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A NOTE ON TRANSIENT SOLUTIONS FOR CONTINUOUS TIME MARKOV CHAIN

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ABSTRACT. We suggest an improvement of the algorithm in [2] for calculating the transient solutions for continuous time Markov chain with countable state space.

1. Introduction

Hsu and Yuan [2] suggested an algorithm for calculating the transient solutions for continuous time Markov chain with countable state space and arbitrary initial distribution by the uniformization technique. They [3] also presented an algorithm for the first passage time for continuous time Markov chain with countable state space based on the results of [2].

In this note we state the algorithm in [2] and then suggest an improvement of the algorithm and finally check its effectiveness by the numerical example.

Consider the continuous time Markov chain $\{X(t); t \geq 0\}$ on the state space $E = \{0, 1, 2, \cdots\}$ with infinitesimal generator $Q = (q_{ij})$. Suppose $0 < c = \sup_{i \in E}\{-q_{ii}\} < \infty$, and let $P = I + \frac{1}{c}Q$, where I is the identity matrix. Let $p_i(t) = P(X(t) = i)$, $i \in E$ and $p(t) = (p_0(t), p_1(t), p_2(t), \cdots)$. Then by uniformization technique (for example see, Ross[4, p. 286]), $p(t)$ is given by

$$p(t) = e^{-ct} \sum_{n=0}^{\infty} \pi(n) \frac{(ct)^n}{n!}, \quad t \geq 0,$$

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where

$$\pi(0) \equiv (\pi_0(0), \pi_1(0), \pi_2(0), \cdots) = p(0)$$
$$\pi(n) \equiv (\pi_0(n), \pi_1(n), \pi_2(n), \cdots) = \pi(n-1)P, \ n = 1, 2, \cdots.$$

To calculate $p(t)$, Hsu and Yuan [2] suggested the following algorithm.

• Algorithm

Step 1. For given error $\delta > 0$, $T > 0$ and $c > 0$, compute $L = \max\{1, \frac{(cT)^2}{2}\} + e^{cT} + cT$;

Step 2. Set $\epsilon = \delta/L$, and $M = \max(2\lfloor ceT \rfloor, \lfloor \log_2(1/\epsilon) \rfloor)$, where $\lfloor x \rfloor$ is the integral part of x;

Step 3. Generate $N_0, N_1, N_2, \cdots, N_M$ by the following recursive formulas:

$$\sum_{j=N_0+1}^{\infty} \pi_j(0) < \epsilon$$
$$\sum_{j=N_n+1}^{\infty} p_{ij} < \epsilon, \quad 0 \le i \le N_{n-1}, \ n = 1, 2, \cdots, M,$$

where p_{ij} is the (i,j)th entry of the matrix P;

Step 4. Compute $\pi_j(n, N_n), n = 0, 1, 2, \cdots, M$ by the following formulas;

$$\pi_j(0; N_0) = \begin{cases} \pi_j(0) & \text{if } 0 \le j \le N_0 \\ 0 & \text{if } j \ge N_0 + 1 \end{cases}$$
$$\pi_j(n; N_n) = \begin{cases} \sum_{i=0}^{N_{n-1}} \pi_i(n-1; N_{n-1}) p_{ij} & \text{if } 0 \le j \le N_n \\ 0 & \text{if } j \ge N_n + 1; \end{cases}$$

Step 5. To approximate $p_j(t)$, $t \in [0, T]$, compute $p_j(t|M)$, where

$$p_j(t|M) = e^{-ct} \sum_{n=0}^{M} \pi_j(n; N_n) \frac{(ct)^n}{n!}.$$

Then the following is obtained

$$0 \le p_j(t) - p_j(t|M) < L\epsilon = \delta, \quad t \in [0, T], \ j \in E.$$

REMARK. Note that the amount of operations in calculating $p_j(t|M)$ severely depends on the constants $N_0, N_1, \cdots, N_M$. Precisely speaking, to calculate $\pi(n; N_n)$, $n = 1, 2, \cdots, M$, the necessary number of multiplications is $N_0N_1 + N_1N_2 + \cdots + N_{M-1}N_M$. For a given error $\delta > 0$, we see from the definition of the sequence $\{N_n, n \geq 0\}$, that as L increases, $\epsilon > 0$ decreases and hence N_n and M increase. Hence it is very important to reduce the constants L and M.

In the following section, we give better constants L and M, and present a numerical example for its effectiveness.

2. Improvements and numerical example

Let $\epsilon > 0$ and $cT > 0$ be given and M is a constant satisfying $\frac{(cT)^{M+1}}{(M+1)!} < \epsilon$. In the procedure of determining the constant L, Hsu and Yuan [2] used the estimation

$$e^{-ct} \sum_{k=M+1}^{\infty} \frac{(ct)^k}{k!} \leq \epsilon e^{cT}, \text{ for all } t \in [0, T]. \tag{1}$$

Here we derive tighter estimation than (1). It is well known that when we expand the function e^{ct} by Taylor series, for any integer n, there is a $\xi \in (0, t)$ such that $\sum_{k=n+1}^{\infty} \frac{(ct)^k}{k!} = \frac{(ct)^{n+1}}{(n+1)!} e^{\xi t}$. Hence we have that for each $t \in [0, T]$,

$$e^{-ct} \sum_{n=M+1}^{\infty} \frac{(ct)^n}{n!} = \frac{(ct)^{M+1}}{(M+1)!} e^{-c(t-\xi)} < \frac{(cT)^{M+1}}{(M+1)!} < \epsilon. \tag{2}$$

Using (2) instead of (1) in the proof Theorem 2.4 of [2], we can obtain smaller number $L^* = \max\{1, \frac{(cT)^2}{2}\} + cT + 1$ than $L = \max\{1, \frac{(cT)^2}{2}\} + e^{cT} + cT$.

Now we suggest an improvement for M. For a constant M with $\frac{(cT)^{M+1}}{(M+1)!} < \epsilon$, Hsu and Yuan [2] take $M(\epsilon) = \max(2\lfloor ceT \rfloor, \lfloor \log_2(1/\epsilon) \rfloor)$ by the Stirling's formula for $n!$. Let $M^*(\epsilon) = \min\{n; \frac{(cT)^{n+1}}{(n+1)!} < \epsilon\}$. Then $M^*(\epsilon)$ can be easily calculated by the formula

$$M^*(\epsilon) = \min\{n - 1; \sum_{k=2}^{n} \log k > -\log \epsilon + n \log(cT)\}$$

and clearly $M^*(\epsilon) \leq M(\epsilon)$.

For the numerical example, we consider the $M^X/M/1$ queue with queue length dependent arrival rates and service rates, whose special case is the $M/M/1$ queue with balking and $M/M/1$ queue with reneging (for example, see Gross and Harris [1]). When there are n customers in the system, customers arrive in batches according to a Poisson process with rate λ_n and the service rates of the server is μ_n. Let $a_i = P(X = i)$, $i \geq 1$ be the distribution of the batch size. Then the queue length process $\{X(t), t \geq 0\}$ is a Markov chain with infinitesimal generator

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 a_1 & \lambda_0 a_2 & \lambda_0 a_3 & \lambda_0 a_4 & \cdots \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 a_1 & \lambda_1 a_2 & \lambda_1 a_3 & \cdots \\ 0 & \mu_2 & -(\mu_2 + \lambda_2) & \lambda_2 a_1 & \lambda_2 a_2 & \cdots \\ 0 & 0 & \mu_3 & -(\mu_3 + \lambda_3) & \lambda_3 a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

When $a_n = (1-a)a^{n-1}, n \geq 1$ and $\lambda_n = \frac{\lambda}{2}(1 + e^{-n\alpha/\mu})$, $\mu_n = \mu(2 - e^{-n\alpha/\mu}), n \geq 0$ with $\mu_0 = 0$, we will compare (L, M) and (L^*, M^*) for the transient distribution of $X(t)$. Taking $c = 2\mu + \lambda \geq \sup\{\lambda_n + \mu_n, n = 0, 1, 2, \cdots\}$, the (i,j)th entry of the matrix $P = I + \frac{1}{c}Q$ are given by

$$p_{ij} = \begin{cases} 0 & \text{if } j < i-1 \\ \frac{\mu}{c}(2 - e^{-i\alpha/\mu}) & \text{if } j = i-1 \\ 1 - \frac{1}{c}[(\lambda/2 + 2\mu) + (\lambda/2 - 2\mu)e^{-i\alpha/\mu}] & \text{if } j = i \\ \frac{1}{c}\frac{\lambda}{2}(1 + e^{-i\alpha/\mu})(1-a)a^{j-i-1} & \text{if } j > i \end{cases}$$

and we have for $k > i$,

$$\sum_{j=k}^{\infty} p_{ij} = \frac{\lambda}{2c}(1 + e^{-i\alpha/\mu})a^{k-i-1}.$$

Hence $\sum_{j=k}^{\infty} p_{ij} < \epsilon$ is equivalent to

$$k > i + 1 + \frac{1}{\log a} \log\left(\frac{2\epsilon c}{\lambda(1 + e^{-i\alpha/\mu})}\right). \tag{3}$$

Let $\lambda = 2, \mu = 0.5, a = 0.5, \alpha = 0.05$. Then we get $c = 3$ and the condition (3) becomes

$$k > i + 1 - \log_2 \epsilon + \log_2 \left(\frac{1 + e^{-0.1i}}{3} \right)$$

Since $-1.6 < \log_2 \left(\frac{1+e^{-0.1i}}{3} \right) < -0.5$, we take for each i, $k = i + \lceil 0.5 - \log_2 \epsilon \rceil$, where $\lceil x \rceil$ is the smallest integer not smaller than x. Let $R(\epsilon) = \lceil 0.5 - \log_2 \epsilon \rceil$ then we have from the structure of the matrix P that $N_k = N_0 + kR(\epsilon), k = 0, 1, 2, \cdots$.

Numerical values for (L^*, M^*) and (L, M) are given in table. Table shows that the significant improvement L^* of L save a lot of computational task and memory.

TABLE. Numerical Results
($\delta = 0.0001, \lambda = 2, \mu = 0.5, a = 0.5, \alpha = 0.05$. and $c = 3$)

	L L^*	$\epsilon = \delta/L$ $\epsilon^* = \delta/L^*$	$M(\epsilon)$ $M(\epsilon^*)$	$M^*(\epsilon)$ $M^*(\epsilon^*)$	$R(\epsilon)$ $R(\epsilon^*)$
$T = 1$	27.5855 8.5	3.62509×10^{-6} 1.17647×10^{-5}	18 16	15 14	19 17
$T = 5$	3.6291×10^{6} 128.5	3.0589×10^{-11} 7.7821×10^{-7}	81 81	58 50	36 21
$T = 10$	1.0686×10^{13} 481	9.35762×10^{-18} 2.079×10^{-7}	163 163	112 92	58 23
$T = 15$	3.4934×10^{19} 1058.5	2.86252×10^{-24} 9.44733×10^{-8}	244 244	166 134	79 24
$T = 20$	1.1420×10^{26} 1861	8.75651×10^{-31} 5.37346×10^{-8}	326 326	219 175	101 25

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