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EQUIVARIANT ALGEBRAIC APPROXIMATIONS OF G MAPS

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ABSTRACT. Let f be a smooth G map from a nonsingular real algebraic G variety to an equivariant Grassmann variety. We use some G vector bundle theory to find a necessary and sufficient condition to approximate f by an entire rational G map. As an application we algebraically approximate a smooth G map between G spheres when G is an abelian group.

0. Introduction

Throughout this paper we let G be a compact Lie group. A *real algebraic G variety* in an orthogonal representation Ω is the common zeros of polynomials $p_1, \dots, p_m : \Omega \to \mathbb{R}$, which is invariant under the action of G on Ω. We also say that G acts algebraically on V.

Let X and Y be nonsingular real algebraic G varieties. The question we are interested in here is the following: when can a given G map $f : X \to Y$ be approximated by polynomial G maps or entire rational G maps? If X is compact and Y is an orthogonal representation, then the classical Stone-Weierstrass theorem can be extended equivariantly. Namely, using Theorem 1.1 which is a version of Stone-Weierstrass theorem and the averaging operator we have Theorem 1.3 and Corollary 1.4. However if Y is not an orthogonal representation, then approximation of f by polynomial maps or entire rational maps are not always possible, see [Wo] and [BK2]. On the other hand, in nonequivariant case (i.e.

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$G = 1$) any map from S^n to S^1, S^2, or S^4 can be approximated by entire rational maps, see [BK1].

In this paper we use Corollary 1.4 and some G vector bundle theory to find a necessary and sufficient condition for a G map from X to Grassmann G variety to be approximated by entire rational G maps. Namely, we have the following theorem which is an equivariant generalization of Lemma 14 of [Iv].

THEOREM 2.4. *Let X be a compact nonsingular real algebraic G variety. Then a $\mathcal{C}^r$ G-map $f : X \to G_\Lambda(\Xi, k)$ can be approximated by entire rational G maps in $\mathcal{C}^\rho$ topology for $0 \leq \rho \leq r \leq \infty$ and $\rho < \infty$ if and only if the G Λ-vector bundle ξ having f as the classifying map is $\mathcal{C}^r$ G isomorphic to a strongly algebraic G vector bundle.*

We apply Theorem 2.4 to complex G vector bundles over the unit spheres of unitary representations of abelian group G to have the following:

THEOREM 3.3. *Let G be an abelian group. Let E and W be any unitary representation of G. Then $\mathcal{R}(S(E), G_{\mathbb{C}}(W, k))^G$ is dense in $\mathcal{C}^r(S(E), G_{\mathbb{C}}(W, k))^G$ with $\mathcal{C}^\rho$ topology for $0 \leq \rho \leq r \leq \infty$ and $\rho < \infty$.*

By considering complex line bundles we have the following extension of the results in [BK1] as a special case of Theorem 3.3.

THEOREM 3.5. *Let G be an abelian group. Let E be any unitary representation of G and V a unitary 1-dimensional representation of G. Then $\mathcal{R}(S(E), S(V \oplus \mathbb{R}))^G$ is dense in $\mathcal{C}^r(S(E), S(V \oplus \mathbb{R}))^G$ with $\mathcal{C}^\rho$ topology for $0 \leq \rho \leq r \leq \infty$ and $\rho < \infty$.*

1. Equivariant polynomial approximations

Let Ξ and Ω be representations of G. A polynomial G map $f : \Xi \to \Omega$ is a G map of the form $f = (f_1, \dots, f_m)$ where each f_i is a polynomial for $0 \leq i \leq m$ and $m = \dim \Omega$. For $0 \leq r \leq \infty$ let $\mathcal{C}^r(\Xi, \Omega)$ denote the set of all C^r maps from Ξ to Ω. Let $\mathcal{P}(\Xi, \Omega)$ be the subset of all polynomial maps from Ξ to Ω. The $\mathcal{C}^\rho$ topology on $\mathcal{C}^r(\Xi, \Omega)$ for $0 \leq \rho \leq r$ is the topology defined as follows. For a multi index $s = (s_1, \dots, s_n)$ with

$n = \dim \Xi$ and $f \in \mathcal{C}^r(\Xi, \Omega)$ let

$$D^s = \frac{\partial^{|s|}}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}}$$

where $|s| = s_1 + \cdots + s_n$. For $n \in \mathbb{N}$ let K_n denote the disk of radius n in Ξ centered at the origin. Let $N_{K_n,\rho}$ be the semi-norm on the set $\mathcal{C}(\Xi, \Omega)$ defined by

$$N_{K_n,\rho}(f) = \sup_{x \in K_n, 0 \le |s| \le \rho} |D^s(f)|.$$

Then the C^ρ topology on $\mathcal{C}(\Xi, \Omega)$ is the smallest topology so that $N_{K_n,\rho}$ are continuous for all K_n. The following theorem is well known.

THEOREM 1.1. *Let $0 \le r \le \infty$. Then $\mathcal{P}(\Xi, \Omega)$ is dense in $\mathcal{C}^r(\Xi, \Omega)$ with C^ρ topology for $0 \le \rho \le r$.*

PROOF. See Corollary 4 of Theorem 15.3 of [Tr]. □

We now generalize this theorem for equivariant case. We first need the following averaging operator. Since G is compact there is the Haar measure of G which is denoted by dg. For any $f : \Xi \to \Omega$ define

$$A(f)(x) = \int_G g^{-1} f(gx)\, dg$$

for $x \in \Xi$. If G is a finite group, this averaging operator is nothing but

$$A(f)(x) = \frac{1}{|G|} \sum_{g \in G} g^{-1} f(gx).$$

Here are some basic properties of the operator A.

PROPOSITION 1.2. *Let G be a compact Lie group, and let Ξ and Ω be representations of G. Let $A : \mathcal{C}^r(\Xi, \Omega) \to \mathcal{C}^r(\Xi, \Omega)$ be defined as above for $0 \le r \le \infty$. Then*

(1) *For any $f \in \mathcal{C}^r(\Xi, \Omega)$ the map $A(f)$ is a G map. If f is a G map then $A(f) = f$.*
(2) *If f is a polynomial map, then so is $A(f)$.*
(3) *If $\Omega = \mathbb{R}^1$ and $f \ge 0$, then $A(f) \ge 0$.*

PROOF. See [DMP]. □

If we define a G action on $\mathcal{C}^r(\Xi,\Omega)$ by $g\cdot f(x) = gf(g^{-1}x)$ for $g \in G$ and $f \in \mathcal{C}^r(\Xi,\Omega)$, then the fixed point set $\mathcal{C}^r(\Xi,\Omega)^G$ is the subspace of all G equivariant $\mathcal{C}^r$ maps, and $\mathcal{P}(\Xi,\Omega)^G$ is the subset of all polynomial G maps. We now have the following equivariant generalization of Theorem 1.1.

THEOREM 1.3. *$\mathcal{P}(\Xi,\Omega)^G$ is dense in $\mathcal{C}^r(\Xi,\Omega)^G$ with $\mathcal{C}^\rho$ topology for $0 \le \rho \le r \le \infty$ and $\rho < \infty$.*

PROOF. Note first that for any $h \in \mathcal{C}^r(\Xi,\Omega)$

$$D^s(A(h))(x) = \int_G D^s(g^{-1}fg(x)\, dg.$$

From the formula for higher order partial derivatives of composition if

$$\sup_{x\in K_n, 0\le|s|\le\rho} |D^s(h)| \le \epsilon$$

then

$$\sup_{x\in K_n, 0\le|s|\le\rho} |D^s(g^{-1}hg)| \le C_\rho\epsilon$$

for some constant C_ρ which depends only on ρ. Therefore

$$\sup_{x\in K_n, 0\le|s|\le\rho} |D^s(A(h))| \le C_\rho\epsilon.$$

Let $f : \Xi \to \Omega$ be a G equivariant $\mathcal{C}^r$ map. Theorem 1.1 implies that for a given $\epsilon > 0$ and K_n there exists a polynomial map $p : \Xi \to \Omega$ such that

$$\sup_{x\in K_n, 0\le|s|\le\rho} |D^s(f-p)| \le \epsilon.$$

From the argument above and Proposition 1.2 (2) we have

$$\sup_{x\in K_n, 0\le|s|\le\rho} |D^s(f-A(p))| = \sup_{x\in K_n, 0\le|s|\le\rho} |D^s(A(f-p))| \le C_\rho\epsilon.$$

Since $A(p) \in \mathcal{P}(\Xi,\Omega)^G$ we are done. □

Let X and Y be two nonsingular real algebraic G varieties in orthogonal representations Ξ and Ω respectively. Since X and Y are nonsingular they are $\mathcal{C}^\infty$ G manifolds. For $0 \le r \le \infty$ let $\mathcal{C}^r(X,Y)^G$ be the set of all $\mathcal{C}^r$ G maps from X to Y. Since any $\mathcal{C}^r$ G map $f : X \to Y$ can be extended to $\mathcal{C}^r$ G maps $\tilde{f} : \Xi \to \Omega$ we can identify $\mathcal{C}^r(X,Y)^G$ with the subspace

$$\mathcal{C}^r(\Xi,\Omega)^G_{(X,Y)} = \{\tilde{f} : \Xi \to \Omega \mid \tilde{f} \text{ is a chosen extension of } f \in \mathcal{C}^r(X,Y)^G\}$$

of $\mathcal{C}^r(\Xi,\Omega)$. Then for $0 \le \rho \le r$ the $\mathcal{C}^\rho$ topology on $\mathcal{C}^r(X,Y)^G$ is the topology induced from the $\mathcal{C}^\rho$ topology on $\mathcal{C}^r(\Xi,\Omega)$ via the identification $\mathcal{C}^r(X,Y)^G$ with $\mathcal{C}^r(\Xi,\Omega)^G_{(X,Y)}$.

A G map $f : X \to Y$ is called a polynomial G map if there exists a polynomial map $p : \Xi \to \Omega$ such that $p|_X = f$. By Proposition 1.2 (2) we can assume that p is a polynomial G map. Let $\mathcal{P}(X,Y)^G$ denote the subset of all polynomial G maps from X to Y. We say that a $\mathcal{C}^r$ G map $f : X \to Y$ is approximated by polynomial G maps in $\mathcal{C}^\rho$ topology for $0 \le \rho \le r \le \infty$ if for a given $\epsilon > 0$ there exists $p \in \mathcal{P}(X,Y)^G$ such that

$$\sup_{x \in X, 0 \le |s| \le \rho} |D^s(f - p)| \le \epsilon.$$

An immediate consequence of Theorem 1.3 is the following.

COROLLARY 1.4. *Let X be a compact nonsingular real algebraic G variety, and let Ω be a representation of G. The any $\mathcal{C}^r$ G map from X to Ω can be approximated by polynomial maps in $\mathcal{C}^\rho$ topology for $0 \le \rho \le r \le \infty$ and $\rho < \infty$.* □

2. Entire rational G maps

Let X be a subspace of Euclidean space $\mathbb{R}^n$. A map $f : X \to \mathbb{R}^m$ is said to be *entire rational* if there exist polynomial maps $P : \mathbb{R}^n \to \mathbb{R}^m$ and $Q : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = P(x)/Q(x)$ for all $x \in X$ and $Q^{-1}(0) \cap X = \emptyset$. Let Ξ and Ω be representations of G, and let X be a G invariant subspace of Ξ. If a G map $f : X \subset \Xi \to \Omega$, viewed as the underlying non-equivariant map, is entire rational, we call f an *entire rational G map*.

The following proposition is an interesting observation.

PROPOSITION 2.1. *Let X be a real algebraic G variety in Ξ. If $f : X \subset \Xi \to \Omega$ is an entire rational G map in Ξ, then there exist polynomial G maps $P' : \Xi \to \Omega$ and $Q' : \Xi \to \mathbb{R}$ such that $f(x) = P'(x)/Q'(x)$ for all $x \in X$ and Q' does not vanish on Ξ.*

PROOF. Let $f = P/Q$ where $P : \Xi \to \Omega$ and $Q : \Xi \to \mathbb{R}$ are polynomial maps such that Q does not vanish on X. Here P and Q are not necessarily G equivariant. Since X is a real algebraic variety there exists a polynomial $h : \Xi \to \mathbb{R}$ such that $h^{-1}(0) = X$. We now consider the map $(PQ)/(Q^2 + h^2)$. Since h vanishes on X the map $(PQ)/(Q^2 + h^2)$ is equal to P/Q on X. Take the average

$$P' = A(PQ)$$
$$Q' = A(Q^2 + h^2).$$

Since $Q^2 + h^2 > 0$ on Ξ its average $Q' = A(Q^2 + h^2) > 0$ on Ξ. Thus Q' does not vanish on Ξ, and P' and Q' are G equivariant polynomials. It is clear that $f(x) = P'(x)/Q'(x)$ for all $x \in X$. □

The following proposition asserts that a G map which is locally entire rational is an entire rational G map.

PROPOSITION 2.2. *Let X be a real algebraic G variety on Ξ. Let $U_1, \dots, U_k$ be Zariski open subsets of X such that $X = \bigcup_{i=1}^k U_i$. Let $f : X \to Y \subset \Omega$ be a G map such that $f|_{U_i}$ are entire rational for all $i = 1, \dots, k$. Then f is an entire rational G map.*

PROOF. Let $f|_{U_i} = P_i/Q_i$ where $P_i : \Xi \to \Omega$ and $Q_i : \Xi \to \mathbb{R}$ are polynomials such that Q_i does not vanish on U_i. Since U_i are Zariski open subset of X there exists an algebraic subvariety $V_i \subset X$ such that $U_i = X - V_i$. Let $g_i : \Xi \to \mathbb{R}$ be a polynomial map such that $g_i^{-1}(0) = V_i$. Let $P_i' = g_i P_i$ and $Q_i' = g_i Q_i$. Then $Q_i'(0) \cap X = V_i$ and

$$f|_{U_i} = (\frac{P_i}{Q_i})|_{U_i} = (\frac{P_i'}{Q_i'})|_{U_i}.$$

Therefore by replacing P_i and Q_i by P_i' and Q_i' respectively, if necessary, we may assume that $Q_i^{-1}(0) \cap X = V_i$. Note that if $\lambda = a_1/b_1 = \cdots =$

a_l/b_l then $\lambda = (a_1 + \cdots + a_l)/(b_1 + \cdots + b_l)$. Since $f(x) = P_i(x)/Q_i(x) = P_i(x)Q_i(x)/Q_i^2(x)$ for all $x \in U_i$ it follows that for all $x \in U_i$

$$f(x) = \frac{\sum_{i=1}^{l} P_i(x)Q_i(x)}{\sum_{i=1}^{l} Q_i^2(x)}.$$

However the righ-hand side of the above equation is independent of the index i. Therefore we can see that the

$$f(x) = \frac{\sum_{i=1}^{l} P_i(x)Q_i(x)}{\sum_{i=1}^{l} Q_i^2(x)}$$

for all $x \in X$. Since the denominator of the right-hand side does not vanish on X the map f is an entire rational G map. □

Let Λ stand for $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let Ξ be a representation of G over Λ, in particular, its underlying space is Λ^n for some n. We assume that the action of G preserves the standard bilinear form on Λ^n over Λ. Let $\mathrm{End}_\Lambda(\Xi)$ denote the set of endomorphisms of Ξ over Λ. It is a representation of G with the action given by

$$G \times \mathrm{End}_\Lambda(\Xi) \to \mathrm{End}_\Lambda(\Xi) \quad \text{with} \quad (g, L) \mapsto gLg^{-1}.$$

Let k be a natural number. We set

$$G_\Lambda(\Xi, k) = \{L \in \mathrm{End}_\Lambda(\Xi) \mid L^2 = L,\ L^* = L,\ \mathrm{trace}\, L = k\}$$
$$E_\Lambda(\Xi, k) = \{(L, u) \in \mathrm{End}_\Lambda(\Xi) \times \Xi \mid L \in G_\Lambda(\Xi, k),\ Lu = u\}.$$

Here L^* denotes the adjoint of L. If one chooses an orthonormal (resp. unitary or symplectic) basis of Ξ, then $\mathrm{End}_\Lambda(\Xi)$ is canonically identified with the set of $n \times n$ matrices Λ^{n^2}, and L^* is obtained by transposing L and conjugating its entries. Since each element of $G_\Lambda(\Xi, k)$ is an orthogonal projection of Ξ onto a k dimensional subspace of Ξ, we can identify $G_\Lambda(\Xi, k)$ with the Grassmann manifold of k dimensional subspaces of Ξ. This description specifies $G_\Lambda(\Xi, k)$ and $E_\Lambda(\Xi, k)$ as real algebraic G varieties. Define $p : E_\Lambda(\Xi, k) \to G_\Lambda(\Xi, k)$ as projection on the first factor. This defines a G vector bundle, which is called the *universal bundle* over $G_\Lambda(\Xi, k)$, and which is denoted by $\gamma_\Lambda(\Xi, k)$.

DEFINITION. *A strongly algebraic G vector bundle over Λ is a pair (X,μ) where X is a real algebraic G variety and $\mu : X \to G_\Lambda(\Xi,k)$ is an entire rational G map. Assuming that Ξ is a summand of a representation Ξ' of G, we have an embedding $i : G_\Lambda(\Xi,k) \to G_\Lambda(\Xi',k)$. In this sense we identify the strongly algebraic G vector bundles (X,μ) and $(X,i\mu)$.*

Let $V_\Lambda(\Xi,k)$ denote the subset of $(\Xi)^k$ which consists of k linearly independent vectors of Ξ. The Stiefel manifold $V_\Lambda(\Xi,k)$ has the obvious G action induced from the G action on Ξ. Let $\pi : V_\Lambda(\Xi,k) \to G_\Lambda(\Xi,k)$ be the map which associates to each collection of vectors the orthogonal projection of Ξ on to the subspace spanned by the vectors. Then π is clearly a G map.

PROPOSITION 2.3. *The map $\pi : V_\Lambda(\Xi,k) \to G_\Lambda(\Xi,k)$ is an entire rational G map.*

PROOF. Let $V_\Lambda^0(\Xi,k) \subset V_\Lambda(\Xi,k)$ be the set of orthogonal k vectors of Ξ. We first claim that $\pi|_{V_\Lambda^0(\Xi,k)}$ is entire rational. If $(v_1,\dots,v_k) \in V_\Lambda^0(\Xi,k)$, then $\pi(v_1,\dots,v_k)$ is an orthogonal projection $P : \Xi \to \Xi$ which maps Ξ onto the space spanned by $v_1,\dots,v_k$. Then

$$P(x) = \sum_{i=1}^{k} \frac{(x,v_i)}{(v_i,v_i)} v_i$$

for $x \in \Xi$. Therefore π is clearly an entire rational map. Now we perform the Gram-Schmidt orthogonalization process to a collection of linearly independent vectors $(v_1,\dots,v_k)$. Let $GS : V_\Lambda(\Xi,k) \to V_\Lambda^0(\Xi,k)$ be the Gram-Schmidt orthogonalization map. If $GS(v_1 \dots, v_k) = (v_1',\dots,v_k')$, then $v_1' = v_1$ and $v_j' = v_j - \sum_{i=1}^{k-1} a_i v_i$ where $a_i = (v_j, v_i')$ for $j = 2,\dots,k$. This formula shows that GS is entire rational. The map π is nothing but the composition $\pi|_{V_\Lambda^0(\Xi,k)} \circ GS$, which is entire rational. □

For two nonsingular real algebraic G varieties $X \subset \Xi$ and $Y \subset \Omega$ let $\mathcal{R}(X,Y)^G$ denote the subset of all entire rational G maps from X to Y. By Proposition 2.1 any $f \in \mathcal{R}(X,Y)^G$ can be extended to $P'/Q' : \Xi \to \Omega$ where P' and Q' are polynomial G maps and Q' does not vanish on Ξ. Therefore we can embed $\mathcal{R}(X,Y)^G$ as a subspace of $C^r(X,Y)^G$ with C^ρ

topology for $0 \leq \rho \leq r \leq \infty$. We say that a C^r G map $f : X \to Y$ is approximated by entire rational G maps in C^ρ topology for $0 \leq \rho \leq r \leq \infty$ if for a given $\epsilon > 0$ there exists $q \in \mathcal{R}(X, Y)^G$ such that

$$\sup_{x \in X, 0 \leq |s| \leq \rho} |D^s(f - q)| \leq \epsilon.$$

THEOREM 2.4. *Let X be a compact nonsingular real algebraic G variety. Then a C^r G-map $f : X \to G_\Lambda(\Xi, k)$ can be approximated by entire rational G maps in C^ρ topology for $0 \leq \rho \leq r \leq \infty$ and $\rho < \infty$ if and only if the G Λ-vector bundle ξ having f as the classifying map is C^r G isomorephic to a strongly algebraic G vector bundle.*

PROOF. If $f : X \to G_\Lambda(\Xi, k)$ can be approximated by an entire rational G map $f' : X \to G_\Lambda(\Xi, k)$ in C^ρ topology, then f is G homotopic to f'. Therefore $\xi = f^*(\gamma_\Lambda(\Xi, k))$ is C^ρ G isomorphic to $\xi' = f'^*(\gamma_\Lambda(\Xi, k))$ which is a strongly algebraic G vector bundle. Conversely, suppose $\xi = f^*(\gamma_\Lambda(\Xi, k))$ is C^ρ G isomorphic to a strongly algebraic G vector bundle $\eta = g^*(\gamma_\Lambda(\Xi', k))$ where $g : X \to G_\Lambda(\Xi', k)$. Let E_ξ and E_η be the total space of ξ and η respectively. Let $d : E_\eta \to E_\xi$ be the C^ρ G diffeomorphism which is induced from the G isomorphism $\eta \to \xi$. Define a G action on $\mathrm{Hom}(\Xi', \Xi)$ by conjugation: for $g \in G$ and $L \in \mathrm{Hom}(\Xi', \Xi)$ define $g \cdot L = gLg^{-1}$. Then $\mathrm{Hom}(\Xi', \Xi)$ is a representation of G. Define a G map $K : X \to \mathrm{Hom}(\Xi', \Xi)$ by

$$K(x)(y) = pr_2 \circ d(x, g(x)y)$$

for $x \in X$, $y \in Y$ and $pr_2 : X \times \Xi \to \Xi$ is the projection. We note that the rank of $K(x)$ is k for all $x \in X$. By Corollary 1.4 we can approximate K by a polynomial G map $K' : X \to \mathrm{Hom}_\Lambda(\Xi', \Xi)$ in C^ρ topology. Define a G-map $L : X \to \mathrm{Hom}_\Lambda(\Xi', \Xi)$ by

$$L(x)(y) = K'(x)(g(x)y).$$

Since K' is a close approximation of K in C^r topology we can still assume that the rank of $L(x)$ is k for all $x \in X$. Therefore for each $x \in X$ the image $\mathrm{Im}\, L(x)$ is a k dimensional subspace of Ξ. Hence we can define $f' : X \to G_\Lambda(\Xi, k)$ by letting $f'(x)$ to be the orthogonal projection of Ξ

onto $\operatorname{Im} L(x)$. Then f' is a G map which clearly approximates f in C^ρ topology. It remains to prove that f' is entire rational. Let $\{\epsilon_1, \dots, \epsilon_n\}$ be a basis for Ξ'. If $x \in X$ and the vectors $L(x)(\epsilon_{i_1}), \dots, L(x)(\epsilon_{i_k})$ are linearly independent, then $f'(x)$ is the orthogonal projection of Ξ onto the subspace spanned by $L(x)(\epsilon_{i_1}), \dots, L(x)(\epsilon_{i_k})$. Let X_i be the set of those $x \in X$ for which $L(x)(\epsilon_{i_1}), \dots, L(x)(\epsilon_{i_k})$ are linearly independent. Then $f' : X - X_i \to G_\Lambda(\Xi, k)$ is an entire rational map because it is the composition $\pi \circ L$ where π is the map in Proposition 2.3. It is easy to see that for each i the set X_i is an algebraic subvariety of X. Let $U_i = X - X_i$. Then $f'|_{U_i}$ is rational for all i and $\bigcup U_i = X$. Therefore f' is entire rational by Proposition 2.2. Hence f' is an entire rational G map which approximates f in C^ρ topology. □

3. Equivariant entire rational approximations

In this section we apply the result of the previous section to complex vector bundles on spheres, and find some approximation result of G maps between G varieties. We need the following propositions.

PROPOSITION 3.1. *Let G be an abelian group, and let E be a unitary representation of G. Then any complex vector bundles over the unit sphere $S(E)$ is stably trivial.*

PROOF. From Thom isomorphism theorem in K_G-theory we have the exact sequence

$$0 \to K_G^1(S(E)) \to R(G) \to R(G) \xrightarrow{\varphi} K_G^0(S(E)) \to 0.$$

See Corollary 2.7.5 of [At]. Here $\varphi : R(G) \to K_G^0(S(E))$ is induced from the map which assigns to each G representation V the trivial G vector bundle $S(E) \times V$ over $S(E)$. □

PROPOSITION 3.2. *If a G vector bundle ξ over a nonsingular real algebraic variety X is stably trivial, then ξ is C^∞ G isomorphic to a strongly algebraic G vector bundle.*

PROOF. See lemma 2.1.4 of [DMS]. □

As an immediate consequence of Theorem 2.3, Proposition 3.1, and Proposition 3.2 we have the following.

THEOREM 3.3. *Let G be an abelian group. Let E and W be any unitary representation of G. Then $\mathcal{R}(S(E), G_{\mathbb{C}}(W,k))^G$ is dense in $\mathcal{C}^r(S(E), G_{\mathbb{C}}(W,k))^G$ with $\mathcal{C}^\rho$ topology for $0 \le \rho \le r \le \infty$ and $\rho < \infty$.* □

An entire rational G map $f : X \to Y$ between two real algebraic G varieties is called a birational G isomorphism if f has the entire rational G inverse.

LEMMA 3.4. *Let G be an abelian group. For a given unitary 1-dimensional representation V of G there exists a unitary 2-dimensional representation W of G such that the unit sphere $S(V \oplus \mathbb{R})$ is G birationally isomorphic to $G_{\mathbb{C}}(W,1)$. Conversely, for a given unitary 2-dimensional representation W of G there exists unitary 1-dimensional representation V of G such that $G_{\mathbb{C}}(W,1)$ is C birationally isomorphic to the 2 dimensional unit sphere $S(V \oplus \mathbb{R})$.*

PROOF. Let V be a 1-dimensional unitary representation of G. Define $\phi : S(V \oplus \mathbb{R}) \to G_{\mathbb{C}}(\mathbb{C}^2, 1)$ by

$$\phi(u,\alpha) = \frac{1}{2}\begin{pmatrix} 1-\alpha & \overline{u} \\ u & \overline{1+\alpha} \end{pmatrix}$$

for $(u,\alpha) \in S(V \oplus \mathbb{R})$. Here

$$G_{\mathbb{C}}(\mathbb{C}^2,1) = \left\{ \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & 1-\alpha \end{pmatrix} \mid \alpha \in [0,1], \beta \in \mathbb{C}, \|\beta\|^2 = \alpha(1-\alpha) \right\}.$$

Then ϕ is clearly a polynomial map which has the inverse $\psi : G_{\mathbb{C}}(\mathbb{C}^2,1) \to S(V \oplus \mathbb{R})$ such that

$$\psi\left(\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & 1-\alpha \end{pmatrix}\right) = (1-2\alpha, 2\beta).$$

It remains to prove that there exists a unitary representation structure W of G on $\mathbb{C}^2$ such that if $G_{\mathbb{C}}(\mathbb{C}^2, 1) = G_{\mathbb{C}}(W, 1)$ has the induced action of G defined by $g \cdot A = gAg^{-1}$ for $g \in G$ and $A \in G_{\mathbb{C}}(W, 1)$, then the map ϕ and ψ are G-equivariant. Let U be any 2-dimensional unitary representation of G. Since G is abelian U is a direct sum of 1-dimensional irreducible representations. Then for $g \in G$ and $(x, y) \in \mathbb{C}^2$ we may assume that

$$(g \cdot (x, y))^t = \begin{pmatrix} e^{i\theta_1(g)} & 0 \\ 0 & e^{i\theta_2(g)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for some $0 \leq \theta_1(g), \theta_2(g) < 2\pi$. Therefore the induced action of G on $G_{\mathbb{C}}(U, 1)$ is defined as follows: for $g \in G$ and $A = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & 1-\alpha \end{pmatrix} \in G_{\mathbb{C}}(U, 1)$

$$\begin{aligned} g \cdot A &= \begin{pmatrix} e^{i\theta_1(g)} & 0 \\ 0 & e^{i\theta_2(g)} \end{pmatrix} \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & 1-\alpha \end{pmatrix} \begin{pmatrix} e^{-i\theta_1(g)} & 0 \\ 0 & e^{-i\theta_2(g)} \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \overline{e^{i(\theta_1(g)-\theta_2(g))}\beta} \\ e^{i(\theta_1(g)-\theta_2(g))}\beta & 1-\alpha \end{pmatrix} \end{aligned}$$

Since V is a 1-dimensional unitary representation of G there exists $0 \leq \omega(g) < 2\pi$ such that $gv = e^{i\omega(g)}v$ for $g \in G$ and $v \in V$. If we choose a 2-dimensional unitary representation W of G such that $\theta_1(g) - \theta_2(g) = \omega(g)$, where $\theta_1(g)$ and $\theta_2(g)$ are the numbers as above, then with the induced G action on $G_{\mathbb{C}}(W, 1)$ the maps ϕ and ψ are G equivariant. The second statement of the lemma can be proved similarly. □

THEOREM 3.5. *Let G be an abelian group. Let E be any unitary representation of G, and V a unitary 1-dimensional representation of G. Then $\mathcal{R}(S(E), S(V \oplus \mathbb{R})^G$ is dense in $C^r(S(E), S(V \oplus \mathbb{R}))^G$ with C^ρ topology for $0 \leq \rho \leq r \leq \infty$ and $\rho \leq \infty$.*

PROOF. The theorem follows from Theorem 3.3, and Lemma 3.4. □

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