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CENTRAL LIMIT THEOREMS FOR A VECTOR PROCESSES OF NON-LINEAR FUNCTIONALS OF GAUSSIAN SEQUENCES OF VECTORS

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ABSTRACT. We formulate central limit theorems for non-linear functionals of stationary Gaussian vector processes with long-range dependence.

1. Introduction

Limit distributions of non-linear functionals of Gaussian processes of fields with dependence have been studied by many authors and many papers provided with some tools to deal with such problems. The behaviours which can be expressed in terms of non-linear functionals of stationary Gaussian process are observed in diverse areas of engineering and natural sciences.

During the last twenty years it has been studied extensively by many authors. In recent years Arcones[1], Sanches[7] and others considered stationary Gaussian sequence of vectors and generalized the results obtained in [2,3 and 8] into multidimensional cases.

When $\mathbb{R}^d$ valued process or field has dependent structure we encounter with some problem which does not appear when we deal with the sequence of random variables. Among them is the cross correlation between random vectors. In the present paper we formulate central limit theorems for non-linear functionals of stationary Gaussian vector processes with long-range dependence.

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Let $\mathbf{X}_n = (X_{n,1}, \dots X_{n,d}), n \in \mathbb{Z}$ be a stationary Gaussian vector process with $EX_{n,p} = 0, EX_{n,p}X_{n+m,q} = r_{pq}(m)$ for $p, q = 1, \dots, d$ and for all $n, m \in \mathbb{Z}$. Without loss of generality we may assume

$$r_{pq}(0) = \delta_{pq} \quad p, q = 1, \dots, d, \tag{1.1}$$

because otherwise we can take a linear transformation to have the property (1.1).

Let f and g be real valued functions on $\mathbb{R}^d$ with, for all $n \in \mathbb{Z}$,

$$Ef(\mathbf{X}_n) = Eg(\mathbf{X}_n) = 0$$

and

$$E[f(\mathbf{X}_n)^2] < \infty, \quad E[g(\mathbf{X}_n)^2] < \infty. \tag{1.2}$$

Let $H_m(x)$ be the Hermite polynomial of d variables defined by

$$\begin{aligned} &H_m(x_1, \dots, x_d) \\ &= (-1)^{|m|} \exp\Big(\frac{x_1^2 + \dots + x_d^2}{2}\Big) \frac{\partial^{|m|}}{\partial x_1^{m_1} \cdots \partial x_d^{m_d}} \exp\Big(-\frac{x_1^2 + \dots + x_d^2}{2}\Big), \end{aligned}$$

where $\phi(x) = (2\pi)^{-d/2} \exp(-\sum_{i=1}^d x_i^2/2)$. Then f and g have the Hermite expansions

$$\begin{aligned} f(\mathbf{X}_n) &= \sum_{j=0}^{\infty} \sum_{|m|=j} c_m H_m(\mathbf{X}_n) \\ g(\mathbf{X}_n) &= \sum_{j=0}^{\infty} \sum_{|m|=j} d_m H_m(\mathbf{X}_n) \end{aligned} \tag{1.3}$$

with the Hermite coefficients, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$\begin{aligned} c_m &= m_1! \cdots m_d! \int \cdots \int H_m(x) f(x) \phi(x) dx \\ d_m &= m_1! \cdots m_d! \int \cdots \int H_m(x) g(x) \phi(x) dx, \end{aligned}$$

where $m = (m_1, \dots, m_d), |m| = m_1 + \cdots + m_d$.

We have, by orthogonality of Hermite polynomials,

$$\sum_{j=0}^{\infty} \sum_{|m|=j} c_m^2 m_1! \cdots m_d! < \infty$$

$$\sum_{j=0}^{\infty} \sum_{|m|=j} d_m^2 m_1! \cdots m_d! < \infty.$$

A function f satisfying (1.2) is said to have Hermite rank ν if the Hermite coefficient $c_m = 0$ for $|m| < \nu$ and $c_m \neq 0$ for some $|m| = \nu$. Assume that f has Hermite rank ν_1 and g has ν_2. Let

$$Z_f^N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} f(\mathbf{X}_n)$$

$$Z_g^N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} g(\mathbf{X}_n).$$

We will investigate the limiting distribution of the vector process (Z_f^N, Z_g^N) as N tends to infinity. We state a theorem.

THEROEM 1. *Let f and g be real valued function on $\mathbb{R}^d$ which satisfy condition (1.2). Let f and g have Hermite rank ν_1 and ν_2, respectively. Let $\nu = \max\{\nu_1, \nu_2\}$ and $r(s) = \max_{1 \le p \le d} r_{pp}(s)$. Suppose that*

$$\sum_{s=-\infty}^{\infty} |r(s)|^{\nu} < \infty.$$

Then, for all $j \geq \nu$, there exist the following limits

$$\lim_{N \to \infty} E(Z_f^N Z_g^N) = \rho < \infty$$

$$\lim_{N \to \infty} E(Z_f^N)^2 = \sigma_1^2 < \infty$$

$$\lim_{N \to \infty} E(Z_g^N)^2 = \sigma_2^2 < \infty.$$

Moreover

$$(1.4) \qquad (Z_f^N, Z_g^N) \xrightarrow{d} (Z_f^*, Z_g^*),$$

where (Z_f^*, Z_g^*) *is a jointly Gaussian random vector with*

$$E(Z_f^*)^2 = \sigma_1^2 \quad E(Z_g^*)^2 = \sigma_2^2 \quad E(Z_f^* Z_g^*) = \rho.$$

2. Proof of Theorem 1

In this seciton we will prove Theorem 1 using several Lemmas. Among them is the diagram formula which is very useful in computations of the expectation for a product of Hermite polynomials of standard Gaussian random vectors[2].A diagram G of order $(l_1, \dots, l_\alpha)$ is a set of vertices $\{(j,l) | 1 \le j \le \alpha, 1 \le l \le l_j\}$ and a set of pairs of vertices $\{((j,l),(k,m)) | 1 \le j < k \le \alpha, 1 \le l \le l_j, 1 \le m \le l_k\}$, called edges, such that every vertex is of degree one. We denote by $\Gamma(l_1, \dots, l_\alpha)$ the set of diagrams of order $(l_1, \dots, l_\alpha)$. Observe that $\Gamma(l_1, \dots, l_\alpha)$ is empty if $l_1 + \cdots + l_\alpha$ is an odd number. The set $L_j = \{(j,l) | 1 \le l \le l_j\}$ is called the j-th level of the graph G. Observe that the edges connect vertices of different levels. We will denote the set of edges of the diagram G by $E(G)$. Givern an edge $\omega = ((j,l),(k,m))$, let $d_i(\omega) = j$ and $d_2(\omega) = k$.

We will state the following without proof. See [2] for the detatil.

LEMMA 1 (DIAGRAM FORMULA). *Let* $(X_1, \dots, X_p)$ *be a Gaussian vector with* $EX_j = 0, EX_j^2 = 1$ *and* $EX_jX_k = r(j,k)$ *for each* $1 \le j,k \le p$. *Then*

$$E\Big[\prod_{j=1}^{p} H_{l_j}(\mathrm{X}_j)\Big] = \sum_{G \in \Gamma(l_1, \dots, l_p)} \prod_{\omega \in E(G)} r(d_1(\omega), d_2(\omega)).$$

Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ be two mean-zero Gaussian random vectors on $\mathbb{R}^d$. Assume that

$$E[X_jX_k] = E[Y_jY_k] = \delta_{jk}$$

for each $1 \leq j, k \leq d$. We define and abuse the notation $r(j,k) = r_{jk}$

$$r_{jk} = E[X_j Y_k].$$

Then

$$\begin{aligned}
E(Z_f^N Z_g^N) &= E\Big(\frac{1}{\sqrt{N}} \sum_{n=n=1}^{N} f(\mathbf{X}_n) \frac{1}{\sqrt{N}} \sum_{n=n=1}^{N} g(\mathbf{X}_n)\Big) \\
&= \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} E(f(\mathbf{X}_{n_1}) g(\mathbf{X}_{n_2})) \\
&= \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} E\left[\Big(\sum_{j_1=\nu_1}^{\infty} \sum_{|m|=j_1} c_m H_m(\mathbf{X}_{n_1})\Big)\Big(\sum_{j_2=\nu_2}^{\infty} \sum_{|l|=j_2} d_l H_l(\mathbf{X}_{n_2})\Big)\right] \\
&= \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \sum_{j=\nu}^{\infty} \sum_{|m|=|l|=j} c_m d_l \sum_{G \in \Gamma(m,l)} \prod_{\omega \in E(G)} \beta(\omega),
\end{aligned}$$

where

$$\beta(\omega) = \begin{cases} r_{d_1(\omega), d_2(\omega)-d}(n_1 - n_2) & \text{if } d_1(\omega) \leq d < d_2(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

Let f and g be functions on $\mathbb{R}^d$ with finite second moment and rank ν_1, ν_2 with respect to $\mathbf{X}_n$, respectively.

Let, for $p, q = 1, \ldots, d$, $z_G(p,q)$ be the number of vertices of the graph G joining the levels p and q. If $z_G(p,q) > 0$ for some p, q such that either $1 \leq p, q \leq d$ or $d+1 \leq p, q \leq 2d$, then $\prod_{\omega \in E(G)} \beta(\omega) = 0$. So we only have to consider the graph G so that $z_G(p,q) = 0$, for $1 \leq p, q \leq d$ and for $d+1 \leq p, q \leq 2d$. Let $z_G(p, q+d) = a(p,q)$ for $1 \leq p, q \leq d$. Then

$$\sum_{q=1}^{d} a(p,q) = l_p \tag{2.1}$$

and

$$\sum_{p=1}^{d} a(p,q) = m_q. \tag{2.2}$$

For a graph like that

$$\prod_{\omega\in E(G)}\beta(\omega)=\prod_{1\le p,q\le d}(r_{pq})^{a(p,q)}.$$

Let $(a(p,q))_{1\le p,q\le d}$ be a matrix of nonnegative numbers satisfying (2.1) and (2.2). Then

$$\begin{aligned}
E(Z_f^N Z_g^N) &= \sum_{j=\nu}^{\infty}\sum_{|m|=|l|=j} c_m d_l \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\sum_{G\in\Gamma(m,l)}\prod_{\omega\in E(G)}\beta(\omega)\\
&= \sum_{j=\nu}^{\infty}\sum_{|m|=|l|=j} c_m d_l \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\sum_{(a(p,q))\in A(m,l)}\prod_{p,q=1}^{d}\frac{1}{a(p,q)!}(r_{p,q}(n_1-n_2))^{a(p,q)},\\
&\le \sum_{j=\nu}^{\infty}\sum_{|m|=|l|=j}\frac{c_m^2+d_l^2}{2}\frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\sum_{A(m,l)}\prod_{p,q=1}^{d}\frac{1}{a(p,q)!}|r_{p,q}(n_1-n_2)|^{a(p,q)}
\end{aligned}$$

Let

$$\begin{aligned}
&C(m,l,j)\\
&= \sum_{|m|=|l|=j} c_m^2\frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\sum_{A(m,l)}\prod_{p,q=1}^{d}\frac{1}{a(p,q)!}|r_{p,q}(n_1-n_2)|^{a(p,q)}.
\end{aligned}$$

Then

$$\begin{aligned}
&C(m,l,j)\\
&= \sum_{|m|=j} c_m^2\frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\sum_{|l|=j}\sum_{(a(p,q))\in A(m,l)}\prod_{p,q=1}^{d}\frac{1}{a(p,q)!}|r_{p,q}(n_1-n_2)|^{a(p,q)}\\
&= \sum_{|m|=j} c_m^2\frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\sum_{\substack{\sum_{q=1}^{d}a(p,q)=m_p\\ p=1,\dots,d}}\prod_{p,q=1}^{d}\frac{1}{a(p,q)!}|r_{p,q}(n_1-n_2)|^{a(p,q)}\\
&= \sum_{|m|=j} c_m^2\frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\prod_{p=1}^{d}\Big(\sum_{\sum_{q=1}^{d}a(p,q)=m_p}\prod_{q=1}^{d}\frac{1}{a(p,q)!}|r_{p,q}(n_1-n_2)|^{a(p,q)}\Big)\\
&= \sum_{|m|=j} c_m^2\frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\frac{1}{m_1!\cdots m_d!}\prod_{p=1}^{d}\Big(\sum_{q=1}^{d}|r_{p,q}(n_1-n_2)|\Big)^{m_p}.
\end{aligned}$$

Since, for $1 \leq p, q \leq d$,

$$|r_{pq}(s)|^2 \leq r_{pp}(s) r_{qq}(s) \ \ and \ \ r_{pp}(s) \leq r(s),$$

we have

$$\begin{aligned}
C(m,l,j) &\leq \sum_{|m|=j} c_m^2 \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \frac{d^j}{m_1! \cdots m_d!} |r(n_1 - n_2)|^j \\
&= \sum_{|m|=j} c_m^2 m_1! \cdots m_d! \frac{d^j}{(m_1! \cdots m_d!)^2} \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} |r(n_1 - n_2)|^j \\
&\leq \sum_{|m|=j} c_m^2 m_1! \cdots m_d! \sum_{|s|<N} (1 - \frac{|s|}{N}) |r(s)|^j \\
&\leq \sum_{|m|=j} c_m^2 m_1! \cdots m_d! \sum_{|s|<N} |r(s)|^\nu \\
&\leq C_1 \sum_{|m|=j} c_m^2 m_1! \cdots m_d!,
\end{aligned}$$

for some $C_1 > 0$, and similarly we have, for some $D_1 > 0$,

$$D(m,l,j) \leq D_1 \sum_{|l|=j} d_l^2 l_1! \cdots l_d!,$$

where $D(m,l,j)$ is an analogous term to $C(m,l,j)$ with exception of the constant d_l^2 in place of c_m^2. Therefore we have
(2.3)

$$|E(Z_f^N Z_g^N)| \leq \sum_{j=\nu}^{\infty} \Big\{ C_1 \sum_{|m|=j} c_m^2 m_1! \cdots m_d! + D_1 \sum_{|l|=j} d_l^2 l_1! \cdots l_d! \Big\} < \infty.$$

We will state the following lemma without proof. It is not hard to show.

Lemma 2. *$\{E(Z_f^N Z_g^N)\}_N$ is a Cauchy sequence*

By Lemma 2 the following limits exist:

$$\lim_{N\to\infty} E(Z_f^N Z_g^N) = \rho, \quad \lim_{N\to\infty} E(Z_f^N)^2 = \sigma_1^2, \quad \lim_{N\to\infty} E(Z_g^N)^2 = \sigma_2^2.$$

We may assume that the expansions in (1.3) have only finite terms. To see this let f_T, g_T be defined by, for a large number T,

$$f_T(\mathbf{X}_n) = \sum_{j=T}^{\infty} \sum_{|m|=j} c_m H_m(\mathbf{X}_n)$$

$$g_T(\mathbf{X}_n) = \sum_{j=T}^{\infty} \sum_{|m|=j} d_m H_m(\mathbf{X}_n).$$

Let

$$Z_T^N(f) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} f_T(\mathbf{X}_n)$$

$$Z_T^N(g) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} g_T(\mathbf{X}_n).$$

Then by (2.3), for $\varepsilon > 0$, we can find a sufficiently large T such that

$$|E(Z_T^N(f) Z_T^N(g))| < \varepsilon. \tag{2.4}$$

By (2.4) we can restrict the theorem to the special case when f and g are polynomials. Thus we will show that

$$(Z_T^N(f), Z_T^N(g)) \xrightarrow{d} (Z_T^*(f), Z_T^*(g)), \tag{2.5}$$

when

$$f(\mathbf{X}_n) = \sum_{j=\nu}^{T} \sum_{|m|=j} c_m H_m(\mathbf{X}_n)$$

$$g(\mathbf{X}_n) = \sum_{j=\nu}^{T} \sum_{|m|=j} d_m H_m(\mathbf{X}_n).$$

To show (2.5) we use the method of moments and show the following, for $a, b \in \mathbb{R}$,

$$aZ_T^N(f) + bZ_T^N(g) \xrightarrow{d} aZ_T^*(f) + bZ_T^*(g).$$

We consider the moment

$$(2.6)\quad E(aZ_T^N(f)+bZ_T^N(g))^k=\sum_{t=0}^{k}\binom{k}{t}a^tb^{k-t}E[(Z_T^N(f))^t(A_T^N(g))^{k-t}].$$

To compute (2.6) we need the following expressions.
(2.7)

$$(Z_T^N(f))^t=N^{\frac{-t}{2}}\sum_{\mathbf{n}\in A(t,N)}\sum_{\mathbf{j}\in E(t,T)}\sum_{\mathbf{m}\in B(t,\mathbf{j})}c(t,\mathbf{m})\prod_{i=1}^{t}H_{m_i}(\mathbf{X}_{n_i}),$$

$$(Z_T^N(g))^{k-t}$$

$$=N^{\frac{-(k-t)}{2}}\sum_{\mathbf{n}'\in A(k-t,N)}\sum_{\mathbf{j}'\in E(k-t,T)}\sum_{\mathbf{m}\in B(k-t,\mathbf{j}')}d(t,\mathbf{m})\prod_{i=1}^{k-t}H_{m_{t+i}}(\mathbf{X}_{n'_{t+i}}).$$

Multiplying these two terms in (2.7) and taking expectation we have

$$E((Z_T^N(f))^t(Z_T^N(g))^{k-t})$$

$$=N^{\frac{-k}{2}}\sum_{\mathbf{n}\in A(k,N)}\sum_{\mathbf{j}\in E(k,T)}\sum_{\mathbf{m}\in B(k\,\mathbf{j})}c(t,\mathbf{m})d(t,\mathbf{m})E\Big[\prod_{i=1}^{k}H_{m_i}(\mathbf{X}_{n_i})\Big]$$

$$=N^{\frac{-k}{2}}\sum_{\mathbf{n}\in A(k,N)}c(t,\mathbf{m})d(t,\mathbf{m})\sum_{\mathbf{j}\in E(k,T)}\sum_{\mathbf{m}\in B(k\,\mathbf{j})}\sum_{G\in\Gamma(\mathbf{j},\mathbf{m})}\prod_{\omega\in E(G)}\beta(\omega),$$

where
(2.8)

$$\beta(\omega)=\begin{cases} r_{d_1(\omega)-(s-1)d,\,d_2(\omega)-(s'-1)d}(n_s-n_{s'}) & \text{if } (s-1)d<d_1(\omega)\le sd \\ & \quad (s'-1)d<d_2(\omega)\le s'd, \\ 0 & \text{otherwise}\end{cases}$$

and

$$A(k,N)=\{(n_1,\dots,n_k)|1\le n_i\le N,i=1,\dots,k\}$$
$$E(k,T)=\{(j_1,\dots,j_k)|\nu\le j_i\le T,i=1,\dots,k\}$$
$$B(k,\mathbf{j})=\{(m_1,\dots,m_k)|\sum_{p=1}^{d}m_i^{(p)}=j_i,0\le m_i^{(p)}\le j_i$$
$$i=1,\dots,k,p=1,\dots,d\}.$$

We will denote the set of levels $\{L_{sd+1}, \ldots, L_{(s+1)d}\}$ by sector S_s, $s = 1, \ldots, k$. We shall call a diagram regular if its sectors can be paired in such a way that no edge passes between sectors in different pairs. Let $\Gamma^*(\mathbf{j}, \mathbf{m})$ be the set of all regular diagrams.

For fixed $\mathbf{j}, \mathbf{m}$ we define

$$T_G(\mathbf{j}, \mathbf{m}, N) = N^{-k/2} \sum_{\mathbf{n} \in A(k,N)} \prod_{\omega \in E(G)} \beta(\omega),$$

where $\beta(\omega)$ is defined as in (2.8).

LEMMA 3. *If $G \in \Gamma(\mathbf{j}, \mathbf{m})$ is not a regular diagram, then*

$$\lim_{N \to \infty} T_G(\mathbf{j}, \mathbf{m}, N) = 0.$$

See [6] for the proof of Lemma 3. If k is odd, by Lemma 2, then

$$\lim_{N \to \infty} E(Z_T^N(f))^t (Z_T^N(g))^{k-t} = 0.$$

Therefore, for any $a, b \in \mathbb{R}$,

$$\lim_{N \to \infty} E(aZ_T^N(f) + bZ_T^N(g))^k = 0.$$

When k is even, say $k = 2s$, then

$$\lim_{N \to \infty} E(Z_T^N(f))^t (Z_T^N(g))^{2s-t}$$
$$= \sum_{\mathbf{j} \in E(k,T)} \sum_{\mathbf{m} \in B(k\,\mathbf{j})} c(t, \mathbf{m}) d(t, \mathbf{m}) \sum_{G \in \Gamma(\mathbf{j},\mathbf{m})} \lim_{N \to \infty} T_G(\mathbf{j}, \mathbf{m}, N),$$

where $c(t, \mathbf{m}) = \prod_{i=1}^{t} c_{m_i}$, and $d(t, \mathbf{m}) = \prod_{i=t+1}^{2s-t} d_{m_i}$.

LEMMA 4.

$$\lim_{N \to \infty} E(Z_T^N(f))^t (Z_T^N(g))^{2s-t} = E[(Z_T^*(f))^t (Z_T^*(g))^{2s-t}],$$

where $(Z_T^*(f), Z_T^*(g))$ is a Gaussian random vector with

$$E(Z_T^*(f))^2 = \lim_{N \to \infty} E(Z_T^N(f))^2 = \sigma_T^2(f)$$
$$E(Z_T^*(g))^2 = \lim_{N \to \infty} E(Z_T^N(g))^2 = \sigma_T^2(g)$$
$$E(Z_T^*(f) Y_T^*(g)) = \lim_{N \to \infty} E(Z_T^N(f) Z_T^N(g)) = \rho_T(f, g).$$

PROOF.

$$
\begin{aligned}
&E(Z_T^N(f))^t(Z_T^N(g))^{2s-t} \\
&= \sum_{\mathbf{j}\in E(2s,T)} \sum_{\mathbf{m}\in B(2s\mathbf{j})} c(t,\mathbf{m})d(t,\mathbf{m}) \sum_{G\in\Gamma(\mathbf{j},\mathbf{m})} T_G(\mathbf{j},\mathbf{m},N) \\
&= \sum_{\mathbf{j}\in E(2s,T)} \sum_{\mathbf{m}\in B(2s\mathbf{j})} c(t,\mathbf{m})d(t,\mathbf{m}) \Big\{ \sum_{G\in\Gamma^*(\mathbf{j},\mathbf{m})} T_G(\mathbf{j},\mathbf{m},N) + \\
&\qquad \sum_{G\in\Gamma(\mathbf{j},\mathbf{m})-\Gamma^*(\mathbf{j},\mathbf{m})} T_G(\mathbf{j},\mathbf{m},N)\Big\}.
\end{aligned}
$$

If G is not a regular diagram, then there is no paring on $\{1,2,\ldots,2s\}$ such that the edges connected only between the paired sectors. Consequently we have

$$
\begin{aligned}
&\lim_{N\to\infty} E(Z_T^N(f))^t(Z_T^N(g))^{2s-t} \\
&= \sum_{\mathbf{j}\in E(2s,T)} \sum_{\mathbf{m}\in B(2s,\mathbf{j})} c(t,\mathbf{m})d(t,\mathbf{m}) \sum_{G\in\Gamma^*(\mathbf{j},\mathbf{m})} \lim_{N\to\infty} T_G(\mathbf{j},\mathbf{m},N) \\
&\sum_{\tau\in P}\prod_{p=1}^{s}\{ \sum_{j_{\tau(2p-1)}=\nu}^{T} \sum_{j_{\tau(2p)}=\nu}^{T} \sum_{|m_{\tau(2p-1)}|=j_{\tau(2p-1)}} \sum_{|m_{\tau(2p)}|=j_{\tau(2p)}} \alpha_{m_{\tau(2p-1)}}\alpha_{m_{\tau(2p)}} \\
&\qquad \sum_{G\in\Gamma_p^*(j_{\tau(2p-1)},j_{\tau(2p)},m_{\tau(2p-1)},m_{\tau(2p)})} \lim_{N\to\infty}\frac{1}{N}\sum_{n_{\tau(2p-1)}}^{N}\sum_{n_{\tau(2p)}}^{N}\prod_{\omega\in E(G)}\beta(\omega)\} \\
&= \sum_{\tau\in P}\prod_{p=1}^{s}\{ \sum_{j_1=\nu}^{T}\sum_{j_2=\nu}^{T}\sum_{|m_1|=j_1}\sum_{|m_2|=j_2} \alpha_{m_1}\alpha_{m_2} \\
&\qquad \sum_{G\in\Gamma^*(j_1,j_2),m_1,m_2)} \lim_{N\to\infty}\frac{1}{N}\sum_{n_1}^{N}\sum_{n_2}^{N}\prod_{\omega\in E(G)}\beta(\omega)\} \\
&= \sum_{\tau\in P}\prod_{p=1}^{s} h_p(\tau),
\end{aligned}
$$

where P is the set of all rearrangements $\{\tau(1),\ldots,\tau(2s)\}$ such that edges are connected only between sectors $S_{\tau(2p-1)}$ and $S_{\tau(2p)}$, $p=1,\ldots,s$, and

$\alpha_{m_j} = c_{m_j}$, if $1 \le j \le t$, $\alpha_{m_j} = d_{m_j}$ if $t+1 \le j \le 2s$, and

$$h_p(\tau) = \begin{cases} \sigma_T^2(f) = \lim_{N\to\infty} E(Z_T^N(f))^2 & \text{for } \tau(2p-1), \tau(2p) \le t \\ \rho_T(f,g) = \lim_{N\to\infty} E(Z_T^N(f) Z_T^N(g)) & \text{for } \tau(2p-1) \le t < \tau(2p) \\ \sigma_T^2(g) = \lim_{N\to\infty} E(Z_T^N(g))^2 & \text{for } \tau(2p-1), \tau(2p) > t. \end{cases}$$

This completes the proof of Lemma 4.

By Lemma 4 we conclude

$$(Z_T^N(f), Z_T^N(g)) \xrightarrow{d} (Z_T^*(f), Z_T^*(g)).$$

(1.4) follows from (2.5). This completes the proof of Theorem 1.

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