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〈연구논문〉

평면 응력 상태에서 이등방성탄-소성 재료의 전단띠 형성

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Shear Band Formation in an Elasto - Plastic Orthotropic Material Under Plane Stress Deformation

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요 약

본 논문에서는 전단띠 형성에 있어서 전단변형의 집중화 현상을 이방성 탄-소성 재료에 대해서 해석하였고 소성스핀과 비등방성이 전단띠 형성에 미치는 영향를 연구하였다. 평면응력 상태에서 소성스핀을 갖고있는 이방성 탄-소성 재료에 대해서 재료 불안정 해석을 수행하여 변형률 집중화의 시작에 미치는 소성스핀과 비등방성의 효과를 연구하였다. 해석 결과 이방성 재료에서의 전단띠 형성은 압축 또는 인장의 하중 형태나 이방성 축의 초기 각도 그리고 소성스핀의 크기에 따라 그 시작이 촉진되거나 지연되었고 전단띠 생성의 방향도 달라졌다.

Abstract—The formation of strain localization in the form of shear banding is examined for an orthotropic elasto-plastic material for the purpose of studying the effect of the plastic spin and the anisotropy upon the strain localization. The material instability analysis is carried out for an orthotropic elasto-plastic material with plastic spin in the state of plane stress. The results show that the plastic spin may hasten or delay the initiation of shear band development depending upon the deformation or loading type - compression or tension, the initial orientation of the orthotropic axes, and the algebraic sign of the plastic spin.
Keywords : Strain localization, orthotropic elasto-plastic materials, plastic spin, material instability analysis

1. Introduction

Strain localization is described by various physical mechanisms, and extensive theoretical, experimental and numerical studies have been performed in order to identify the basic mechanisms of shear banding. Within the material instability framework, Thomas[1], Hill[2], Rice[3] and others elucidated the condition for the onset of localization. Within this framework, the localization of deformation is very sensitive to constitutive features, and the material instability analysis is useful for identifying such constitutive features. Indeed, consideration of some deviations from the von Mises type classical idealization may lead to prediction of realistic strains at the onset of localization : for example, yield surface vertex(Rudnicki and Rice[4], and Rice[5], Needleman and Rice[6], Nemat-Nasser[7]), yield surface with a large curvature(Mear and Hutchinson[8], Tvergaard[9]), deviation from plastic normality, i. e., deviation from an associated flow rule(Rudnicki and Rice[4], Rice and Rudnicki[10], Bigoni and Zaccaria[11]) or dilatancy effect induced by the nucleation and growth of microvoids(Rudnicki and Rice[4], Needleman and Rice[6], Lee[12]).

Among a number of investigations dealing with the phenomenon of shear localization found in the literature, there has been only a little effort to examine the effect of anisotropy upon the onset of instability and the development of shear band for the anisotropic materials. In the large deformation analysis of materials with a substructure, the plastic spin concept was introduced for modeling the reorientation of this substructure, and the effect of plastic spin upon the simple shear as well as upon non-homogeneous deformations has been examined by extensive studies(Loret[13], Dafalias[14-16], Im and Atluri[17]), and the plastic spin was useful in determining the evolution of purely orientational variables for the orthotropic materials (Dafalias[15], [18, 19], Loret and Dafalias[20]). Moreover, in the recent work of Tvergaard and van der Giessen[21] and Zhu et al.[22], it has been shown that the back stress and the plastic spin have a significant influence upon the onset of instability. For a porous ductile material, Tvergaard and van der Giessen [21] found that the localization behavior predicted by a kinematic hardening theory is sensitive to the corotational stress rates used for the finite strain generalization of the material model and a significant delay of final void-sheet fracture is predicted when plastic spin is neglected. Zhu et al.[22] examined the stability of homogeneous deformation of biaxial stretching using a linear stability analysis, and they found the softening effect of the plastic spin causing instability of positive strain hardening for the plane strain deformation with von Mises flow rule.

In this paper, we are concerned with material instability analysis of an orthotropic elastic-plastic materials under plane stress biaxial loading. We employ Hill's orthotropic yield criterion in material instability analysis of a rate-independent orthotropic material, and rely upon the plastic spin concept for representing evolution of orientation for the material orthotropic axes(Loret and Dafalias[20]).

The material instability analysis under a given plane stress loading results in the critical hardening modulus and the corresponding shear band orientation for the rate-independent orthotropic materials. The effect of plastic spin upon the onset of shear band in the orthotropic elastoplastic material is found via this instability analysis. The result of this instability analysis indicates that the initial orientation of orthotropic axes has a significant effect upon the shear band orientation.

The results of analysis show that plastic spin may hasten or delay the initiation of shear band development depending upon the loading type - compression or tension, stress ratio, the initial orientation of the orthotropic axes and the algebraic sign of the plastic spin parameter.

2. Elastoplastic Constitutive Model

Decompositions of rate of deformation tensor $\mathbf{D}$ and material spin tensor $\mathbf{W}$ into elastic and plastic(inelastic) parts may be written as

$$\mathbf{D}=\mathbf{D}^E+\mathbf{D}^P, \ \mathbf{W}=\boldsymbol{\omega}+\mathbf{W}^p \tag{1}$$

where $\mathbf{D}^e$ and $\mathbf{D}^p$ are elastic and plastic parts of $\mathbf{D}$ respectively, $\boldsymbol{\omega}$ is rigid body spin of substructure and $\mathbf{W}^p$ is the plastic spin. $\mathbf{W}^p$ expresses the rate of rotation of the continuum with respect to its substructure in the process of inelastic deformations. The corotational rate $\mathring{\mathbf{a}}$ of a material state variable tensor $\mathbf{a}$ with respect to $\boldsymbol{\omega}$ is defined by

$$\mathring{\mathbf{a}} = \dot{\mathbf{a}} - \boldsymbol{\omega}\mathbf{a} + \mathbf{a}\boldsymbol{\omega} = \dot{\mathbf{a}}^J - \mathbf{a}\mathbf{W}^p + \mathbf{W}^p\mathbf{a} \tag{2}$$

where $\dot{\mathbf{a}}$ is the time rate of $\mathbf{a}$ and the superscript "J" on $\mathbf{a}$ indicates the Jaumann rate.

With the state variables defined as the Cauchy stress $\boldsymbol{\sigma}$ and a set of structure variables consisting of a second order tensor $\mathbf{a}$ and a scalar k, the rate form of the constitutive relations for large deformation elasto-plasticity can be expressed as(see Dafalias[18])

$$\mathbf{D}^E=\mathbf{L}^{-1}:\boldsymbol{\sigma}^o,\ \mathbf{D}^p=\langle\lambda\rangle\mathbf{N}^p,\ \mathbf{W}^p=\langle\lambda\rangle\boldsymbol{\Omega}^p,$$
$$\dot{\mathbf{a}}^o=\langle\lambda\rangle\bar{\mathbf{a}}_i,\ \dot{k}=\langle\lambda\rangle\bar{k}_i \tag{3}$$

where summation is implied over i, $\mathbf{L}$ is elasticity tensor, $\langle\ \rangle$ is the Macauley bracket, λ is a loading index linear in $\dot{\boldsymbol{\sigma}}$, and $\mathbf{N}^p$, $\boldsymbol{\Omega}^p$, $\bar{\mathbf{a}}_i$ and $\bar{k}_i$ define the direction of $\mathbf{D}^p$, $\mathbf{W}^p$, $\dot{\mathbf{a}}^o$ and $\dot{k}$. Invariance require ments under any superposed rigid-body rotation render the $\mathbf{D}^p$, $\mathbf{W}^p$, $\dot{\mathbf{a}}^o$ and $\dot{k}$ isotropic functions of $\boldsymbol{\sigma}$, $\mathbf{a}$ and k. If $\mathbf{a}$ is a purely orientational variable, then $\dot{\mathbf{a}}^o\equiv\mathbf{0}$ for any value of $\langle\lambda\rangle$, which implies $\bar{\mathbf{a}}_i\equiv\mathbf{0}$.

Consider an orthotropic material with orthonormal basis $\mathbf{n}_i$ (i=1, 2, 3) along the axes of orthotropy. Then the purely orientational structure variables for orthotropic materials are written as(Liu[23])

$$\boldsymbol{\alpha}_i=\mathbf{n}_i\times\mathbf{n}_i\ \text{(no sum over i)} \tag{4}$$

and its corotational rate associated with the rigid body spin $\boldsymbol{\omega}$ of substructure becomes always zero since $\boldsymbol{\alpha}_i$ specify only the orientation of the orthotropic axes.

For Hill's orthotropic material, a yield criterion for an orthotropic plastically incompressible solid is given by

$$f=J-\sigma_y=0 \tag{5}$$
$$\text{with } J(\boldsymbol{\sigma},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)=[A(\hat{\sigma}_{11}-\hat{\sigma}_{22})^2+B(\hat{\sigma}_{22}-\hat{\sigma}_{33})^2+C(\hat{\sigma}_{33}-\hat{\sigma}_{11})^2+2D\hat{\sigma}_{23}^2+2E\hat{\sigma}_{31}^2+2F\hat{\sigma}_{12}^2]^{1/2}$$

where A, B, C, D, E and F are material constants and for the isotropic case A=B=C=1, D=E=F=3, and $J=\sqrt{2}(\frac{3}{2}\sigma_{ij}'\sigma_{ij}')^{1/2}$; moreover σ_y is the reference yield stress such that $\sigma_y=\sqrt{A+C}\,\hat{\sigma}_{11}^Y=\sqrt{A+B}\hat{\sigma}_{22}^Y=\sqrt{A+B}\hat{\sigma}_{33}^Y$ with $\hat{\sigma}_{11}^Y$, $\hat{\sigma}_{22}^Y$, $\hat{\sigma}_{33}^Y$ being the uniaxial yield stress along each of the orthotropic axes. A superposed $\wedge$ denotes tensor components in reference to $\mathbf{n}_i$ and $\hat{\sigma}_{ij}$ is represented in terms of σ_{ij} and purely orientational variable $\mathbf{n}_i$ (or α_i) as follows :

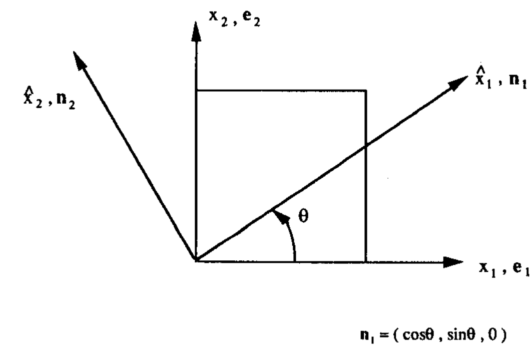

Fig. 1. Schematic illustration of coordinate system in the orthotropic material.

$$\hat{\sigma}_{11}=\mathbf{n}_1\cdot\sigma\cdot\mathbf{n}_1,\ \hat{\sigma}_{22}=\mathbf{n}_2\cdot\sigma\cdot\mathbf{n}_2,\ \hat{\sigma}_{12}=\mathbf{n}_1\cdot\sigma\cdot\mathbf{n}_2,\ \text{etc.} \tag{6}$$

From Eqns (3), we employ an elasto-plastic constitutive model with plastic spin as follows :

$$\overset{\cdot J}{\boldsymbol{\sigma}}=\mathbf{L}:\mathbf{D}-\langle\lambda\rangle\mathbf{L}:\mathbf{N}^P-\mathbf{W}^P\boldsymbol{\sigma}+\boldsymbol{\sigma}\mathbf{W}^P,$$
$$\overset{\cdot J}{\boldsymbol{\sigma}_i}=\boldsymbol{\alpha}_i\mathbf{W}^P+\mathbf{W}^P\boldsymbol{\alpha}_i, \tag{7}$$
$$\dot{\mathbf{n}}_i^J=-\mathbf{W}^P\mathbf{n}_i$$

Here a superposed "J" implies the classical Jaumann rate as before and $\mathbf{N}_p$ is defined by $\partial J/\partial\boldsymbol{\sigma}$(associative flow rule).

Based on the representation theorem, Dafalias 15,16,18 and Loret and Dafalias 20 proposed the expression for the evolution of plastic spin

$$\boldsymbol{\Omega}^P=\eta_1(\boldsymbol{\alpha}_1\boldsymbol{\sigma}-\boldsymbol{\sigma}\boldsymbol{\alpha}_1)+\eta_2(\boldsymbol{\alpha}_2\boldsymbol{\sigma}-\boldsymbol{\sigma}\boldsymbol{\alpha}_2)+\eta_3(\boldsymbol{\alpha}_1\boldsymbol{\sigma}\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_2\boldsymbol{\sigma}\boldsymbol{\alpha}_1). \tag{8}$$

Here η_i may be scalar-valued functions of state variables and are considered as material constants measuring the plastic spin. In the subsequent development, however, the plastic spin parameter η_i is assumed to be a constant for simplicity. Note that $\boldsymbol{\alpha}_3$ is not included in (8) since $\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2+\boldsymbol{\alpha}_3$ becomes the identity tensor. From Eqns (3), (8) and $N_p=\partial J/\partial\boldsymbol{\sigma}$, the plastic spin components in reference to $\mathbf{n}_i$ can be expressed as follows :

$$\hat{W}^p_{12}=\hat{\eta}_3\hat{D}^p_{12},\ \hat{W}^p_{12}=\hat{\eta}_1\hat{D}^p_{23},\qquad \hat{W}^p_{13}=\hat{\eta}_2\hat{D}^p_{13} \tag{9}$$

where $\hat{\eta}_3=\frac{1}{2F}(\eta_1-\eta_2+\eta_3),\ \hat{\eta}_2=\frac{1}{2E}\eta_1,\ \hat{\eta}_1=\frac{1}{2D}\eta_2.$

Throughout the analysis, we assume that the axis $\hat{x}_3$ is identical to the axis x_3 of a Cartesian coordinate system x_i fixed in space, while the axes $\hat{x}_1$ and $\hat{x}_2$ form an angle θ with x_1 and x_1, respectively(Fig. 1).

3. Bifurcation Analysis for the Rate Independent Solids

For quasi-statically and isothermally deforming rate independent solids, shear band instabilities can be characterized in terms of bifurcation of the equilibrium path and the bifurcation analysis is useful for determining the critical strain or stress and the corresponding direction of bands at the onset of instability.

In the present section, the bifurcation analysis for the orthotropic rate independent solids is used to examine the effect of plastic spin and initial orthotropic orientation upon the onset of shear localization.

Constitutive relations for plastic solids are generally expressed as a relation between some objective rate of Cauchy or Kirchhoff stress and the rate of deformation tensor. The plastic flow rule for orthotropic elastoplastic solids can be written as

$$D^p_{ij}=\lambda\frac{\partial f}{\partial\sigma_{ij}} \tag{10}$$

where λ is a loading index and "f" is an orthotropic yield criterion in (5). A loading index λ is obtained by employing the following consistency condition and the definition of equivalent plastic strain rate.

$$\frac{df}{dt}=\dot{f}=\frac{\partial f}{\partial\sigma_{ij}}\dot{\sigma}_{ij}+\frac{\partial f}{\partial\alpha_{1ij}}\dot{\alpha}_{1ij}+\frac{\partial f}{\partial\alpha_{2ij}}\dot{\alpha}_{2ij}$$

$$-\frac{\partial\sigma_y}{\partial\bar{\varepsilon}^p}\dot{\bar{\varepsilon}}^p=\frac{\partial f}{\partial\sigma_{ij}}\dot{\sigma}^o_{ij}-\frac{\partial\sigma_y}{\partial\bar{\varepsilon}^p}\dot{\bar{\varepsilon}}^p=0 \tag{11}$$

Use is made of the following relation

$$\frac{\partial f}{\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}}+\frac{\partial f}{\partial\boldsymbol{\alpha}_1}:\dot{\boldsymbol{\alpha}}_1+\frac{\partial f}{\partial\boldsymbol{\alpha}_2}:\dot{\boldsymbol{\alpha}}_2=\frac{\partial f}{\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}}^o. \tag{12}$$

This is apparent from the physics that the yield function is affected only by the change in stress tensor observed by an observer attached to the orthotropic axes $\hat{x}_i$, and it is a straightforward matter to confirm this via algebra. From Eqns (10-12) the plastic flow rule can be rewritten as

$$D^p_{ij}=\frac{N^p_{rs}\dot{\sigma}^o_{rs}}{hX}N^p_{ij}$$

$$\text{where } N^p_{ij}=\frac{\partial f}{\partial\sigma_{ij}}=\frac{\partial J}{\partial\sigma_{ij}}=\frac{\partial J}{\partial\hat{\sigma}_{rs}}\frac{\partial\hat{\sigma}_{rs}}{\partial\sigma_{ij}}, \tag{13}$$

$$X=\sqrt{\frac{2}{3}N^p_{ij}N^p_{ij}},\ h=\frac{\partial\sigma_y}{\partial\bar{\varepsilon}^p}.$$

In Eqn (13), "h" denotes the strain hardening modulus. Then the rate of deformation tensor $\mathbf{D}$ is written as

$$D_{ij}=D^e_{ij}+D^p_{ij}=L^{-1}_{ijkl}\dot{\sigma}^o_{kl}+\frac{N^p_{kl}\dot{\sigma}^o_{kl}}{hX}N^p_{ij}. \tag{14}$$

The Eqn (14) may be inverted easily, then the constitutive relations for orthotropic elastoplastic solids are as follows :

$$\dot{\boldsymbol{\sigma}}^o=\mathbf{L}^{tan}:\mathbf{D} \tag{15}$$

$$\dot{\boldsymbol{\sigma}}^J=\mathbf{L}^{tan}:\mathbf{D}-\mathbf{W}^p\boldsymbol{\sigma}+\boldsymbol{\sigma}\mathbf{W}^p$$

$$\text{with } L^{tan}_{ijkl}=L_{ijkl}-\frac{L_{ijrs}N^p_{rs}N^p_{pq}L_{pqkl}}{hX+N^p_{pq}L_{pqrs}N^p_{rs}}$$

The component of plastic spin tensor can be expressed in terms of the rate of deformation tensor and the direction of initial orthotropic axes(Eqn (9)). For the deformation in the x_1-x_2 plane(Fig. 1), the plastic spin components are as follows :

$$W^p_{12}=\hat{W}^p_{12}=\hat{\eta}_3\hat{D}^p_{12}=\hat{\eta}_3\Big[\frac{D^p_{22}-D^p_{11}}{2}\sin 2\theta$$

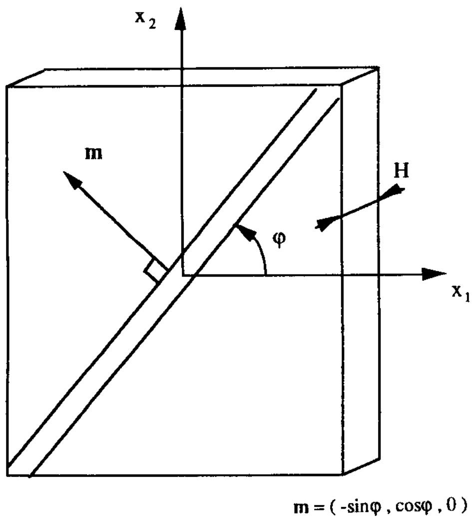

Fig. 2. A material element with a shear band.

$$+ D^p_{12}\cos 2\theta],\ \hat{W}^p_{23} = \hat{W}^p_{31} = 0 \tag{16}$$

where $W^p_{12} = \hat{W}^p_{12}$ because the $\hat{x}_3$ and x_3 are identical, and cosθ, sinθ are the components of $\mathbf{n}_1$ in reference to x_α (α=1, 2).

Within a thin band with the unit normal vector **m** in the current configuration(Fig. 2), the compatibility and equilibrium must be satisfied across the band interfaces(Rudnicki and Rice[4]).

$$\Delta\left(\frac{\partial v_j}{\partial x_j}\right) = g_i m_j \tag{17}$$

$$\mathbf{m} \cdot \Delta\dot{\boldsymbol{\sigma}} = \mathbf{0} \tag{18}$$

where Δ denotes the difference between the field inside the band and the field outside the band and **g** is the vector representing discontinuity across the band. The differences of the spin tensor and the rate of deformation tensor then can be written

$$\Delta D_{ij} = \frac{1}{2}(g_i m_j + m_i g_j),$$

$$\Delta W_{ij} = \frac{1}{2}(g_i m_j - m_i g_j) \tag{19}$$

The difference of plastic spin tensor are given as

$$\Delta W^p_{ij} = \hat{\eta}_3(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})\left(\frac{\Delta D^p_{22} - \Delta D^p_{11}}{2}\sin 2\theta + \Delta D^p_{12}\cos 2\theta\right) \tag{20}$$

where $\Delta D^p_{ij} = \frac{N^p_{kl} L^{tan}_{klpq} \Delta D_{pq}}{hX} N^p_{ij}$ under the assumption of continuous bifurcation(see Rice and Rudnicki[10]). Rice and Rudnicki[10] showed that localization first occurs as a continuous bifurcation, in which the material inside and outside the localized zone is assumed to continue to be under elastic-plastic loading at the inception of localization. Then the material at hand is conceived of being replaced by the so called "linear comparison solid," (Hill[24], Raniecki and Bruhns[25]) so that the tangent modulus L^{tan}_{ijkl} is not dependent upon the strain rate, that is, upon loading or unloading branch. Hence the values L^{tan}_{ijkl} remain the same inside and outside the band at the point of bifurcation, and the following difference can be found.

$$\Delta\dot{\sigma}^J_{ij} = L^{tan}_{ijkl}\Delta D_{kl} - \Delta W^p_{ik}\sigma_{kj} + \sigma_{ik}\Delta W^p_{kj} \tag{21}$$

$$\Delta\dot{\sigma}^J_{ij} = L^{tan}_{ijkl}\Delta D_{kl} - \Delta W^p_{ik}\sigma_{kj} + \sigma_{ik}\Delta W^p_{kj} + \Delta W_{ik}\sigma_{kj} - \sigma_{ik}\Delta W_{kj} \tag{22}$$

From the Eqns (18) through (22), we can obtain the following characteristic equation.

$$[m_i L^{tan}_{ijkl} m_l + S_{jk} + R_{jk}]g_k = 0 \tag{23}$$

where $S_{jk} = \hat{\eta}_3 m_i(\sigma_{i1}\delta_{k2} - \sigma_{i2}\delta_{j1} + \delta_{i2}\sigma_{1j} - \delta_{i1}\sigma_{2j}) \times \left\{\frac{N^p_{22} - N^p_{11}}{2}\sin 2\theta + N^p_{12}\cos 2\theta\right\}\frac{N^p_{pq}L^{tan}_{pqkl}m_l}{hX}$,

$$R_{jk} = \frac{1}{2}(m_j\sigma_{ij}m_k - m_j\sigma_{ik}m_i - \sigma_{kj} + m_p\sigma_{pq}m_q\delta_{jk})$$

S_{jk} and R_{jk} are terms which arise due to the plastic spin and the difference between $\dot{\sigma}$ and $\dot{\sigma}^J$, respectively. For the existence of a non-trivial solution, the determinant of coefficient matrix in Eqn (23) must vanish, and the condition for localization is written as

$$\det[m_i L^{tan}_{ijkl} m_l + S_{jk} + R_{jk}] = 0 \tag{24}$$

In plane stress($\sigma_{i3} = \hat{\sigma}_{i3} = 0$) Eqn (15) reduces to

$$\overset{\circ}{\sigma}_{\alpha\beta} = L^{tan}_{\alpha\beta\gamma\delta} D_{\gamma\delta} \ (\alpha, \beta, \gamma, \delta = 1, 2)$$
$$D_{33} = M_{\gamma\delta} D_{\gamma\delta} \tag{25}$$

where the moduli $L^{tan}_{\alpha\beta\gamma\delta}$ and $M_{\gamma\delta}$ are usually derived from the general three-dimentional equations by requiring $\sigma_{i3} = 0$ and solving for D_{i3} in terms of $D_{\alpha\beta}$. From the incompressibility condition the plastic rate of deformation D^p_{33} and the direction of plastic flow N^p_{33} are written as

$$D^p_{33} = -(D^p_{11} + D^p_{22}),\ N^p_{33} = -(N^p_{11} + N^p_{22}) \tag{26}$$

and then X in Eqn (13) is written as

$$X = \sqrt{\frac{4}{3}\left(N^{p\,2}_{11} + N^{p\,2}_{22} + N^{p\,2}_{12} + N^p_{11} + N^p_{11}\right)} \tag{27}$$

Consider a homogeneous plane sheet with thickness H (Fig. 2). Within a thin band with the unit normal vector $m_\alpha (\alpha = 1, 2)$, the difference in the rate of deformation across the band interface is

$$\Delta\left(\frac{\partial v_\alpha}{\partial x_\beta}\right) = g_\alpha m_\beta \tag{28}$$

and the equilibrium in the plane of the sheet must be satisfied across the band as follows (Stören and Rice[5])

$$m_\alpha (\Delta\dot{\sigma}_{\alpha\beta} + \sigma_{\alpha\beta}\Delta D_{33}) = 0 \tag{29}$$

Then, using Eqns (25), (28) and (29), the characteristic Eqn (24) for the onset of localization is rewritten as

$$\det[m_\alpha \bar{L}^{tan}_{\alpha\beta\gamma\delta} m_\delta + S_{\beta\gamma} + R_{\beta\gamma} + m_\alpha \sigma_{\alpha\beta} M_{\gamma\delta} m_\delta] = 0 \tag{30}$$

For initially specified orthotropic axes and the plastic spin coefficient η_i, the solution to Eqn (24) gives the critical strain hardening modulus h and the corresponding shear band orientation **m** for a given stress state. As the deformation progresses, however, the orthotropic axes is rotated according to the evolution equation of $\mathbf{n}_i$ (7) for the nonzero value of η_i. Determination of the orthotropic axis orientation needs the complete solution for an elasto-plastic boundary value problem under an actual hardening behavior. However, we here limit ourselves to seeking the shear band orientation simply by carrying out material instability analysis for a fixed orientation of the orthotropic axes of the type of materials as given by Eqns (5) and (7) with a hardening behavior not specified and without taking into account the rotation of orthotropic axes associated with the plastic spin.

4. Numeriacl Results and Discussion

In section 3, we obtained the characteristic Eqn (30) for the condition of a localization bifurcation under plane stress state. In this section, the solution of Eqn (30) is numerically examined in order to obtain the critical hardening modulus and the plane of localization at the onset of localization. Because the hardening modulus "h" is a decreasing function of strain in general, we seek the orientation **m** for which the value of h is maximum so that the condition for the onset of the shear band is first met. In the calculations, the loading type is specified as biaxial plane stress state, and all material parameters are kept fixed except the angle of initial orthotropic axes and that we introduce η such that $\eta = \hat{\eta}_3$ (Eqn (9)) for convenience, which is employed for representing the plastic spin instead of $\hat{\eta}_3$.

Based on the results of Stickels and Mould[26]'s experiments, we used the following material properties for carbon steel.

$E_1 = 210.85$ GPa, $\nu_{12} = 0.3$, $E_2 = 213.95$ GPa, $\nu_{23} = 0.25$,
$E_3 = 220.64$ GPa, $\nu_{31} = 0.3333$, $G_{12} = 84.75$ GPa, $G_{23} = 83.70$ GPa,
$G_{31} = 81.15$ GPa, $A = 0.551$, $B = 0.7247$, $C = 0.333$, $F = 0.633$

where A, B, C and F are calculated from the plastic strain ratio R in their experimental results. Rather than we seek for critical stress level or critical ha-

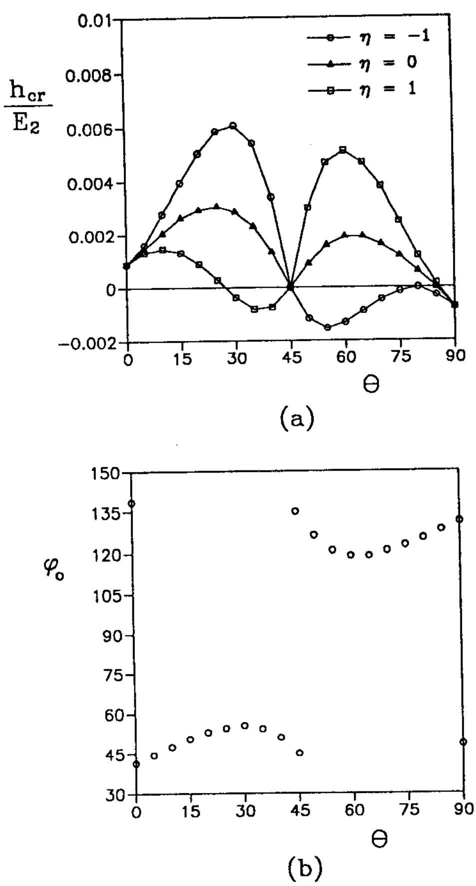

Fig. 3. (a) The critical strain hardening modulus versus the orientation of orthotropic axes, (b) The corresponding angle of shear band versus the orientation of orthotropic axes (plane stress biaxial loading $\sigma_{11}/\sigma_{22}=-1$).

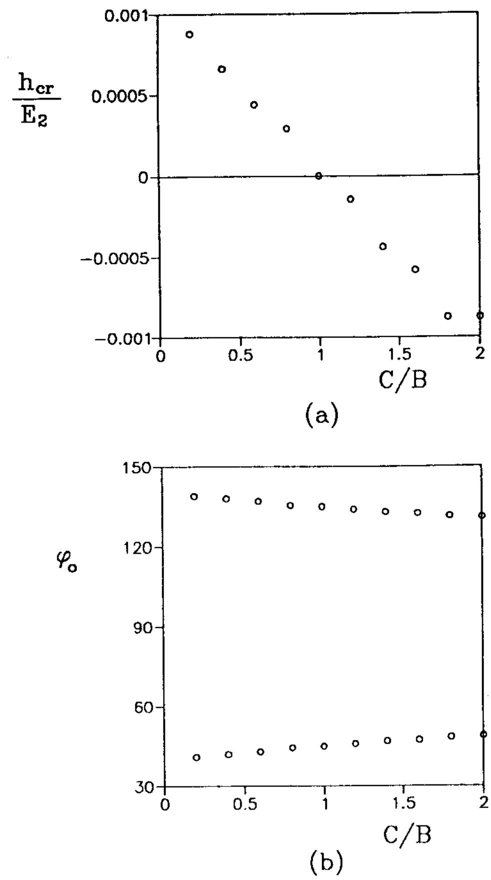

Fig. 4. Dependence of h_{cr}/E_2 and ϕ_o upon the combination of B and C (plane stress biaxial loading $\sigma_{11}/\sigma_{22}=-1$ with $\theta=0°$, $\eta=0$).

rdening modulus for a given material, i. e., for a given hardening behavior, we leave the detailed hardening behavior $\sigma_y=\sigma_y(\bar{\varepsilon}^P)$ not specified and we seek for the critical hardening modulus h_{cr}/E_2 for a given stress state.

For the case of plane stress biaxial loading with $\sigma_{22}/E_2=-\sigma_{11}/E_2=0.00215$ and σ_{12} zero, Fig. 3 (a) and 3 (b) show the critical hardening modulus h_{cr}/E_2 and the corresponding angle of shear band ϕ_o for the various angles of orthotropic axes θ. In general, the stress level in continuum is much less than Young's modulus because the true stress at the point of ultimate strength is order of $10^{-3}\sim10^{-6}$ times Young's modulus for most of the materials. As will be explained, the change of stress level within such a range, i. e., σ_{11} or $\sigma_{22} \ll E_2$ does not influence the critical hardening modulus h_{cr}/E_2 and the corresponding angle of shear band.

As shown in these figures, the plastic spin and the initial orientation of orthotropic axes have a significant influence upon the onset of shear band and the corresponding band orientation for the present constitutive model at hand. Particularly the figures show that the plastic spin hastens or delays the onset of shear band depending upon the initial orientation of orthotropic axes ; in the range of $0°<\theta<45°$, the hastening occurs for the negative value of $\eta=-1$ while delay occurs for the positive value

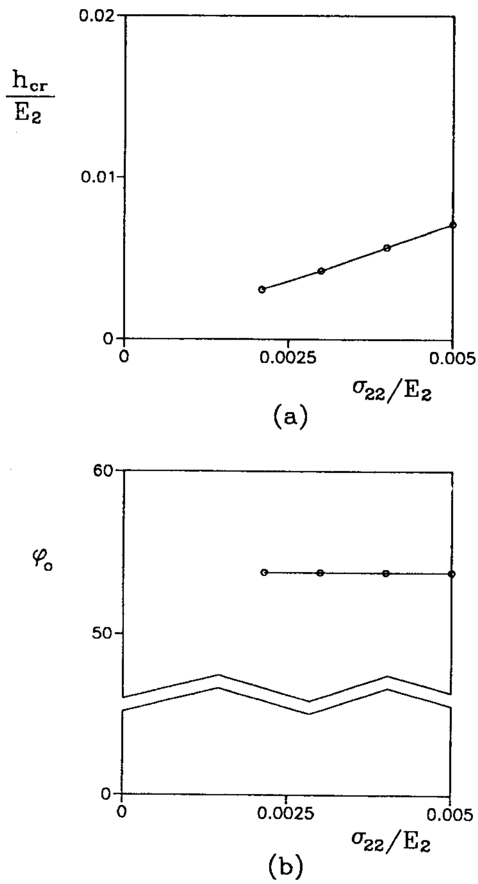

Fig. 5. Dependence of h_{cr}/E_2 and ϕ_o upon stress level (plane stress biaxial loading $\sigma_{11}/\sigma_{22}=-1$ with $\theta=22.5°$, $\eta=0$).

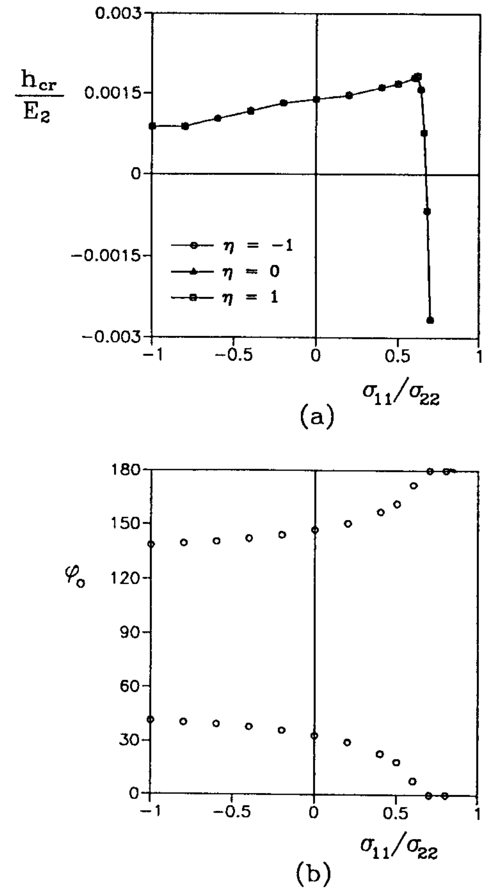

Fig. 6. (a) The critical strain hardening modulus versus stress ratio (b) The corresponding angle of shear band versus stress ratio (plane stress biaxial loading with $\theta=0°$).

of $\eta=1$ in the same range of θ. As might be expected, however, there are no influence of plastic spin parameter η near $\theta=0°$ and $90°$ because $\hat{\sigma}_{12}$ and accordingly $\hat{\Omega}^p_{12}\,\hat{\eta}_3\hat{\sigma}_{12}$ disappear from Eqn (8) for these values of θ under plane stress biaxial loading. Moreover, it is noticed that for $\theta=45°$, which is corresponding to a pure shear stress state in reference to $\hat{x}_i$, there are no hastening nor delay due to the plastic spin.

However, the critical hardening modulus h_{cr}/E_2 and the corresponding angle of shear band ϕ_o for $\theta=0°$ are different from those for $\theta=90°$. This is due to the fact that in plane stress state the direction of plastic flow $N^p_{\alpha\beta}$ for $\theta=0°$ is generally different from that for $\theta=90°$ except for B=C in yield criterion (5). Fig. 4 shows the effect of the combination of B and C on the localization mode for the plane stress biaxial $\sigma_{11}/\sigma_{22}=-1$ with $\theta=0°$, $\eta=0$.

Fig. 5 shows the effect of stress level on the critical hardening modulus h_{cr}/E_2 and the corresponding angle of shear band ϕ_o for the special case of biaxial stress state with $\theta=22.5°$, $\eta=0$. This shows that the changes of h_{cr}/E_2 and ϕ_o can be neglected for the realistic stress level(σ_{11} or $\sigma_{22} \ll E_2$) at the onset of localization.

For the case of plane stress problem, localization mode is strongly dependent upon the stress ratio

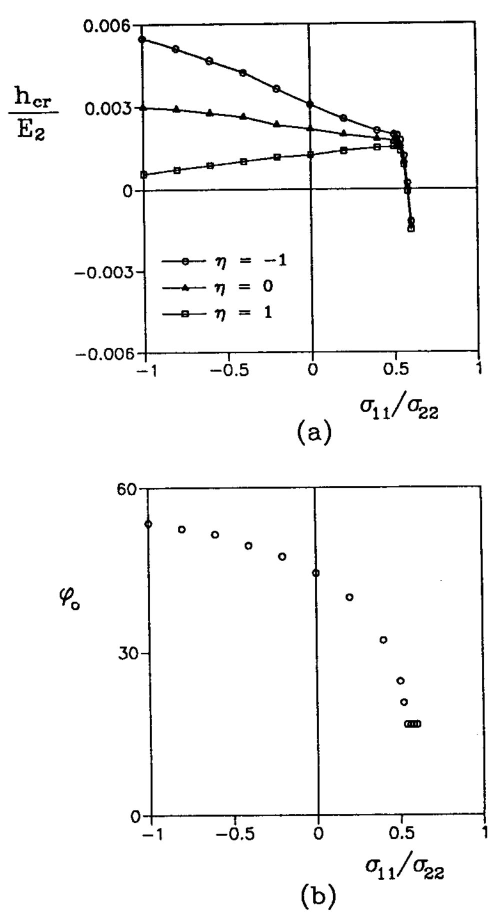

Fig. 7. (a) The critical strain hardening modulus versus stress ratio (b) The corresponding angle of shear band versus stress ratio (plane stress biaxial loading with $\theta=22.5°$).

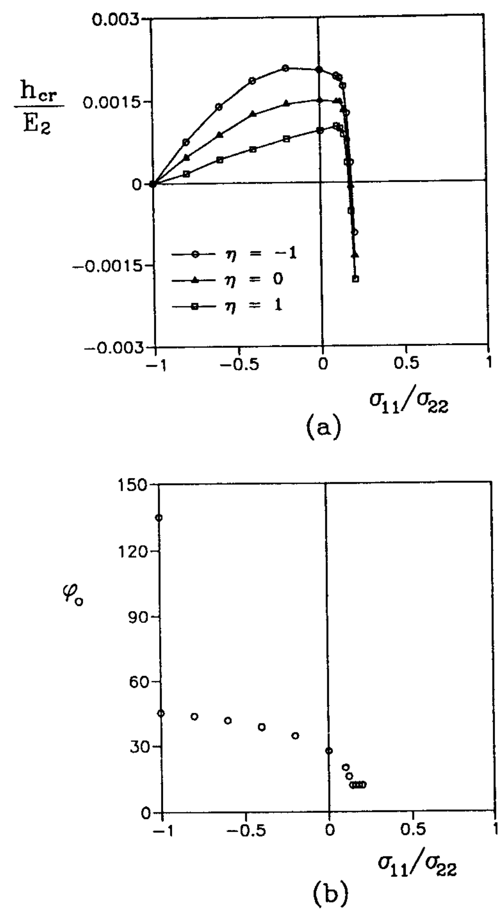

Fig. 8. (a) The critical strain hardening modulus versus stress ratio (b) The corresponding angle of shear band versus stress ratio (plane stress biaxial loading with $\theta=45°$).

(or the strain ratio) even for the isotropic material. For a isotropic, rigid-plastic, Mises material, Hill [27] predicted the critical band orientation as a function of strain ratio and he showed that no localized necking is predicted for biaxial stretching, i. e., $d\varepsilon_1>0$ and $d\varepsilon_2>0$. However, Stören and Rice[5] confirmed that realistic necking can be predicted for biaxial stretching with introducing vertices on the yield surface. Fig. 6 through 8 show the plot for h_{cr}/E_2 and the critical band orientation versus stress ratio σ_{11}/σ_{22} for the various angles of orthotropic axes θ. These results are obtained for $\sigma_{12}=0$. For $\theta=0°$, there are no influences of plastic spin as shown in Fig. 3. However, the plastic spin hastens or delays the onset of shear band for $\theta=22.5°$ and $\theta=45°$ because $\hat{\sigma}_{12}$ does not disappear from Eqn (8) except for $\sigma_{11}/\sigma_{22}=-1$ and $\theta=45°$. As the stress ratio σ_{11}/σ_{22} increases from the negative value to the positive value, the orientation of shear band approaches the direction normal to maximum principal strain, i. e., $\Phi_o=0°$ for $\theta=0°$, $\phi_o=16.5°$ for $\theta=22.5°$ and $\phi_o=12°$ for $\theta=45°$. Moreover, there is an abrupt critical hardening modulus drop that is associated with the shear band orientation normal to maximum principal strain, and no localized necking is allowed beyond a critical value of σ_{11}/σ_{22}.

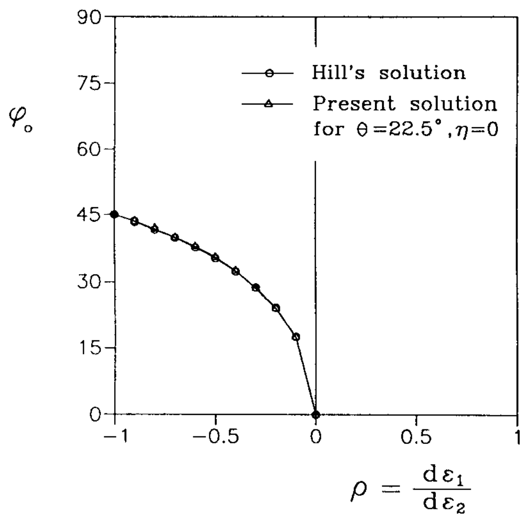

Fig. 9. The angle of shear band versus strain ratio.

For a given strain ratio $d\varepsilon_1/d\varepsilon_2=\rho$ and $d\varepsilon_{12}/d\varepsilon_2=0$, the direction of localized necking for the orthotropic material with $\theta=22.5°$, $\eta=0$ is shown in Fig. 9 and is compared with Hill's solution arctan $(\sqrt{-\rho})$. The predicted solution are very close to the zero-extension direction for $\rho\leq 0$ and localized necking is not allowed for $\rho>0$.

5. Conclusion

From the foregoing bifurcation analysis, we may draw the following conclusions for the shear band formationin the present orthotropic material :

1) Since the stress state in reference to $\hat{x}_i$ coordinate depends upon the orientation of the orthotropic axes, the formation of shear band in an orthotropic material is strongly dependent upon the orientation of the initial orthotropic axes.

2) The plastic spin, one of the constitutive features, has a substantial hastening or delay effect upon the shear band formation depending upon the initial orientation of the orthotropic axes and the deformation mode(or stress state).

3) For plane stress state, a Mises type material without the assumption of imperfections does not allow localized necking for a stress ratio or strain ratio beyond the value associated with the shear band orientation normal to maximum principal strain.

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Nomenclature

$\mathbf{a}$	: second order tensor valued structure variable
$\bar{\mathbf{a}}_i$	: direction of the corotational rate $\dot{\mathbf{a}}^o$
$\mathbf{D}$	: rate of deformation tensor
$\mathbf{D}^e$	: elastic part of $\mathbf{D}$
$\mathbf{D}^p$	: plastic part of $\mathbf{D}$
f	: yield criterion function
$\mathbf{g}$	: vector representing discontinuity across the band
h	: strain hardening modulus
h_{cr}	: critical hardening modulus corresponding to the condition for the onset of localization
J	: orthotropic yield surface
k	: scalar valued structure variable
$\bar{k}_i$	: direction of the corotational rate $\dot{k}^o$
$\mathbf{L}$	: elasticity tensor
$\mathbf{L}^{tan}$	: tangent stiffness matrix
$\bar{\mathbf{L}}^{tan}$	: tangent stiffness matrix for the case of plane stress state
$\mathbf{M}$	: modulus derived from the general three-dimensional equation by solving for D_{33} in terms of $D_{\alpha\beta}(\alpha, \beta=1, 2)$
$\mathbf{m}$	: unit vector normal to the shear band
$\mathbf{N}^p$	: direction of $\mathbf{D}^p$
$\mathbf{n}_i$	: orthonormal basis along the axes of orthotropy
$\mathbf{v}$	: velocity vector
$\mathbf{W}$	: material spin tensor

$\mathbf{W}^p$	: plastic spin tensor
A, B, C, D, E, F	: material constants in an orthotropic yield surface
$\boldsymbol{\alpha}_i$	: purely orientational structure variables for orthotropic material
δ_{ij}	: Kronecker delta
$\bar{\varepsilon}^p$	: equivalent plastic strain
$\dot{\bar{\varepsilon}}^p$	: the time rate of the equivalent plastic strain
η_i	: plastic spin parameters
λ	: loading index
$\boldsymbol{\sigma}$	: Cauchy stress tensor
σ_y	: reference yield stress
$\hat{\sigma}^Y_{11}$, $\hat{\sigma}^Y_{22}$, $\hat{\sigma}^Y_{33}$	: uniaxial yield stress along each of the orthotropic axes
$\boldsymbol{\Omega}^p$	: direction of $\mathbf{W}^p$
$\boldsymbol{\omega}$	: rigid body spin of substructure
ϕ_o	: angle between the shear band and the x_1 axis

Superscripts

•	: time rate
o	: corotational rate
J	: Jaumann rate
−1	: inverse matrix
′	: deviatoric part of a second order tensor
∧	: tensor component in reference to $\mathbf{n}_i$

References

1. T. Y. Thomas, "Plastic Flow and Fracture in Solids", *Academic Press*, 1961.
2. R. Hill, *J. Mech. Phys. Solids,* **10**, 1 (1962).
3. J. R. Rice, "The localization of plastic deformation", Proc. 14th *Int. Congr. Theoret. Appl. Mech.*, 1977, p.207.
4. J. W. Rudnicki and J. R. Rice, *J. Mech. Phys. Solids*, **23**, 371 (1975).
5. S. Stören and J. R. Rice, *J. Mech. Phys. Solids*, **23**, 421 (1975).
6. A. Needleman and J. R. Rice, "Limits to ductility set by plastic flow localization. Mechanics of Sheet Metal Forming", *Plenum Press*, 1978, p.237
7. S. Nemet-Nasser, *Appl. Mech. Reviews*, **45**, 19 (1992).
8. M. E. Mear and J. W. Hutchinson, *Mech. Mater.*, **4**, 395 (1985).
9. V. Tvergaard, *J. Mech. Phys. Solids*, **35**, 43 (1987).
10. J. R. Rice and J. W. Rudnicki, *Int. J. Solids Struct.*, **16**, 597 (1980).
11. D. Bigoni and D. Zaccaria, *Int. J. Plasticity,* **9**, 21 (1993).
12. Y. K. Lee, *Int. J. Plasticity*, **5**, 197 (1989).
13. B. Loret, *Mech. Mater.*, **2**, 287 (1983).
14. Y. F. Dafa;oas, *J. Appl. Mech.*, **50**, 561 (1983).
15. Y. F. Dafalias, *Mech. Mater.*, **3**, 223 (1984).
16. Y. F. Dafalias, *J. Appl. Mech.*, **52**, 865 (1985).
17. S. Im and S. N. Atluri, *Int. J. Plasticity*, **3**, 163 (1987).
18. Y. F. Dafalias, "Orientational, rate and temperature effects on the plastic response of anisotropic solids. Yielding, Damage and Failure of Anisotropic Solids", Mechanical Engineering Publications Ltd., London, 1990, p.359
19. Y. F. Dafalias, *Int. J. Solids Struct.*, **26**, 149 (1990).
20. B. Loret and Y. F. Dafalias, *J. Mech. Phys. Solids*, **40**, 417 (1992).
21. V. Tvergaard and Van Der Giessen, *J. Mech. Phys. Solids*, **39**, 763 (1991).
22. H. **Zhu**, H. M. **Zbib** and E. C. **Aifantis**, *Appl. Mech. Reviews*, **45**, 110 (1992).
23. I. S. Liu, *Int. J. Engng Sci.*, **20**, 1099 (1982).
24. R. Hill, *J. Mech. Phys. Solids*, **6**, 236 (1958).
25. B. Raniecki and O. T. Bruhns, *J. Mech. Phys. Solids*, **29**, 153 (1981).
26. C. A. Stickels and P. R. Mould, *Met. Trans.*, **1**, 1303 (1970).
27. R. Hill, *J. Mech. Phys. Solids*, **1**, 19 (1952).