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A Perturbation Based Method for Variational Inequality over Convex Polyhedral

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Abstract

This paper provides a locally convergent algorithm and a globally convergent algorithm for a variational inequality problem over convex polyhedral. The algorithms are based on the B (ouligand)-differentiability of the solution of a nonsmooth equation derived from the variational inequality problem. Convergences of the algorithms are achieved by the results of Pang[3].

1. Introduction

In the papers[1, 3, 5, 6, 9], the authors present systems of nonsmooth equations derived from variational inequality problems. Robinson[11] suggested a 'normal map' equation which is equivalent to variational inequality problems. The normal map could be understood by two ways. First, it is a composite map of $f \circ g$ where f is F(rechet)-differentiable and g is Lipschitz continuous. Robinson[9] described a Newton's method for the composite map equation and Park[6] developed a Newton-Mysovskii type local convergence and a globally convergent continuation method based on the local convergence. Robinson[10] obtained conditions of homomorphism of the normal map which is essential to the globally convergent continuation method. Second, the normal map is B(ouligand)-differentiable. No methods using B-differentiability for the normal map equation are known to us. But, Pang extended the classical Newton's method to B-differentiable systems of equations and he applied the Newton's method to the systems derived from variational inequality problems. Pang[4] also briefly discussed the continuation method of the Newton's method.

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In this paper we apply the Newton's method of Pang[3] to the normal map equation, suggested by Robinson[11], which is equivalent to variational inequality problems. In specializing the Newton method to a given B-differentiable equation, we need to solve a system of nonlinear equations, called as a generalized Newton equation, for finding a direction at each iteration. To solve the generalized Newton equation derived from the normal map, we use the B-differentiability of the solution of the perturbed variational inequality problem defined over a polyhedral convex set. The B-differentiability of the solution of generalized equation is given by Robinson[8] and is rephrased by Kyparisis for variational inequality problem over polyhedral convex set in [2].

The rest of the paper is organized in three sections. In the next section, we review some mathematical programming problems and their relationship. We also discuss the normal map and B-differentiability. In section 3, we perturb the problem and obtain some useful results in our method. Section 4 contains the algorithms and their convergences. Numerical examples are in section 5.

2. Preliminaries

The variational inequality problem defined over convex polyhedral is to find $y \in C \subset R^n$ such that

$$\langle z-y, f(y) \rangle \geq 0 \text{ for all } z \in C = \{z \in R^n \mid Az \leq b\}$$

where $f: R^n \to R^n$ is continuously F-differentiable, $A \in R^{m \times n}$, and $b \in R^m$. $\langle \cdot , \cdot \rangle$ denotes the inner product. Through this paper $VI(C, f)$ denotes the variational inequality problem defined by C and f.

The variational inequality problem $VI(C, f)$ has close relationship with several mathematical programming problems. If $C = R^n_+$, $VI(C, f)$ is the nonlinear complementary problem to find $y \in R^n$ such that

$$y \geq 0, \ f(y) \geq 0, \quad \langle f(y), y \rangle = 0.$$

Define the normal cone of C at x by

$$N_C(x) = \begin{cases} \{z \mid \langle y-x, z\rangle \le 0, \ \forall y \in C\}, & \text{if } x \in C \\ \phi, & \text{if } x \notin C \end{cases}$$

Then the generalized equation of

$$0 \in f(y) + N_C(y)$$

is equivalent to $VI(C, f)$.

Consider the nonlinear programming problem and let y^* be an optimal solution.

$$\begin{aligned} \text{Minimize} \quad & f(y) \\ \text{Subject to} \quad & h(y) = 0 \\ & g(y) \le 0. \end{aligned}$$

Then the following necessary optimality condition is satisfied at y^*;

$$\langle z-y^*, \nabla f(y^*)\rangle \ge 0 \quad \forall \ z \in C$$

where $C = \{z \in R^n \mid \langle z-y^*, \nabla g_I(y^*)\rangle \le 0, \langle z-y^*, \nabla h(y^*)\rangle = 0\}$ is the set of feasible directions and the subscript I of ∇g_I denotes the index set of binding constraint g at y^*. The above necessary optimality condition is a $VI(C, \nabla f)$ where ∇ f denotes the F-derivative of f.

We now introduce B-differentiability and directional differentiability of functions. Let $G: R^n \to R^m$ be Lipschtz continuous. Then G is B-differentiable if there is a positive homogeneous function $DG: R^n \to R^m$ such that

$$\lim_{h \to 0} \{G(x+h) - G(x) - DG(x)h\} / \|h\| = 0.$$

The directional derivative of $G(x)$ at x in the direction v is defined to the limit

$$G'(x;v) = \lim_{\tau \downarrow 0} \{G(x+\tau \cdot v) - G(x)/\tau.$$

$P_C(x)$ denotes the projection of x on C and $G(x) = f(P_C(x)) + x - P_C(x)$ is called the normal map.

Lemma 1 Let f: $R^n \to R^m$ be continuously F-differentiable. Then

$G(x)=f(P_c(x))+x-P_c(x)$ is B-differentiable.

Proof. For given $x\in R^n$, $v\in R^n$ and sufficiently small $\tau>0$, there exists $d\in R^n$ such that $P_c(x+\tau\cdot v)=P_c(x)+\tau\cdot d$ and $P'_c(x;v)=d$. clearly $P'_c(x;v)$ is positively homogeneous in v. For each v, w in R^n and $\tau>0$, from the Lipschtz continuity of $P_c(\cdot)$

$$\|P_c(x+\tau\cdot v)-P_c(x+\tau\cdot w)\| \leq \tau\|v-w\|.$$

By dividing by τ and taking $\tau\downarrow 0$ on both sides of inequality, we obtain

$$\|P'_c(x;v)-P'_c(x;w)\| \leq \|v-w\|.$$

Hence $P'_c(x;\cdot)$ is Lipschtz continuous. Now if we show $DP_c(x)v=P'_c(x;v)$, then $P_c(x)$ is B-differentiable and so is $G(x)$. For any sequence $\{h_n\}$ that converges to 0, choose sequences $\{\tau_n\}$ and $\{d_n\}$ such that $h_n=\tau_n\cdot d_n$ where $\tau_n\downarrow 0$ and $\|d_n\|=1$ for $n=1, 2, \cdots$. Then there exists a limit point d^* of $\{d_n\}$. Choose a subsequence $\{d_{n_j}\}$ such that $d_{n_j}\to d^*$. For the subsequence $\{n_j\}$,

$$\begin{aligned}
&\lim_{j\to\infty} \{P_c(x+h_{nj})-P_c(x)-P'_c(x;h_{nj})\}/\|h_{nj}\| \\
&=\lim_{j\to\infty} \{P_c(x+\tau_{nj}\cdot d_{nj})-P_c(x)-P'_c(x;\tau_{nj}-d_{nj})\}/\|h_{nj}\| \\
&=\lim_{j\to\infty} \left[\frac{P_c(x+\tau_{nj}\cdot d_{nj})-P_c(x)}{\tau_{nj}} - P'_c(x;d_{nj})\right] \\
&=\lim_{j\to\infty} \left[\frac{P_c(x+\tau_{nj}\cdot d^*)-P_c(x)}{\tau_{nj}} - P'_c(x;d^*)\right] \\
&=P'_c(x;d^*)-P'_c(x;d^*) \\
&=0 \quad \blacksquare
\end{aligned}$$

Pang[3] presented another B-differentiable equation arised from $VI(C, f)$. *Let* $\nabla H(y)=f(y)$. Then the $VI(C, f)$ is the necessary optimality condition of the following nonlinear program with linear constraints;

$$\begin{array}{ll} \text{Minimize} & H(y) \\ \textit{Subject to} & Az \leq b. \end{array}$$

With constraint qualification, the Kuhn-Tucker conditions are

$$f(y) + u^{\prime}A = 0$$

$$u \geq 0, \ <u, \ Az - b> \ = \ 0, \ Az \ \leq \ b.$$

Hence he has the B−differentiable equations equivalent to the Kuhn−Tucker conditions;

$$f(y) \ + \ u^T \ A \ = \ 0$$
$$\min \{u, \ b \ - \ Az\} \ = \ 0.$$

Lemma 2 Suppose y^* solves the $VI(C, f)$. Then $x^* = y^* - f(y^*)$ solves the normal map equation $f(P_c(x)) + x - P_c(x) = 0$. Conversely, if x^* solves the the normal map equation, then $y^* = P_c(x^*)$ *solves* $VI(C, f)$.

Proof. For any solution y^* of $VI(C, f)$

$$\langle z - y^*, \ - f(y^*) \rangle \ = \ \langle z - y^*, \ x^* - \ y^* \rangle \ \leq \ 0 \quad \forall \ z \in C.$$

From the condition of normal cone, $-f(y^*) = x^* - y^* \in N_c(y^*)$. Hence $P_c(x^*) = y^*$ and we have $x^* = y^* - f(y^*) = P_c(x^*) - f(P_c(x^*))$. Conversely from the definition of normal cone of C at $P_c(x^*)$, we have $x^* - P_c(x^*) \in N_c(P_c(x^*))$. Now et $y^* = P_c(x^*)$. Then the normal map equation is $f(y^*) + x^* - y^* = 0$. Hence for all $z \in C$,

$$\begin{aligned} \langle z - y^*, \ - f(y^*) \rangle \ &= \ \langle z - y^*, \ x^* - \ y^* \rangle \\ &= \ \langle z - P_c(x^*), \ x^* - P_c(x^*) \rangle \\ &\leq \ 0. \end{aligned}$$ ■

3. Perturbed Variational Inequality Problem

For any $x \in R^n$, $VI(C, -e + f)$ denotes the perturbed variational inequality problem; find $y \in R^n$ such that $\langle z - y, -e + f(y) \rangle \geq 0$ for all $z \in C$.

Lemma 3 For any $x \in R^n$, let $e = f(P_c(x)) + x - P_c(x)$. Then $y = P_c(x)$ solves $VI(C, -e + f)$.

Proof. Since x solves the equation of $-e + f(P_c(x)) + x - P_c(x) = 0$, by Lemma 2, $y = P_c(x)$ is a solution of $\langle z - y, -e + f(y) \rangle \geq 0$ for all $z \in C$. ■

Through this paper we use the notations. For given x and e, let $F(x, e) = f(P_c(x)) - e + x - P_c(x)$. For a given x, let $e(x) = f(P_c(x)) + x - P_c(x)$ and for a given e, let $x(e)$ be the solution of $F(x,e) = 0$. For a given e, let $y(e)$

be the solution of $VI(C, -e+f)$ and hence $y(e) = P_c(x(e))$.

Now we introduce the perturbation analysis of $VI(C, -e+f)$ by applying the result of generalized equation given by Robinson[8] to the variational inequality problem. This application to the variational inequality problem over a polyhedral convex set is also found in [2, 4, 7].

Theorem 1 For a given $e^k \in R^n$, suppose that $\nabla f(y(e^k))$ is positive definite on $C_k - C_k$ where $C_k = \{z \mid \langle f(y^k), z\rangle \doteq \langle e^k, z\rangle, A^k z \le 0\}$ and A^k denotes the rows of the matrix A corresponding to the binding constraints of $Az \le b$ at $y(e^k)$. Then there exist neighborhoods U of e^k, V of $y(e^k)$, and Lipschtiz continuous functions $y: U \to V$ with the following properties

(a) for each $e \in U$, $y(e)$ is the unique solution of $VI(C, -e+f(y))$ in V;

(b) for each $e \in U$, $-y(e) \in C_k$;

(c) the function $y(\cdot)$ is B-differentiable at e^k with B-derivative $v = Dy(e^k)u$ given as the unique solution of $VI(C_k, g_k)$ where $g_k = \nabla f(y(e^k))y - u$.

Proof. It is clear from Theorem 3.2 of [8]. ■

In Theorem 1, $C_k - C_k$ denotes the smallest subspace containing C_k and it is defined by

$$C_k - C_k = \{\, x-y \quad x \in C_k \; y \in C_k \,\}.$$

For the algebra of convex sets, please see sections 2 & 3 of [12].

By using Theorem 1, we obtain the directional derivative of the solution $x(e)$ of $F(x,e) = 0$ in the direction $-e^k$ at e^k.

Lemma 4 For a given x^k, let $e^k = f(P_c(x^k)) + x^k - P_c(x^k)$ and let $v^k = Dy(e^k)(-e^k)$. Then the directional derivative of $x(\cdot)$ in the direction $-e^k$ at e^k is given by $d^k = Dx(e^k)(-e^k) = v^k - \nabla f(P_c(x^k))v^k - e^k$.

Proof. For each e, let $y(e)$ be a solution of $VI(C, -e+f)$. Then from Lemma 2,
$x(e) = y(e) - f(y(e)) + e$.

By B-differentiating on both sides of the equation in the direction $-e^k$ at e^k, we have

$$\begin{aligned} d^k &= Dx(e^k)(-e^k) \\ &= Dy(e^k)(-e^k) - \nabla f(y(e^k))\, Dy(e^k)(-e^k) - e^k \\ &= v^k - \nabla f(y(e^k))v^k - e^k. \end{aligned}$$ ■

Recall the generalized Newton equation for B−differentiable function introduced in [3]. Let $G: R^n \to R^n$ *be B−differentiable. Then*

$$G(x^k) + DG(x^k)d = 0$$

is called the generalized Newton equation at x^k for $G(x)=0$.

Lemma 5 Let $G(x)=f(P_c(x)) + x-P_c(x)$ and let $e^k=G(x^k)$. Then $d^k=Dx(e^k)(-e^k)$ solves the generalized Newton equation at x^k for $G(x)=0$.

Proof. Let $F(x,e) = -e + G(x)$. Since $d^k=Dx(e^k)(-e^k)$,

$$\begin{aligned} 0 &= F'(x^k, e^k; -e^k) \\ &= D_e F(x^k, e^k) + D_x F(x^k, e^k)d^k \\ &= e^k + D-x\ F(x^k, e^k)\ d^k \\ &= G(x^k) + DG(x^k)d^k. \quad \blacksquare \end{aligned}$$

We now define a function m: $R^n \to R^n$ by

$$m(x)=(1/2)\ \| e(x) \|^2=(1/2)\ \| f(P_c(x)) + x-P_c(x) \|^2.$$

Then x solves $G(x) = 0$ if and only if $m(x)=0$.

Lemma 6 d^k is a descent direction of $m(\cdot)$ at x^k.

Proof. From the proof of Lemma 5, $D_x F(x^k, e^k)d^k = -e^k$. Consider the equation $e(x)=e + F(x, e)$. By directional differentiating in the direction d^k at x^k on both sides of the equation, we obtain $D_xe(x^k)d^k=D_xF(x^k, e^k)d^k$. And hence $D_xe(x^k)d^k = -e^k$. Therefore

$$\begin{aligned} m'(x^k; d^k) &= e(x^k)\ De(x^k)d^k \\ &= e^k(-e^k) \\ &= -\| e^k \|^2 \\ &< 0 \quad \text{if } e^k \neq 0. \quad \blacksquare \end{aligned}$$

4. Algorithms and Convergences

In this section we develop a locally convergent algorithm and a globally convergent algorithm.

Algorithm I

(step 0) Let x^o be the initial guess of x^* and $k=0$.

(step 1) Compute $e^k = f(P_c(x^k)) + x^k - P_c(x^k)$.

(step 2) Compute the solution v^k of $VI(C_k, g_k)$ where

$C_k = \{ z \mid \langle f(P_c(x^k)), z \rangle = \langle e^k, z \rangle, A^k z \le 0 \}$ and $g_k = \nabla f(P_c(e^k))z + e^k$.

(step 3) Compute $d^k = v^k - \nabla f(P_c(x^k))v^k - e^k$.

(step 4) $x^{k+1} = x^k + d^k$ and $k=k+1$. Go to (step 1).

In the global algorithm, (step 4) of Algorithm I is replaced by a line search step. For this step we use Armijo-Goldstein step-size rule. Let $\gamma > 0$, $\beta \in (0,1)$ and $\sigma \in (0, 0.5)$. Then $x^{k+1} = x^k + \alpha_k \cdot d^k$ where $\alpha_k = \beta^{j(k)} \cdot \gamma$, $j(k)$ is the first nonnegative integer exponent such that $m(x^k) - m(x^k + \beta^k \cdot \gamma \cdot d^k) \ge -\sigma \cdot \beta^k \cdot m'(x^k ; d^k)$.

Algorithm II

(step 0) Let x^o be the initial guess of x^* and $k=0$.

(step 1) Compute $e^k = f(P_c(x^k)) + x^k - P_c(x^k)$.

(step 2) Compute the solution v^k of $VI(C_k, g_k)$ where

$C_k = \{ z \mid \langle f(P_c(x^k)), z \rangle = \langle e^k, z \rangle, A^k z \le 0 \}$ and $g_k = \nabla f(P_c(e^k))z + e^k$.

(step 3) Compute $d^k = v^k - \nabla f(P_c(x^k))v^k - e^k$.

(step 4) (Line Search) $x^{k+1} = x^k + \alpha_k \cdot d^k$ and $k=k+1$. Go to (step 1).

Theorem 2 In Algorithm I & II, if $v^k=0$, then $P_c(x^k)$ solves $VI(C, f)$.

Proof. Since $v^k=0$ is a solution of $VI(C_k, g_k)$, we have

$\langle z-0, \nabla f(P_c(x^k)) \cdot 0 + e^k \rangle \ge 0$, $\forall z \in C_k$ where

$C_k = \{ z \mid \langle f(P_c(x^k)), z \rangle = \langle e^k, z \rangle, A^k z \le 0 \}$. That is, $\langle z, e^k \rangle \ge 0$, $\forall z \in C_k$.

Hence for all $z \in \{ z \mid A^k z \le 0 \}$, $\langle z, e^k \rangle = \langle z, f(P_c(x^k)) \rangle \le 0$. Now let $z = P_c(x^k) + z$, then we have $\langle z - P_c(x^k), f(P_c(x^k)) \rangle \le 0$, $\forall z \in z \mid A^k z \le b \}$. ■

We now have a locally quadratic convergence of Algorithm I. Let $G(x)=f(P_c(x)) + x-P_c(x) =0$.

Theorem 3 Let x^* be a solution of $G(x)=0$. Suppose that G is continuously F-differentiable at x^* and $\nabla G(x^*)$ is nonsingular. Then there exists a neighborhood N of x^* such that for any initial guess x^0 in N, the sequence $\{ x^k \}$ generated by Algorithm I converges to x^*. If $DG(\cdot)$ is Lipschtz continuous at x^*, then the rate of convergence is quadratic.
Proof. It is clear from Lemma 5 of this paper and Theorem 3 of [3]. ■

We also have a global convergence of Algorithm II.

Theorem 4 Let x^0 be any initial point in R^n. Assume that
(a) $\{ x \mid \| e(x) \| \leq \| e(x^0) \| \}$ is bounded and
(b) for each k, $\nabla f(P_c(x^k))$ is positive definite on $C_k - C_k$.
Let $\{ x^k \}$ be any sequence generated by Algorithm II. Assume that $G(x^k) \neq 0$ for all k. Then,
(i) $\| e(x^k+1) \| \leq \| e(x^k) \|$,
(ii) $\{ x^k \}$ is bounded and
(iii) if x^* is any accumulation point such that
(c) $\| e(\cdot) \|^2$ is F-differentiable at x^* and
(d) there exists a neighborhood N of x^* and a real number $c > 0$ such that for all $z \in N$ and all $v \in R^n$, $\| De(x)v \| \geq c \cdot \| v \|$. Then $P_c(x^*)$ is a solution of $VI(C, f)$.
Proof. It is clear from Lemma 6 of this paper and Theorem 4 of [3]. ■

5. Numerical Examples

We consider the following linear variational inequality problem defined over convex polyhedral($VI(C, f)$).

$$VI(C, f) \quad \langle z-y, f(y) \rangle \geq 0 \text{ for all } z \in C$$

where $f(y) = \begin{matrix} 2y_1 - 2 \\ 4y_2 - 4 \end{matrix}$ and $C = \{ (z_1, z_2) \in R^2 \mid z_1 + z_2 \leq 1, z_1 \geq 0, z_2 \geq 0 \}$. Then as presented in the Preliminaries this problem is equivalent to the quadratic programming problem:

$$\text{Minimize} \quad y_1^2 + 2y_2^2 - 2y_1 - 4y_2$$
$$\text{Subject to} \quad y_1 + y_2 \le 1$$
$$y_1 \ge 0,\ y_2 \ge 0.$$

We can solve this problem directly by applying Lemke's algorithm. In fact, we suggest Lemke's alorithm for the linear variational inequality $VI(C_k, g_k)$ in our Algorithm I & II. Now we apply our Algorithm I to the linear variational inequality problem $VI(C, f)$ though our algorithms are developed for nonlinear variational inequality problems.

[Initialization]

(step 0) $x^o = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ & $k = 0$

[Iteration 1]

(step 1) $y^o = P_c(x^o) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$e^o = f(y^o) + x^o - y^o = \begin{pmatrix} -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}.$$

(step 2)

$$C_o = \{ z \in R^2 \mid \langle \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rangle = \langle \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rangle,\ z_1 \ge 0,\ z_2 \ge 0 \}$$
$$= \{ z \in R^2 \mid z_1 \ge 0,\ z_2 \ge 0 \}$$
$$g_o = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + e^o = \begin{pmatrix} 2z_1 - 2 \\ 4z_2 - 4 \end{pmatrix}.$$

$VI(C_o, g_o)$ is equivalent to the quadratic programming problem(QP_o):

$$QP_o \qquad \text{Minimize} \quad y_1^2 + 2y_2^2 - 2y_1 - 4y_2$$
$$\text{Subject to} \quad y_1 \ge 0,\ y_2 \ge 0.$$

The solution of QP_o is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, this is v^o.

(step 3)

$$d^o = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(step 4)

$$x^1 = x^o + d^o = \begin{matrix} 0 \\ 0 \end{matrix} + \begin{matrix} 1 \\ 1 \end{matrix} = \begin{matrix} 1 \\ 1 \end{matrix} .$$

[Iteration 2]

(step 1)

$$y^1 = P_c(x^1) = \begin{matrix} 1/2 \\ 1/2 \end{matrix} ,$$

$$e^1 = f(y^1) + x^1 - y^1 = \begin{matrix} -1 \\ -2 \end{matrix} + \begin{matrix} 1 \\ 1 \end{matrix} - \begin{matrix} 1/2 \\ 1/2 \end{matrix} = \begin{matrix} -1/2 \\ -3/2 \end{matrix} .$$

(step 2)

$$C_1 = \{ z \in R^2 | \langle \begin{matrix} -1 \\ -2 \end{matrix} , \begin{matrix} z_1 \\ z_2 \end{matrix} \rangle = \langle \begin{matrix} -1/2 \\ -3/2 \end{matrix} , \begin{matrix} z_1 \\ z_2 \end{matrix} \rangle, z_1 + z_2 \leq 0 \}$$

$$= \{ z \in R^2 \mid z_1 + z_2 = 0 \}$$

$$g_1 = \begin{matrix} 2 & 0 \\ 0 & 4 \end{matrix} \begin{matrix} z_1 \\ z_2 \end{matrix} + e^1 = \begin{matrix} 2z_1 - 1/2 \\ 4z_2 - 3/2 \end{matrix} .$$

$VI(C_1, g_1)$ is equivalent to the quadratic programming problem(QP_1):

$$QP_1 \qquad \text{Minimize} \quad y_1^2 + 2y_2^2 - (1/2)y_1 - (3/2)y_2$$
$$\text{Subject to} \quad y_1 + y_2 = 0.$$

The solution of QP_1 is $\begin{matrix} -1/6 \\ 1/6 \end{matrix}$, this is v^1.

(step 3)

$$d^1 = \begin{matrix} -1/6 \\ 1/6 \end{matrix} - \begin{matrix} 2 & 0 \\ 0 & 4 \end{matrix} \begin{matrix} -1/6 \\ 1/6 \end{matrix} - \begin{matrix} -1/2 \\ -3/2 \end{matrix} = \begin{matrix} 2/3 \\ 1 \end{matrix} .$$

(step 4)

$$x^2 = x^1 + d^1 = \begin{matrix} 1 \\ 1 \end{matrix} + \begin{matrix} 2/3 \\ 1 \end{matrix} = \begin{matrix} 5/3 \\ 2 \end{matrix} .$$

[Iteration 3]

(step 1)

$$y^2 = P_c(x^2) = \begin{matrix} 1/3 \\ 2/3 \end{matrix} ,$$

$$e^2 = f(y^2) + x^2 - y^2 = \begin{matrix} -4/3 \\ -4/3 \end{matrix} + \begin{matrix} 5/3 \\ 2 \end{matrix} - \begin{matrix} 1/3 \\ 2/3 \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix} .$$

By the proof of Lemma 6, $e^k=0$ implies $d^k=0$. Since $I-\nabla f(P_c(x^k))=\begin{matrix}-1 & 0\\ 0 & -3\end{matrix}$ is not zero in this problem, $v^k=0$. By Theorem 2 the current point $y=\begin{matrix}1/3\\ 2/3\end{matrix}$ is the solution of $VI(C, f)$.

6. Conclusions

This paper provides a locally convergent algorithm and a globally convergent algorithm for a variational inequality problem over convex polyhedral. The algorithms are based on the B (ouligand)−differentiability of the solution of a nonsmooth equation, which is a normal map equation derived from the variational inequality problem. The algorithms need to solve a linearized variation inequality problem and the solution can be obtained by Lemke's algorithm. Even though the convergences of the algorithms are achieved in the case of F−differentiability at the solution point by using the results of Pang[3], the algorithms are based on the B−differentiability of solutions. The convergences of the algorithms without F−differentiability at the solution point are expected to be possible and this will be the future research.

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