(n\pi)^2}{3R(B^0)^2}\right] > 0,$$

$$A_1 = B^0\left[\frac{(n\pi)^2}{R(B^0)^2} - w\xi(c_{12}^0)^2\right] > 0,$$

$$\text{or} \quad \frac{w\xi}{3}(c_{12}^0)^2 < \frac{(n\pi)^2}{3R(B^0)^2} < 1, \tag{19}$$

where in eqs.(11) we see that c^0_{12} is monotonically increasing with respect to Γ^0, and B^0 is shown in eqs.(14). After reaching the maximum at $\Gamma^0 = \sqrt{6/\xi}$ or at $c_{12}^0 = \sqrt{3/(2\xi)}$, $\hat{\sigma}$ decreases but c_{12}^0 increases, hence the inequality (19) is again violated in the decreasing branch of solutions.

Thus, using this method, we proved that the solutions corresponding to the decreasing branches of the flow curves are unstable when the inertia term is taken into consideration. These unstable branches occur for the Giesekus model with $\alpha > 1/2$ and $\Gamma^0 > (2\alpha-1)^{-2}$ and for the Larson model with $0 < \xi \leq 1$ and $\Gamma^0 > \sqrt{6/\xi}$. The result obtained for the Giesekus model corresponds to a result by Schleiniger and Weinacht [7].

2.3. Instability in creep shear flow

As mentioned previously, all three models have a common characteristic of boundedness of steady shear stress. A more severe type of 1-D instabilities in shear flow incurred by this boundedness can be manifested even in inertialess approximation. If the step stress, greater than the achievable maximum stress, is applied for each models, there exists a "blow-up" instability, which means the solution of the start-up problem for the shear rate Γ, goes to infinity within a finite time. This type of instability is illustrated by solving a step stress problem for the above-mentioned three differential models.

The solutions (10) and (11) for steady shear flow show that the shear stresses have either a maximum value or a certain upper bound $\hat{\sigma}^m$, depending on the numerical parameters α or ξ of the CE's. The values of $\hat{\sigma}^m$ for three differential CE's under study are

$$\hat{\sigma}^m = \begin{cases} 1/(2\alpha) & \text{for the Giesekus CE} \\ 1 & \text{for the Leonov CE} \\ \sqrt{3/(8\xi)} & \text{for the Larson CE.} \end{cases} \tag{20}$$

Now, we consider the solution of CE's (6)~(9) under the condition of a step stress

$$\hat{\sigma} = \hat{\sigma}^* H(\hat{t}), \quad \hat{\sigma}^* > \hat{\sigma}^m, \tag{21}$$

where $H(\hat{t})$ is a Heaviside step function.

the Giesekus model

In this case, eqs.(6) result in the following set of equations:

$$\frac{\partial c_{11}}{\partial \hat{t}} - 2\alpha(\hat{\sigma}^*)^2 \frac{c_{11} + c_{22} - 2 + 1/\alpha}{c_{22}} + \alpha\left[c_{11}^2 + (\hat{\sigma}^*)^2\right] + (1-2\alpha)c_{11} + \alpha - 1 = 0,$$

$$c_{22} = \frac{2b\alpha + \left[1 - 2\alpha(\hat{\sigma}^*)^2\right] \cdot \tan(b\alpha\hat{t})}{2b\alpha + \tan(b\alpha\hat{t})},$$

$$\Gamma(\hat{t}) = \hat{\sigma}^* \delta(\hat{t}) + \alpha\hat{\sigma}^* \frac{c_{11} + c_{22} - 2 + 1/\alpha}{c_{22}}. \tag{22}$$

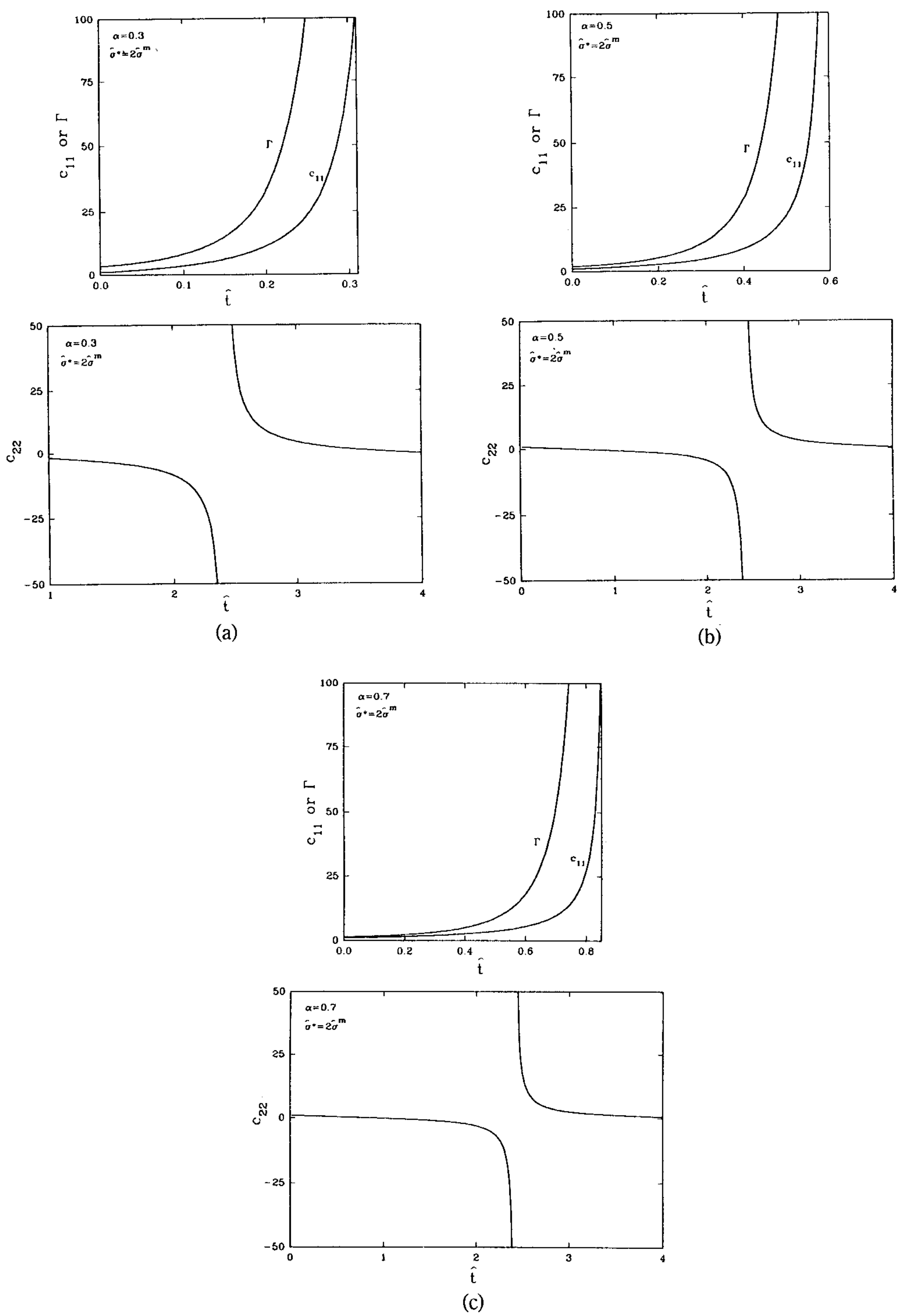

Fig. 3. (a) Blow-up instability of various rheological variables exhibited by the Giesekus model in creep shear flow (α=0.3).
(b) Blow-up instability of various rheological variables exhibited by the Giesekus model in creep shear flow (α=0.5).
(c) Blow-up instability of various rheological variables exhibited by the Giesekus model in creep shear flow (α=0.7).

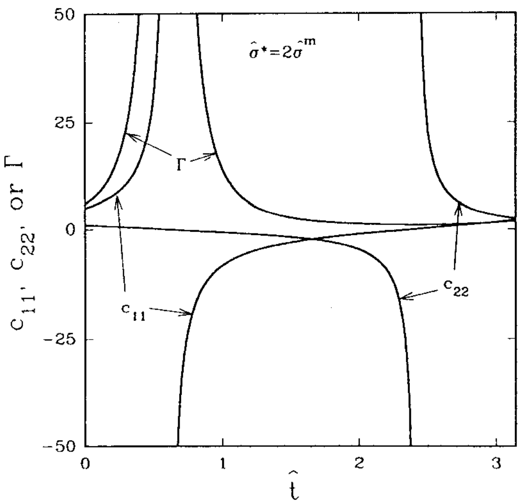

Fig. 4. Blow-up instability of various rheological variables exhibited by the Leonov model in creep shear flow.

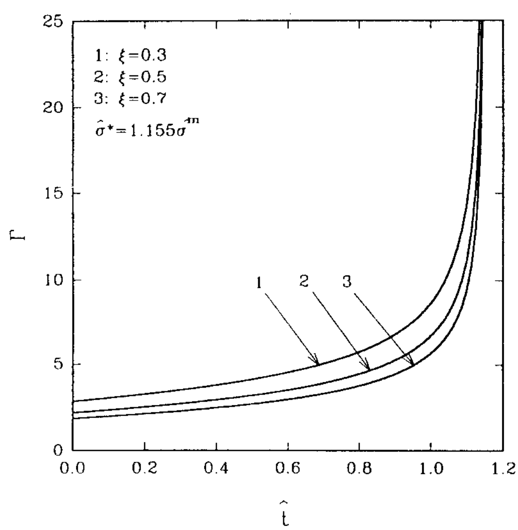

Fig. 5. Blow-up instability of dimensionless shear rate exhibited by the Larson model in creep shear flow.

Here $\delta(\hat{t})$ is a Dirac delta function. For several values of a parameter α, by solving numerically the above equations, the severe instability is demonstrated in Fig. 3, which shows clearly that the solution goes to infinity in a finite time.

<u>the Leonov model</u>

The case of $\alpha=1/2$ in eqs.(6) under the constant stress (21) yields an explicit solution

$$c_{11}=(2+b^2)\frac{1+b^{-1}\tan(b\hat{t}/2)}{1-b\cdot\tan(b\hat{t}/2)},$$

$$c_{22}=\frac{1-b\cdot\tan(b\hat{t}/2)}{1+b^{-1}\cdot\tan(b\hat{t}/2)},$$

$$\Gamma(\hat{t})=\hat{\sigma}^*\delta(\hat{t})+\frac{\sqrt{1+b^2}}{2}\left\{(2+b^2)\left[\frac{1+b^{-1}\tan(b\hat{t}/2)}{1-b\cdot\tan(b\hat{t}/2)}\right]^2+1\right\}. \tag{23}$$

The above solution is plotted in Fig.4 against a dimensionless time.

<u>the Larson model</u>

For this model, we meet two separate cases. One of them corresponds to the case when the applied stress is in the range of $\sqrt{3/(8\xi)}<\hat{\sigma}^*\le\sqrt{3/(4\xi)}$, which yields

$$\frac{\partial c_{11}}{\partial\hat{t}}=\frac{3}{\xi}\left\{\frac{1+\xi(c_{11}-1)/3}{1-2\xi(\hat{\sigma}^*)^2\left[1+\xi(c_{11}-1)/3\right]/3}-\left[1+\xi(c_{11}-1)/3\right]^2\right\},\quad c_{12}=\left[1+\xi(c_{11}-1)/3\right]\hat{\sigma}^*,$$

$$\Gamma(\hat{t})=b\cdot\delta(\hat{t})+\hat{\sigma}^*\frac{1+\xi(c_{11}-1)/3}{1-2\xi(\hat{\sigma}^*)^2\left[1+\xi(c_{11}-1)/3\right]/3},$$

$$c_{11}(\hat{t}=0^+)=b^2+1,\quad c_{12}(\hat{t}=0^+)=b, \tag{24}$$

where the solution of the dimensionless shear rate is computed in Fig. 5 for several values of ξ. On the other hand, when $\hat{\sigma}^*>\sqrt{3/(4\xi)}$, there exists no solution.

Through eqs.(22)~(24), the supercriticity constant b for each CE is

$$b=\begin{cases}\sqrt{(\hat{\sigma}^*)^2-(\hat{\sigma}^m)^2} & \text{for the Giesekus and the Leonov}\\ \left[1-\sqrt{1-4\xi(\hat{\sigma}^*)^2/3}\right]/(2\xi\hat{\sigma}^*/3) & \text{for the Larson model.}\end{cases} \tag{25}$$

All three differential CE's under study showed a severe "blow-up" instability or sometimes even non-existence of solutions presented in eqs.(22)~(24) or in Figs. 3~5. Evidently, this pathological behavior has no physical sense, hence under large shear stress the description of high elastic phenomena demonstrated by these single-mode models is unphysical and has to be improved.

3. Discussion

In this work, the dissipative type of 1-D instability for three Maxwell-like differential CE's such as the Giesekus, the simplest Leonov and the Larson models was discussed in order to investigate the fundamental properties of viscoelastic CE's. However, the global analysis of dissipative stability is far away from being completed, if it is generally possible. Nevertheless, one distinct pattern of dissipative instability has been revealed, which is related to the monotonically and unboundedly increasing steady flow curves in simple shear.

In the steady simple shear flow, the Giesekus and the Larson models possess the decreasing branch of flow curves. The instability of the Giesekus CE related to this decreasing branch was first proved by Schleiniger and Weinacht [7, 8]. In this paper, the instability of the Larson as well as the Giesekus CE is shown in a unified form, and the result for the Giesekus model coincides exactly with the previous one [7].

Regarding the boundedness of steady flow curves which exists for all three models, the severe instability was predicted in the paper [11], and it is illustrated here in detail in the case of step loading. This instability appears in a blow-up type even without the inclusion of inertia terms.

There is a tough problem as to how to distinguish the unstable behavior caused by poor modeling of CE's and the observed physical instabilities which those equations should also describe. However, the long history of various branches of continuum mechanics and physics teaches us that occurrence of an ill-posedness in 1-D situations without such important physical reasons as phase transitions, is a sign of inappropriateness in the CE's. Thus, we should treat the dissipative instability demonstrated in this paper as being associated not with the real instabilities observed in flows of polymer melts, but rather with the improper modeling of terms in CE's, and it has to be removed from the further consideration. In the numerical simulation of complex flows with unstable CE's, when the flow rate becomes high enough, occurrence of various unphysical instabilities is inevitable. Even in the range of the moderate Deborah number, the existence of singular points in flow geometry such as the corner singularity in die entrance region, is sufficient to spoil the whole numerical procedure.

In the case of the simplest Leonov model, we can improve its behavior in several ways. The simplest way is to add a small Newtonian viscous term in the third of eqs.(1), but it will prevent the CE from the instantaneous elastic response. Next, we can modify the elastic potential F to satisfy the constraint of convexity already suggested in the original paper [18]. Finally, the dissipative term can be replaced by the other stable functional form within the category of the general Leonov class of CE's [18].

For the Giesekus CE, the first way of improvement can be applied only in the case of $0 \leq \alpha \leq 1/2$. In addition, the second method is also available for the general class of the Giesekus CE's. However, for the Larson model based on the molecular theory, no stabilizing procedure has been found so far.

References

1. M. J. Crochet, A. R. Davies and K. Walters, "Numerical Simulation of Non-Newtonian Flow", Elesevier, Amsterdam, 1984.
2. D. D. Joseph, "Fluid Mechanics of Viscoelastic Liquids", Springer, New York, 1990.
3. A. I. Leonov, *J. Non-Newton. Fluid Mech.*, **42**, 323 (1992).
4. Y. Kwon, and A. I. Leonov, *J. Non-Newton. Fluid Mech.*, **47**, 77 (1993).
5. Y. Kwon, and A. I. Leonov, *J. Non-Newton. Fluid Mech.*, **58**, 25 (1995)
6. C. Truesdell and W. Noll, "The Non-Linear Filed Theories of Mechanics", Springer-Verlag, New York, 1992.
7. G. Schleiniger, and R. J. Weinacht, *J. Rheol.*, **35**, 1157 (1991).
8. G. Schleiniger, and R. J. Weinacht, *J. Non-Newton. Fluid Mech.*, **40**, 79 (1991).
9. M. A. Hulsen, *J. Non-Newton. Fluid Mech.*, **38**, 93 (1990).
10. Y. Kwon, and A. I. Leonov, *Rheol. Acta*, **33**, 398 (1994).
11. Y. Kwon, and A. I. Leonov, *J. Rheol.*, **36**, 1515 (1992).
12. U. Akbay, E. Becker, S. Krozer, and S. Sponagel, *Mech. Res. Comm.*, **7**, 1991.
13. M. Ahrens, D. D. Joseph, M. Renardy, and Y. Renardy, *Rheol. Acta*, **23**, 345 (1984).
14. J. Dunwoody, and D. D. Joseph, Arch. Rat. *Mech. Anal.*, **86**, 65 (1984).
15. K. E. P. Adewal, and A. I. Leonov, *J. Non-Newton. Fluid Mech.*, **49**, 133 (1993).
16. H. Giesekus, *Rheol. Acta*, **2**, 50 (1962).
17. H. Giesekus, *J. Non-Newton. Fluid Mech.*, **11**, 69 (1982).
18. A. I. Leonov, *Rheol. Acta*, **15**, 85 (1976).
19. R. G. Larson, *J. Rheol.*, **28**, 545 (1984).
20. J. Y. Yoo, and H. C. Choi, *Rheol. Acta*, **28**, 13 (1991).
21. A. I. Leonov, E. Ch. Lipkina, E. D. Paskhin and A. N. Prokunin, *Rheol. Acta.*, **15**, 411 (1976).