Analysis of Mean Transition Time and Its Uncertainty between the Stable Modes of Water Balance Model

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ABSTRACT : The surface hydrology of large land areas is susceptible to several preferred stable states with transitions between stable states induced by stochastic fluctuation. This comes about due to the close coupling of land surface and atmospheric interaction. An interesting and important issue is the duration of residence in each mode. Mean transition times between the stable modes are analyzed for different model parameters or climatic types. In an example situation of this differential equation exhibits a bimodal probability distribution of soil moisture states. Uncertainty analysis regarding the model parameters is performed using a Monte-Carlo simulation method. The method developed in this research may reveal some important characteristics of soil moisture or precipitation over a large area, in particular, those relating to abrupt changes in soil moisture or precipitation having extremely variable duration.

1. Introduction

The duration of drought condition is interesting in hydrology or hydrometeorology and has an important effect on the human life. A time series of rainfall shows transition between drought and wet states with several or tens of years persistence. The surface hydrology of large land areas is susceptible to several preferred stable states with transitions between stable states induced by stochastic fluctuation. This comes about due to the close coupling of land surface and atmospheric interaction (Rodriguez-Iturbe et al., 1991). It is important to predict the mean transition time between the stable modes, since this would be helpful in predicting the average duration of drought or wet conditions.

In this study, a governing equation for the mean transition time between the dry and wet stable mode is derived based on the stochastic processes and analyzed for different model parameters or climatic types. Also uncertainty analysis with respect to the model parameters is performed using a

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Monte-Carlo simulation method.

2. A Statistical-Dynamical Water Balance Model

Rodriguez-Iturbe et al. (1991a, b) proposed a statistical-dynamical model to represent the surface hydrology of large continental regions. The model has physically based components, and it introduces a parameterization scheme for the local recycling of precipitation. The precipitable water over large continental regions can be obtained from the moisture in the overlying atmosphere. Therefore, the rainfall over large regions is made up of two components: the advective component is the some of precipitation formed from external water vapor and the internal component is the precipitation falling from water vapor of local origin. The water balance model developed by Rodriguez-Iturbe describes the time evolution of soil moisture at large spatial scales and represents the continuous rates of inflow and outflow in the system.

$$\frac{ds(t)}{dt}=a\left(1+\frac{s^c(t)}{\Omega}\right)(1-\varepsilon s^r(t))-bs^c(t) \tag{1}$$

$$a=\frac{Pa}{nZr} \tag{2}$$

$$b=\frac{Ep}{nZr} \tag{3}$$

where s(t) is relative soil saturation which is a function of time; n is soil porosity; Zr is hydrologically active depth of soil(L); P_a is advective precipitation rate(L/T); E_p is potential evapotranspiration rate(L/T); Ω is feedback coefficient; and c, r, ε are non-negative numerical constants.

Rodriguez-Iturbe et al. (1991a) introduced environmental fluctuations into the local precipitation term with the variable $\alpha=1/\Omega$, which is taken to be an uncorrelated Gaussian noise process with mean $\bar{\alpha}$ and standard deviation σ. Therefore, a stochastic differential equation can be obtained from the deterministic water balance equation (1).

$$\begin{aligned} ds(t)=&[a(1-\varepsilon s^r(t))+a\bar{\alpha}s^c(t)(1-\varepsilon s^r(t))-bs^c(t)]dt \\ &+\sigma as^c(t)(1-\varepsilon s^r)(t))dW_t \end{aligned} \tag{4}$$

where dW_t is white noise process (derivative of the Wiener process). This stochastic differential equation can be expressed more simply

$$ds(t)=G[s(t)]dt+\sigma g[s(t)]dW_t \tag{5}$$

In this nonlinear stochastic differential equation, the function G[•] expresses the deterministic evolution of the system without environmental fluctuation, and the function g[•] provides the environmental random fluctuations.

3. Derivation of Governing Equation

Environmental fluctuations were introduced to the model and deterministic differential equation was replaced by the stochastic differential equation. Steady states probability distribution of soil moisture described by Eq. (5) can be obtained from the corresponding Fokker-Planck Equation (FPE). The following FPE is derived from the stochastic differential equation based on the theory of stochastic processes (Gardiner, 1985; Arnold, 1974; Matkowski, 1977; Schuss, 1979).

$$\frac{\partial f(s,t)}{\partial t} = -\frac{\partial}{\partial s}[G(s,t)f(s,t)] + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial s^2}[g^2(s,t)f(s,t)] \tag{6}$$

where f(s,t) is a probability density function of soil moisture at time t.

In the water balance model the dynamics of soil moisture are viewed as a succession of noise-induced transitions. A transition will not occur unless the potential barrier is reached through the movement induced by the fluctuations. How frequently this potential barrier is overcome is a function of the intensity of the fluctuations. The statistical moments of transition time can be derived on the theory of stochastic processes (Gardiner, 1985; Demaree, 1990).

3.1 Absorbing and Reflecting Boundary Conditions of the System

It is important to know how long soil moisture whose position is described by a FPE remains in a certain domain R. The solution of this problem can be achieved by use of the FPE.

Let the value of soil moisture be initially s at time t=0 and let us ask how long it remains in the interval (a,b), which is assumed to contain s:

$$a \leq s \leq b \tag{7}$$

Consider two boundary conditions. First, an absorbing barrier-one can assume that the moment the value of soil moisture reaches the boundary it is removed from the domain R, thus the boundary can be described as an absorbing barrier. Consequently, the probability of being on the boundary is zero and the probability of the value of soil moisture reentering a domain R from the boundary is zero. Consider now the potential barrier. After the value of soil moisture reaches the potential barrier, it will move to the other side of the barrier and will not return to the previous side (assume the value of soil moisture moves from the left to the right of the potential barrier). Therefore, the absorbing barrier can be applied to the potential barrier. Regarding the reflecting barrier, one can assume that the motion of the value of soil moisture can not leave the domain R. Therefore, there is zero net flow of probability across the boundary and the value of soil moisture must be reflected there. Consider the boundary where the value of soil moisture is zero. After the value of soil moisture reaches the boundary, it will not move to the left (negative value of soil moisture) but to the right. Therefore, the reflecting barrier can be applied to the boundary where the value of soil moisture is zero.

3.2 First Passage Times for Homogeneous Process

Assume that at boundaries a and b there are absorbing barriers so that the value of soil moisture is removed from the domain R when it reaches boundary a or b. If it is still in the interval (a,b), it has never left that interval. Under these conditions, the probability Z(s,t) that at time t the value of soil moisture is still in (a,b) is

$$\int_a^b f(s',t \mid s,0)ds' = Z(s,t) \tag{8}$$

Let the time the value of soil moisture leaves the interval (a,b) be T. Then we can rewrite the equation as

$$Prob(T \geq t) = \int_a^b f(s',t \mid s,0)ds' \tag{9}$$

which means that Z(s,t) is the same as Prob $(T \geq t)$. Since the system is stationary, function G(s) does not depend on time and hence Z(s,t) obeys the following equation.

$$\frac{\partial}{\partial t}Z(s,t) = G(s)\frac{\partial}{\partial s}Z(s,t) + \frac{1}{2}\sigma^2 g^2(s)\frac{\partial^2}{\partial s^2}Z(s,t) \tag{10}$$

The boundary conditions are such that

$$Z(s,0)\begin{cases} =1 & a \leq s \leq b \\ =0 & \text{otherwise} \end{cases} \tag{11}$$

Since Z(s, t) is the probability that $T \leq t$, the mean first passage time T(s) and the nth moment T_n (s) are given by

$$T(s) = \int_0^\infty Z(s,t)dt \tag{12}$$

$$T_n(s) = \int_0^\infty t^{n-1}Z(s,t)dt \tag{13}$$

One can derive a simple ordinary differential equation for T(s) by using Eq. (12) and integrating Eq. (10) over $(0, \infty)$

$$G(s)\frac{\partial}{\partial S}T(s)+\frac{1}{2}\sigma^2 g^2(s)\frac{\partial^2}{\partial s^2}T(s)=-1 \tag{14}$$

Similarly, for the nth moment

$$G(s)\frac{\partial}{\partial S}T_n(s)+\frac{1}{2}\sigma^2 g^2(s)\frac{\partial^2}{\partial s^2}T_n(s)=-nT_{n-1}(s) \tag{15}$$

which means that all the moments of the first passage time can be found by repeated integration.

Consider the value of soil moisture still in the interval (a,b), but suppose the barrier at a to be reflecting. This assumption is resonable for the soil moisture model because the change of soil moisture cannot be defined for a negative domain. Therefore, we can assume that there is a reflecting barrier at a=0. The boundary conditions then become

$$\frac{\partial}{\partial s}Z(a,t)=0, \qquad Z(b,t)=0 \tag{16}$$

which follow from the condition on the Fokker-Planck Equation.

4. First Transition Time between Dry and Wet States

4.1 Numerical Solution of Mean Transition Time

The nth moment T_n is obtained by solving governing equation (15) and corresponding boundary value problem. The boundary conditions for a transition from a (dry) to c (wet) through potential barrier b is

$$T_n(s=b)=0 \qquad \frac{T_n}{ds}\Big|_{s=0}=0 \tag{17}$$

and for a transition from c (wet) to a (dry) through potential barrier b is

$$T_n(s=b)=0 \qquad \frac{T_n}{ds}\Big|_{s=1}=0 \tag{18}$$

For n=1, the mean transition time between dry and wet states can be obtained:

$$G(s)\frac{d}{ds}T(s)+\frac{1}{2}\sigma^2 g^2(s)\frac{d^2}{ds^2}T(s)=-1 \quad (19)$$

This second–order ordinary differential equation can be expressed as two first–order ordinary differential equations. This boundary condition problem can be solved using the shooting method and the Runge–Kutta method. To solve these equations, the IVPRK (fifth–to–sixth–order Runge– Kutta integration technique) program in the IMSL (International Mathematical and Statistical Libraries) has been used.

4.2 Mean Transition Times for the Two Different Climatic Conditions

Until now, we have seen that the noise–induced fluctuations lead to two probable values of soil moisture quite different from the equilibrium solution of the deterministic formulation. We have also studied the mean transition time between the two stable modes.

For the humid case, the values of model parameters assigned for calculation of the mean transition time are as follows: $c=0.5$, $\varepsilon=1.0$, $r=6.0$, $\Omega=2.7$ and $\sigma=1.0$ year (Entekhabi, 1990). The advective precipitation, P_a, and potential evapotranspiration, Ep, are taken as 1.0 m/year and 1.5 m /year. The effective depth of soil, nZr, is taken as 1.2 m. Therefore, the normalized climatic forcing coefficients a and b are 0.833/year and 1.25/year, respectively. The parameter $c=0.5$ corresponds approximately to the case of the fully vegetated model suggested by Pierce (1958). The mean value of the generated soil moisture series is 0.54. The term $\bar{S}/\Omega$ is about 0.2, which implies that the feedback component of precipitation is approximately 17% of total precipitation.

The time evolution of soil moisture using the above values is shown in Fig. 1, and the probability density function of soil moisture is also shown in Fig. 2. For the humid case the pdf of soil moisture has two maxima (at $s_a=0.27$, and $s_c=0.94$) and a minimum (at $s_b=0.66$). From the pdf, one finds that the system states are more probable at higher values than at lower values of soil moisture. This means that the transition from dry to wet states takes a relatively short time because it is not difficult to overcome the potential barrier (at $s_b=0.66$). The transition time from low soil moisture(left of the potential barrier) to high soil moisture(right of the potential barrier) is shown in Fig. 3. In Fig. 3, the transition time at soil moisture $s_a=0.27$ (dry condition) is 2.2 years. This means that the time to the potential barrier (at $s_b=0.66$) is 1.1 years and the time to overcome the potential barrier is 2.2 years because it has equal probability to change either toward a dry or wet state.

To check our numerical results, the mean transition time of the generated time series of soil moisture was calculated. To obtain the mean transition time, a simulated sample of 5000 year data was used. Fig. 3 also shows the mean transition time from the data. The results of the two cases agree very well. The standard deviation and skewness coefficient can be obtained from the study of Demaree (1990). Because the standard deviation is of the same order as the mean, the system is characterized by very pronounced variability. The positive value of the skewness coefficient implies that the probability distribution of the transition time is highly asymmetrical, its most probable value

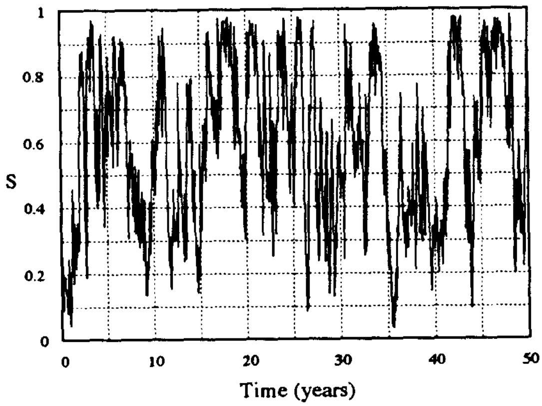

Fig. 1. Time Evolution of Soil Moisture Content for the Humid Case

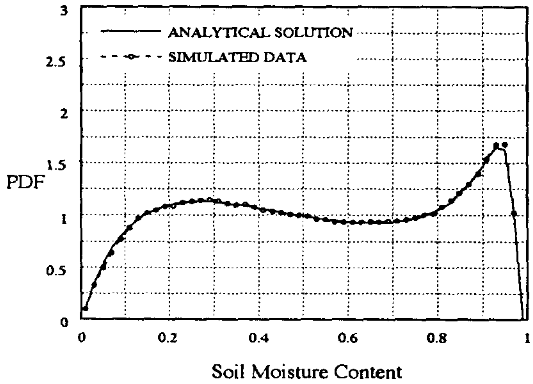

Fig. 2. Probability Distribution for the Humid Case

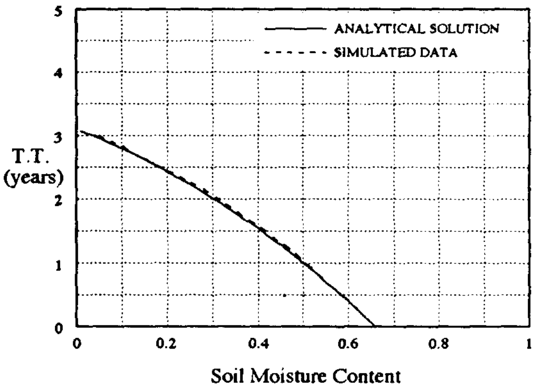

Fig. 3. Predicted Mean Transition Time for the Humid Case

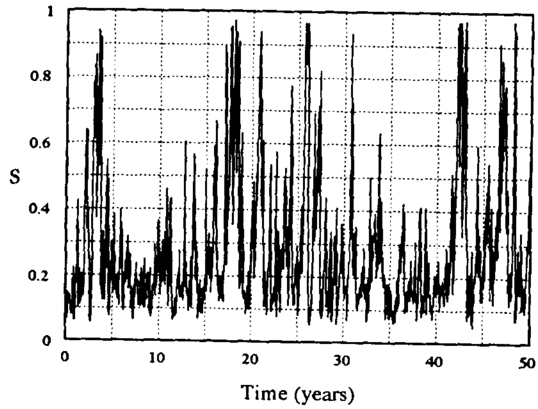

Fig. 4. Time Evolution of Soil Moisture Content for the Arid Case

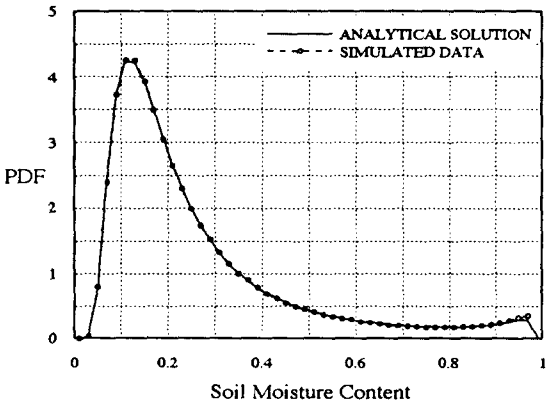

Fig. 5. Probability Distribution for the Arid Case

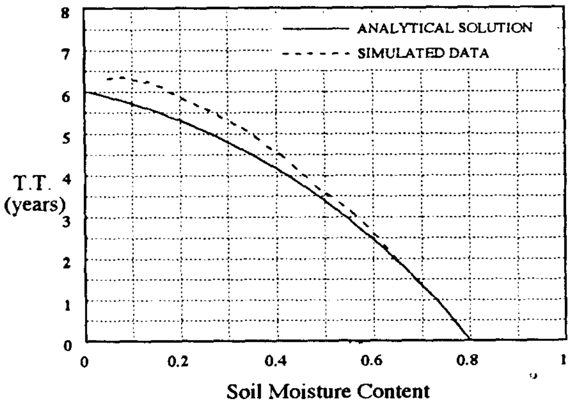

Fig. 6. Predicted Mean Transition Time for the Arid Case

being shifted toward smaller values of the transition time relative to the mean.

For the arid case, the values assigned for calculation of the mean transition time are as follows: $c=1.0$, $\varepsilon=1.0$, $r=6.0$, $\Omega=0.5$ and $\sigma=2.5$ year (Entekhabi, 1990). The advective precipitation, P_a, and potential evapotranspiration, E_p, are taken as 0.4 m/year and 2.2 m/year. The effective depth of soil, nZr, is taken as 0.5 m. The normalized climatic forcing coefficients a and b are 0.8/year and 4.4/year, respectively. The parameter $c=1.0$ corresponds approximately to the case of the linear model suggested by Thornthwaite et al.(1955) and Budyko (1956). The mean value of the generated series of soil moisture is 0.26. The term$\tilde{S}/\Omega$ is about 0.52, which implies that the feedback component of precipitation is approximately 34% of total precipitation. The time evolution of soil moisture using the above values is shown in Fig. 4, and the probability density function of soil moisture is also shown in Fig. 5. In Fig. 5, one can see that the pdf has two maxima (at $s_a=0.12$, and $s_c=0.96$) and a minimum (at $s_b=0.80$). The transition time from dry to wet states is shown in Fig. 6. In this figure, the transition time at soil moisture $s_a=0.12$ is 5.7 years.

5. Uncertainty Analysis of the Mean Transition Time

We have analyzed the mean transition time using the water balance model. There are uncertainties in the model since we do not know the exact values of the parameters. Spatial nonhomogeneities also cause difficulties in precise determination of parameter values. Assume that the randomness in the model is caused by parameter c which relates evapotranspiration to potential evapotranspiration as a function of soil moisture. Parameter c strongly depends on vegetation characteristics. The effect of uncertainty in our knowledge of parameter c (and other parameters as well) can be investigated using a Monte-Carlo approach (Rubinstein, 1981).

In the Monte-Carlo simulation it should be noted that the potential barrier is different for every realization because the probability density of soil moisture is changed for different values of the parameter. Therefore, the lowest potential barrier was selected for the whole realization, and for the remaining realizations the transition times were adjusted to match the lowest potential barrier.

Assume parameter c is uniform with minimum $c=0.45$ and maximum $c=0.9$ for the humid case. The results of the Monte-Carlo simulation are shown in Fig. 7. For this simulation 100 realizations are performed. In the case of the low value of parameter c (≤ 0.5), the transition time is short compared with $c=0.5$. The value of $c\leq 0.5$ imply that the system has vegetated soil and wet climate. Therefore, the transition time is rather short. For $c\geq 0.5$ the system has a less wet climate, and the transition time is long compared with $c=0.5$.

Let us consider other parameters. Fig. 8 shows that the transition time does not changed for parameter ε when uniform distribution U(0,1) is used. This means that ε does not have a great effect on transition time. Parameter r was changed from 3 to 7. The feedback factor, Ω, is 2.7 for the humid case, and the mean value of the generated soil moisture is 0.54. If the effect of feedback in the total precipitation is considered between 15% and 35%, then the range of the feedback factor,

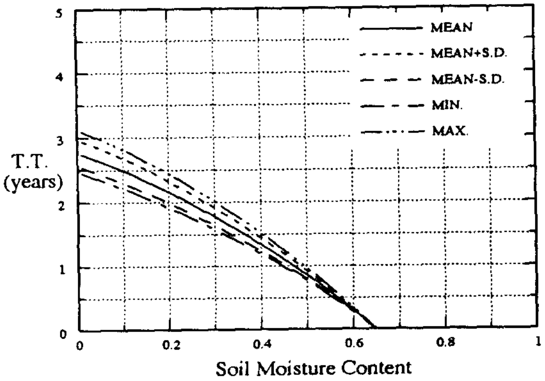

Fig. 7. The Results of Uncertainty Analysis of Parameter c for the Humid Case

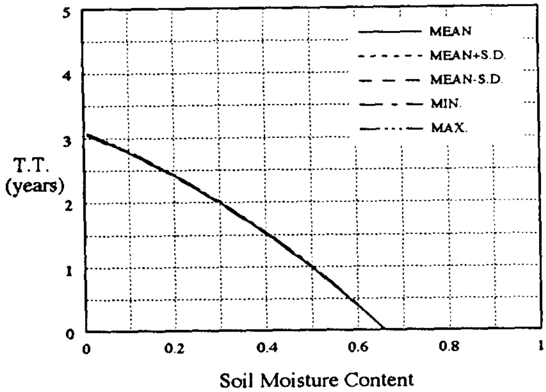

Fig. 8. The Results of Uncertainty Analysis of Parameter ε for the Humid Case

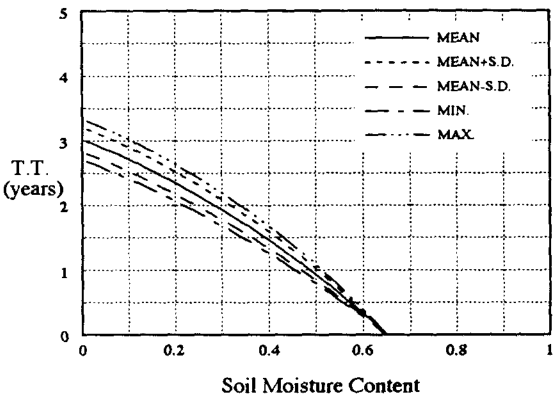

Fig. 9. The Results of Uncertainty Analysis of Parameter σ for the Humid Case

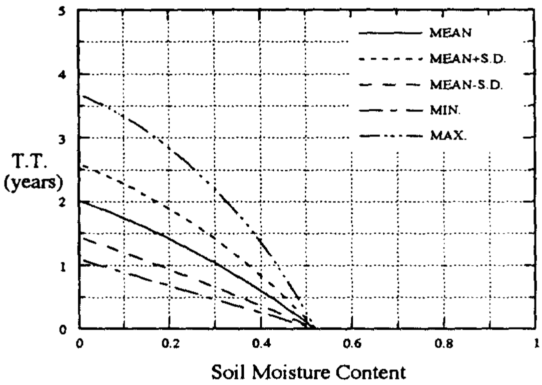

Fig. 10. The Results of Uncertainty Analysis of All Parameter for the Humid Case

Ω, is between 1.0 and 3.0. Therefore, U(1.0, 3.0) is used for the effect of a feedback parameter.

For the normalized climate forcing parameter a, advective precipitation P_a ($=1.0$ m/year) was used, and the effective depth of soil, nZr, was changed from 0.8 to 1.5 m. Therefore, the range of a is from 0.67/year to 1.25/year. Since the model is very sensitive to parameter a, the range between 0.7/year and 1.0/year was used in the Monte-Carlo analysis. Similarly, for the normalized parameter b, potential evapotranspiration, E_p (1.5 m/year), and nZr (0.8-1.5 m) were used, and the range of parameter b from 1.0/year to 1.88/year was obtained. For the Monte-Carlo simulation, the parameter b from 1.0/year to 1.8/year was used. From the results, it can be seen that parameter a and b have a great effect on transition time.

Uncertainty analysis was performed for the intensity of the fluctuation, σ, and the result is shown in Fig. 9. The different values of noise intensity changes the shape of the distribution, shifting the position and weight of the modes as well as their number. The larger the intensity of the fluctuations, the shorter is the mean transition time between stable states through the potential barrier. Fig. 9 confirms this discussion.

Finally, joint uncertainty analysis for all parameters was performed. The following parameter values were used: c(0.45–0.9), ε(0–1.0), r(3–7), Ω(1–3), a(0.7–1.0), b(1.0–1.8), and $\sigma=1.0$. The result is shown in Fig. 10. A similar study was performed for the arid case and the results does not show much difference.

6. Conclusions

The water balance equation in soil moisture, when coupled with a parameterization for local recycling of water over large land areas, exbibits some of the statistical structure often observed in different climatic variables. It has been discussed that the soil moisture tends to lock itself around the value of one mode but, with a strong enough fluctuation in the climate, it may then shifts it back to the previous mode. In the study, transition times between stable modes in the model for different climate types are analyzed to arrive at a statistical prediction of the duration of alternative regimes through the relative stability of two stable states as deduced from mean values and higher moments of residence times.

For the sample data (this data represents the Amazon River Basin), we obtained transition times for the two climatic types. The transition time from dry soil condition ($s_a=0.27$) to potential barrier ($s_b=0.66$) is about 2.2 years for the humid climatic condition and is about 5.7 years from $s_a=0.12$ to $s_b=0.80$ for the arid climatic condition. For the arid case, the transtion time is almost 2.5 times that for the humid case. This means that, for the arid climatic condition, the transition from low soil moisture to high soil moisture is relatively difficult. But for the humid climatic condition, the transition is not so difficult. The uncertainty analysis of the model shows that the transition time is very sensitive to all parameters except parameter ε. Uncertainty analysis was also performed for the environmental fluctuation. It has been shown that different values of fluctuation intensity change the shape of the distribution, shifting the position and weight of the mode as well as their number in its long–term probability density function. The larger the intensity of the fluctuations, the shorter the mean transition time between stable states through the potential barrier.

The method developed in this research may reveal some important characteristics of soil moisture or precipitation over a large area, in particular those relating to abrupt changes in soil moisture or precipitation regimes of extremely variable duration.

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