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광범위한 전단률에서 적용 가능한 수정된 POWER LAW 유체에 대한 GRAETZ문제 해법

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Graetz Problem Solutions for a Modified Power Law Fluid over a Wide Range of Shear Rate

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요 약

층류 열전달에서 작은 Reynolds Number에서의 원통내의 흐름에대한 Power Law 유체의 해법이 사용될수가 없는것은 낮은 전단률(Shear Rate)에서 Power Law 법칙이 적용되지 않기 때문이다. 본 연구에서는 이 문제를 새로운 구성방정식을 이용하여 높은 전단률에서는 Power Law 법칙이, 낮은 전단률에서는 Newtonian 법칙이, 그 중간에서는 천이과정을 모두 포함하는 전범위의 전단률에 대해 해결하였다(Modified Power Law Equation). 이러한 구성방정식의 기본 개념은 실험결과와도 잘 일치하였다. 이 구성방정식을 이용하여 Graetz문제를 해결하였고 경계조건은 일정온도(Constant Wall Temperature)와 일정열속 (Constant Heat Flux)이다. 이용하기 쉬운 전단률변수는 적용범위가 Newtonian인지 Power Law인지 아니면 중간 천이과정인지를 나타내준다.

Abstract—Power law fluid solutions for laminar heat transfer in a circular duct may not be applicable at low Reynolds numbers where the low shear rates are out of the power law region. The present study investigates this problem by using a constitutive equation (Modified Power Law Equation) which predicts Newtonian behavior at low shear rates leading to a rheological transition region at intermediate shear rates and power law behavior at higher shear rates. Such basic concept of the Modified Power Law Equation has shown good agreement with experimental viscometric data. Using this constitutive equation, the Graetz problem has been solved numerically for both the constant wall temperature and constant wall heat flux cases. A convenient parameter specifies the shear rate range under particular operating conditions and thus determines the applicability of the classical Newtonian or power law solutions or the present solution in the transition region.

Keywords: Modified power law equation, Graetz problem, low shear rate, shear rate parameter, circular duct

1. Introduction

The prediction of heat transfer to fluids flowing in circular tubes is important in engineering applications. For fully developed laminar flow of a Newtonian fluid, the solutions are well known for both classical boundary conditions of constant wall temperature(CWT) and constant wall heat flux (CHF)[1]. For power law fluids, such solutions are also available[2, 3]. For situation where the velocity profile is fully developed but the temperature profile is developing (the Graetz problem), solutions have also been reported for Newtonian fluids for both boundary conditions[1] and for power law fluids[4, 5, 6, 7, 8].

However many non-Newtonian fluids have viscous properties as shown in Fig. 1 where at low shear rates the fluid is Newtonian ($\eta_a = \eta_o$) and in the high shear rate range the fluid acts as a power law fluid. Between these two region is a transition zone. Such a rheological behavior poses several problems.

1) It must be determined in which shear rate range the system is operating and if either of the Newtonian or power law solutions can be applied. This is not always simple because there is not a suitable shear rate parameter available since the solutions were obtained independently.

2) If the shear rate range falls within the transition zone, no solution yet exists for this region for the type of non-Newtonian fluid considered here.

3) In the thermally developing region, in order to conserve similitude both the Graetz number (and the flow index for power law fluids) and the shear rate must be considered. At present there is no convenient way that this can be done.

The present paper attempts to correct this situation by presenting a solution for fluids having the rheological characteristics illustrated in Fig. 1. Such a solution should have the characteristics that at low velocities(low shear rates) the Newtonian solution is an asymptote while at large shear rates the power law solution is an asymptote. In addition, the solution should predict the appropriate heat transfer behavior in the transition zone. Finally a parameter is needed to predict the proper solution to be used in terms of the operating characteristics of the system.

2. Analysis

A number of constitutive equations can describe the apparent viscosity-shear rate relation for fluids such as shown in Fig. 1. A convenient and useful equation is the "modified power law model" which to the authors' knowledge was first used by Dunleavy and Middleman[9].

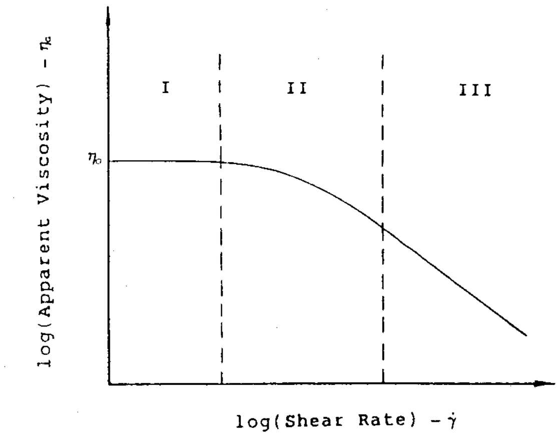

Fig. 1. Typical flow curve for a pseudoplastic fluid. I-Newtonian Region, II-Transition Region, III-Power Law Region

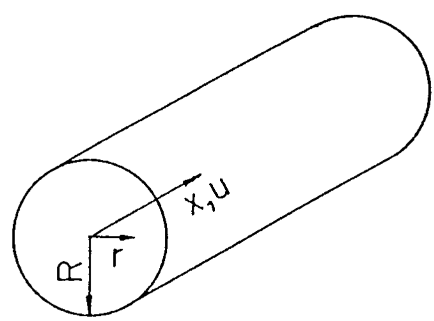

Fig. 2. Coordinate system and nomenclature for the circular duct.

$$\eta_a = \frac{\eta_o}{1 + \frac{\eta_o}{K}(\dot{\gamma})^{1-n}} \tag{1}$$

Inspection of Eq.(1) reveals that at low shear rates ($\eta_o/K(\dot{\gamma})^{1-n} \ll 1$) the apparent viscosity becomes equal to η_o and the fluid is operating in the Newtonian region of Fig. 1. At high shear rates ($\eta_o = K(\dot{\gamma})^{n-1} \gg 1$) the fluid becomes a power law fluid where $\eta_a = K(\dot{\gamma})^{n-1}$. At intermediate shear rates, there is a transition zone. An additional advantage of the modified power law over other constitutive equations such as Ellis, Sutterby, Cross. etc., is that the familiar Newtonian and power law Reynolds and Prandtl numbers are retained in the analysis.

If the circular duct shown in Fig. 2 is considered, the fully developed shear stress field is described by the equation(see nomenclature)

$$\frac{1}{r}\frac{d}{dr}(\gamma\tau)= -\frac{dp}{dx} \tag{2}$$

The following dimensionless quantities may now be defined

$$r^+=\frac{r}{R} \qquad f=\frac{2d\frac{dp}{dx}}{\rho\bar{u}^2}$$

$$u^+=\frac{u}{\bar{u}} \qquad Re_g=\frac{\rho\bar{u}^{2-n}d^n}{K}$$

$$Re=\frac{\rho\bar{u}d}{\eta_o} \qquad \beta=\frac{\eta_o}{K}\left(\frac{\bar{u}}{d}\right)^{1-n}$$

$$\eta^*=\frac{\eta_o}{1+\beta} \qquad \eta^+=\frac{1+\beta}{1+2^{1-n}\beta\left(\frac{du^+}{dr^+}\right)^{1-n}}$$

which allows Eq. (2) to be written in dimensionless form as

$$\frac{1}{r^+}\frac{d}{dr^+}(r^+\eta^+\frac{du^+}{dr^+})=-\frac{f\cdot Re_M}{8} \tag{3}$$

with boundary conditions $u^+=0$ at $r^+=1$ and $\frac{du^+}{dr^+}=0$ at $r^+=0$

$$\text{where } Re_M=\frac{\rho\bar{u}d}{\eta^*} \tag{4}$$

The parameter β is the shear rate parameter that determines whether the fluid system is operating in the Newtonian, transition or power law regions. As β becomes small, Re_M approaches the Newtonian Reynolds number Re and as β becomes large, Re_M approaches the power law Reynolds number Re_g.

The solution of Eq. (3) is described in a previous paper by Brewster and Irvine[10] and thus the appropriate velocity field was available for insertion into the energy equation for the present analysis. In addition, the above reference defines the three regions in Fig.1 in terms of β

Region I-- Newtonian, $\beta<10^{-2}$
Region II--Transition, $10^{-2}\leq\beta\leq10^2$
Region III--Power Law, $\beta>10^2$

The energy equation in dimensionless form for the thermally developing flow can be written as

$$\frac{1}{r^+}\frac{\partial}{\partial r^+}(r^+\frac{\partial T^+}{\partial r^+}) = \frac{u^+}{4}\frac{\partial T^+}{\partial x^+} \tag{5}$$

where the quantities not previously defined are

$$T^+=\frac{T-T_w}{T_o-T_w} \text{ for constant wall temperature}$$

$$T^+=\frac{T-T_o}{q_wR/k} \text{ for constant wall heat flux}$$

$$x^+=\frac{x/d}{Pe} \text{ where } Pe=\frac{\bar{u}d}{\alpha}$$

with inlet and boundary conditions

	(CWT)	(CHF)
i.c.	$T^+(0, r^+)=1$	$T^+(0, r^+)=0$
b.c.	$T^+(x^+, 1)=0$	$T^{+\prime}(x^+, 1)=1$
	$T^{+\prime}(x^+, 0)=0$	$T^{+\prime}(x^+, 0)=0$

Equation(5) was solved numerically by using "finite difference method" over the appropriate β range to determine the local Nusselt numbers from the entrance to the fully developed or asymptotic Nusselt numbers. TriDiagonal Matrix Algorithm(TDMA) and unequally spaced grid structure were employed to improve accuracy of temperature calculation when applicable. In addition, the thermal entrance lengths defined as the dimensionless distance for the local Nusselt number to reach a value of 1.05 of the full developed Nusselt number were calculated. These results will be presented in the following section.

3. Results and Discussion

The fully developed Nusselt numbers as a function of β with n as a parameter are shown in Fig. 3 for constant wall temperature and Fig. 4 for constant wall heat flux. Table 1 lists numerical

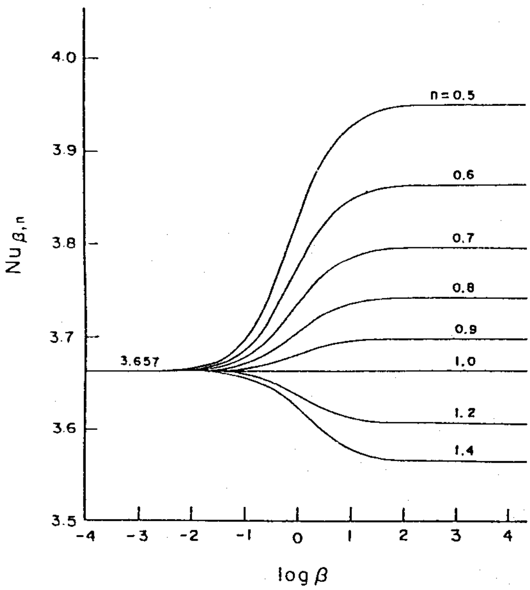

Fig. 3. Variation of the fully developed Nusslt numbers with the shear rate parameter β and n (CWT).

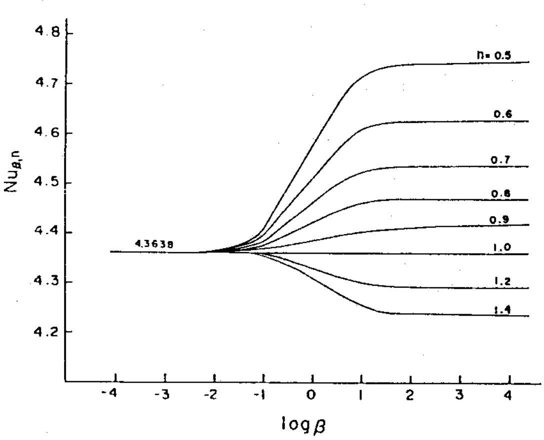

Fig. 4. Variation of the fully developed Nusselt numbers with the shear rate parameter β and n (CHF).

values of the calculated asymptotic Nusselt numbers along with asymptotic values from the literature. In Table 1, the values from Irvine and Karni[11] are only from an approximate correlation. The excellent agreement of the calculated asymptotic values with those previously published lends credence the calculated transition region Nusselt numbers where no comparisons are available.

Table 1. Comparison of calculated fully developed Nusselt numbers with previously published values

	Constant Wall Temperature		Constant Heat Flux	
n	Present Results	Reported by Irvine et al.[11]	Present Results	Reported by Bird et al.[4]
1.4	3.5564	3.568	4.2365	4.2363
1.2	3.5993	3.605	4.2907	4.2905
1.0	3.6568	3.657*	4.3638	4.3636
0.9	3.6934	3.691	4.4109	4.4107
0.8	3.7377	3.732	4.4679	4.4678
0.7	3.7921	3.783	4.5385	4.5384
0.6	3.8605	3.850	4.6281	4.6281
0.5	3.9494	3.949*	4.7456	4.7458
0.33	4.7152	4.715*		

*Bird et al. [4].

The use of tabular results for a solution with so many parameters is quite awkward. Thus it is convenient to have a correlation equation to represent the numerical data. For the fully developed Nusselt numbers for values of $0.5 \leq n \leq 1.4$ and $10^{-4} \leq \beta \leq 10^4$ the following equation represents the numerical data with a maximum error of 1.5%.

$$Nu_{\beta,n} = \frac{A(1+\beta)}{1+A\beta/\{(3n+1)/4n\}^{1/3}} \tag{6}$$

where $A = 3.6568$ for (CWT)
$A = 4.3636$ for (CHF)

Entrance region Nusselt numbers are illustrated in Fig. 5. Because of the large amount of numerical data available it is only possible to show several representative curves. Fig. 5 shows the calculated results for both boundary conditions in the Newtonian and power law regions in comparison with previously reported calculations. Also shown in the figure are the entrance Nusselt numbers in the transition region ($\beta=1$) for a fluid with $n=0.5$: The calculated curves for large and small values of β are in good agreement with previously reported results.

Of particular interest to the heat transfer designer is the extent of the thermal entrance region x^{+}_{ent}. These are shown in Figs. 6 and 7 for the (CWT) and (CHT) cases respectively. Table

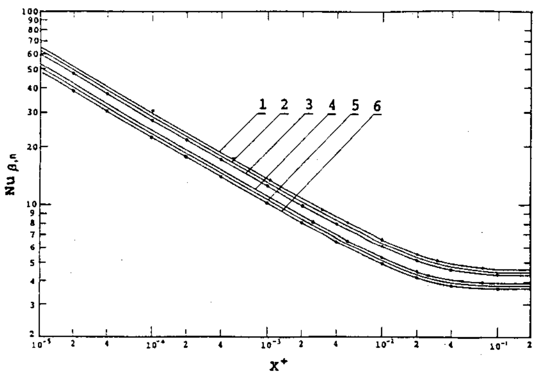

Fig. 5. Entrance region Nusselt number vs. x^+
1. n=0.5, β=10⁴(CHF), 2. n=0.5 β=1(CHF)
3. n=1, β=10⁴(CHF), 4. n=0.5 β=10⁴(CWT)
5. n=0.5, β=1(CWT) 6. n=1, β=10⁴(CWT)
Symbol •, 1-Calculation from Joshi and Bergles (1980), 3 and 6-Calculation from Shah and London(1978), 4-Calculation from Mckillop (1964).

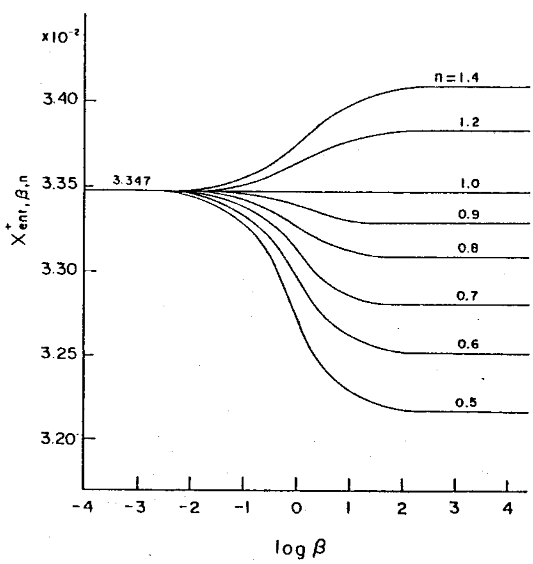

Fig. 6. Thermal entrance lengths vs. shear rate parameter β and n (CWT).

2 lists numerical values of the calculated entrance lengths for Newtonian and power law fluids along with comparisons with previously published data where available. No comparison data are available for intermediate values of β in the transition region and only one in the power law region.

Once again it is convenient to represent the

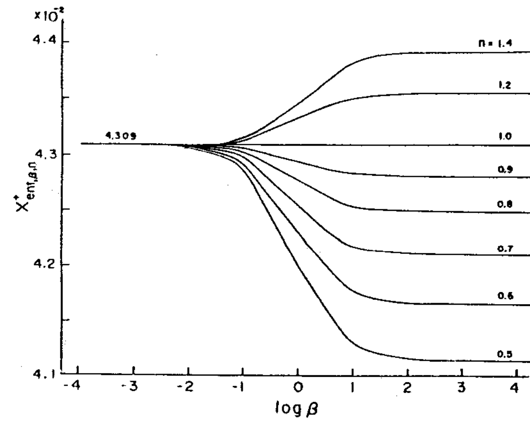

Fig. 7. Thermal entrance lengths vs. shear rate parameter β and n (CHF).

Table 2. Comparison of calculated thermal entrance lengths $x^+_{ent} \times 10^2$ for Newtonian and Power Law fluids with previously published values

	Constant Wall Temperature		Constant Heat Flux	
n	Present Results	Reported by Shah et al.[1]	Present Results	Reported by Shah et al.[1]
1.4	3.411	-	4.392	-
1.2	3.380	-	4.356	-
1.0	3.347	3.347	4.309	4.305
0.9	3.326	-	4.281	-
0.8	3.306	-	4.248	-
0.7	3.279	-	4.210	-
0.6	3.250	-	4.166	-
0.5	3.213	-	4.114	4.105*

*Joshi et al. [8].

thermal entrance length data in Fig.6 and 7 by correlation equations. These are given below and are accurated within 0.5% for 0.5 $\leq$ n $\leq$ 1.4 and $10^{-4} \leq \beta \leq 10^4$

$$x^+_{ent,\beta,n} = \frac{C(1+\beta)}{1 + C\beta / x^+_{ent}} \tag{7}$$

where C=0.03347 for (CWT)
C=0.04309 for (CHF)

Values of x^+_{ent} in Eq.(7) can be obtained from Table 2 for the appropriate boundary condition.

4. Summary Conclusion

The Graetz problem has been solved numerically for a modified power law fluid in a circular duct for the boundary conditions of constant wall temperature and constant wall heat flux. By using the modified power law fluid constitutive equation, solutions were obtained which are applicable over a wide shear rate range for modified power law fluids from Newtonian behavior at law shear rates to power law behavior at large shear rates.

Values of the entrance Nusselt numbers and the thermal entrance lengths are given in tables or graphs and by correlation equations. Whenever possible, calculated asymptotic quantities were compared with previously published data and it shows excellent agreement. As a result of the analysis a shear rate parameter β was identified which allows the prediction of the shear rate range for a given set of operating conditions.

Nomenclature

c_p : Specific heat [J/kg/K]
d : Circular duct diameter [m]
f : Darcy friction factor $(2\frac{dp}{dx}d/\rho\bar{u}^2)$[-]
h : Heat transfer coefficient (q_w/T_w-T_b) [W/m²K]
k : Thermal conductivity [W/mK]
K : Power law consistency $[Ns^n/m^2]$
n : Power law flow index[-]
N_u : Nusselt number (hd/k)[-]
p : Pressure $[N/m^2]$
Pe : Peclet number $(\bar{u}d/\alpha)$ [-]
Pr : Prandtl number (ν/α) [-]
Pr_g : Generalized Prandtl number $(\frac{c_pK}{k}(\frac{\bar{u}}{d})^{n-1})$[-]
q_w : Wall heat flux $[W/m^2]$
Re : Newtonian Reynolds number$(\rho\bar{u}d/\eta_o)$[-]
Re_g : Power law Reynolds number$(\rho\bar{u}^{2-n}d^n/K)$[-]
Re_M : Modified Reynolds number $(\rho\bar{u}d/\eta^*)$[-]
r : Radial coordinate[m]
R : Duct radius[m]
u : Velocity in flow direction [m/s]
$\bar{u}$: Duct average velocity [m/s]
u^+ : Dimensionless velocity in flow direction $(u/\bar{u})$[-]
T : Temperature [K]
x : Coordinate inflow direction [m]
x^+ : Dimensionless coordinate in flow direction $(x/d/P_e)$ [-]

Greek Symbols

α : Thermal diffusivity $[m^2/s]$
β : Shear rate parameter $(\beta=\frac{\eta_o}{K}(\frac{\bar{u}}{d})^{1-n})$ [-]
$\dot{\gamma}$: Shear rate [1/s]
η_a : Apparent viscosity $(\tau/\dot{\gamma})[Ns/m^2]$
η_o : Zero shear rate viscosity $[Ns/m^2]$
η^* : Reference viscosity $(\eta_o/1+\beta)[Ns/m^2]$
η^+ : Dimensionless viscosity (η_a/η^*)[-]
ν : Kinematic viscosity $[m^2/s]$
ρ : Fluid density $[kg/m^3]$
τ : Shear stress$[N/m^2]$

Subscript

b : Refers to bulk temperature
o : Refers to condition at x=0
w : Refers to wall condition

Superscript

+ : Refers to dimensionless quantities
' : Refers to derivative

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