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ON STRUCTURES OF CONTRACTIONS IN DUAL OPERATOR ALGEBRAS

MYUNG JAE KIM

ABSTRACT. We discuss certain structure theorems in the class $\mathbb{A}$ which is closely related to the study of the problems of solving systems concerning the predual of a dual operator algebra generated by a contraction on a separable infinite dimensional complex Hilbert space.

The notation and terminology employed here agree with those in [4] and [9]. We recall nonetheless them for the convenience of the reader.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. Let $\mathcal{C}_1 = \mathcal{C}_1(\mathcal{H})$ be the ideal of trace class operators in $\mathcal{L}(\mathcal{H})$ under the trace norm. Suppose that $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$ and let ${}^{\perp}\mathcal{A}$ denote the preannihilator of $\mathcal{A}$ in $\mathcal{C}_1$. Let $\mathcal{Q}_{\mathcal{A}}$ denote the quotient space $\mathcal{C}_1/{}^{\perp}\mathcal{A}$. One knows that $\mathcal{A}$ is the dual space of $\mathcal{Q}_{\mathcal{A}}$ under the pairing

$$< T, [L] > = \ trace(TL), \ T \in \mathcal{A}, \ [L] \in \mathcal{Q}_{\mathcal{A}}.$$

Furthermore, the weak* topology that accrues to $\mathcal{A}$ by virtue of this duality coincides with the ultraweak operator topology on $\mathcal{A}$.

The structure of contractions in dual algebras is deeply related to the classes $\mathbb{A}$ and $\mathbb{A}_{m,n}$ which will be defined below and the study of the problem of solving systems of simultaneous equations in the predual of a singly generated dual algebra (see [1], [3] and [4]). This theory is applied to the study of invariant subspaces and compression theory. In particular, in [2] and [5] Bercovici and Chevreau proved independently $\mathbb{A}= \mathbb{A}_{1,1}$.

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In this paper, we obtain some structure theorems of contractions in the class $\mathbb{A}$.

For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the dual algebra generated by T. For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in C_1 defined by $(x \otimes y)(u) = (u, y)x, \quad u \in \mathcal{H}$.

Let us recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator (cf. [9]). If T_2 is absolutely continuous or acts on the space (0), T will be called an *absolutely continuous contraction.*

Recall that $T \in C_{\cdot 0}$ if $\|T^{*n}x\| \longrightarrow 0$ for any $x \in \mathcal{H}$. We say $T \in C_{0\cdot}$ if $T^* \in C_{\cdot 0}$.

We shall denote by $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$ and we write $\mathbb{T}$ for the boundary of $\mathbb{D}$. For $1 \le p \le \infty$ we denote the usual Lebesgue function space by $L^p = L^p(\mathbb{T})$. For $1 \le p \le \infty$ we denote by $H^p = H^p(\mathbb{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish. One knows that the preannihilator ${}^{\perp}(H^\infty)$ of H^∞ in L^1 is the subspace H_0^1 consisting of those functions g in H^1 for which analytic extension $\tilde{g}$ to $\mathbb{D}$ satisfies $\tilde{g}(0) = 0$. It is well known that H^∞ is the dual space of L^1/H_0^1.

The following Foiaş-Nagy functional calculus provides a good relationship between the function space H^∞ and a singly generated dual algebra $\mathcal{A}_T$.

THEOREM 1 [4, THEOREM 4.1]. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism*

$$\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$$

defined by $\Phi_T(f) = f(T)$ such that

(a) $\Phi_T(1) = I_{\mathcal{H}}, \quad \Phi_T(\xi) = T$,

(b) $\|\Phi_T(f)\| \le \|f\|_\infty, \quad f \in H^\infty$,

(c) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*

(d) *the range of Φ_T is weak* dense in $\mathcal{A}_T$,*

(e) *there exists a bounded, linear, one-to-one map $\phi_T : Q_T \longrightarrow L^1/H_0^1$ such that $\phi_T^* = \Phi_T$, and*

(f) *if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of Q_T onto L^1/H_0^1.*

Definition 2. Suppose that m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \ 0 \leq i < m, \ 0 \leq j < n,$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m}$, $\{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For brevity, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. The class $\mathbb{A}(\mathcal{H})$ consists of all those absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ is an isometry. Furthermore, we denote by $\mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$. We write simply $\mathbb{A}_{m,n}$ for $\mathbb{A}_{m,n}(\mathcal{H})$ unless we mention otherwise.

Let P_λ be the Poisson kernel function

$$P_\lambda(e^{it}) = \frac{1 - |\lambda|^2}{|1 - \overline{\lambda} e^{it}|^2}, \quad e^{it} \in \mathbb{T}, \tag{1}$$

in L^1, for each $\lambda \in \mathbb{D}$. Then it follows from [4, p34] that

$$\langle f, [P_\lambda] \rangle = \widetilde{f}(\lambda), \quad f \in H^\infty, \tag{2}$$

where $\widetilde{f}$ is the analytic extension of f to $\mathbb{D}$. For a given contraction $T \in \mathbb{A}$, let us denote $\phi_T^{-1}([P_\lambda]) = [C_\lambda]$. Then we have

$$\langle f(T), [C_\lambda] \rangle = \widetilde{f}(\lambda), \quad f \in H^\infty. \tag{3}$$

Throughout this paper, $\mathbb{N}$ is the set of all natural numbers. For $\mathcal{M} \in \text{Lat}(T)$, the class of invariant subspaces for an operator $T \in \mathcal{L}(\mathcal{H})$, we denote $T|\mathcal{M}$ for the restriction of T to $\mathcal{M}$. If $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{K}$ is a semi-invariant subspace for T (i.e., there exist $\mathcal{K}_1, \mathcal{K}_2 \in \text{Lat}(T)$ with $\mathcal{K}_1 \supset \mathcal{K}_2$ such that $\mathcal{K} = \mathcal{K}_1 \ominus \mathcal{K}_2$), we shall write $T_\mathcal{K} = P_\mathcal{K} T|\mathcal{K}$ for the *compression* of T to $\mathcal{K}$, where $P_\mathcal{K}$ is the orthogonal projection whose range is $\mathcal{K}$.

For $T \in \mathbb{A}$ and any nonnegative integer n, we define a linear functional

$$C_T^{(n)} : \mathcal{A}_T \to \mathbb{C}$$

by

$$C_T^{(n)} h(T) = \hat{h}(n) \quad \text{for every } h \in H^{\infty},$$

where $\hat{h}(n)$ is the n-th Fourier coefficient of h (cf. [6]).

Now we are ready to prove the following structure theorem for a contraction in the class $\mathbb{A}$.

THEOREM 3. *Suppose that $T \in \mathbb{A}(\mathcal{H})$. Assume that there exist a vector x and a sequence $\{t_i\}_{i=1}^{\infty}$ of vectors in $\mathcal{H}$ such that*

$$[x \otimes t_i]_T = [C_0^{(i)}]_T, \quad i = 0, 1, 2, \cdots . \tag{4}$$

Then there exists a semi-invariant subspace $\mathcal{N}$ for T such that

(i) *$T_{\mathcal{N}} \in C_{\cdot 0}$ and*

(ii) *there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of vectors in the closed unit ball of $\mathcal{N}$ converging weakly to zero such that*

$$[C_0]_T = [x_n \otimes x_n]_T, \quad n = 1, 2, \cdots \tag{5}$$

PROOF. Let us consider the subspace

$$\mathcal{M} = \bigvee_{k \geq 0} T^k x.$$

Then it is obvious that $\mathcal{M} \in Lat(T)$. According to the method of [6, Theorem 6.2], we can obtain an orthonormal sequence $\{x_n\}_{n=1}^{\infty}$ of vectors in $\mathcal{H}$ converging weakly to zero such that

$$x_n \in Ker(T|\mathcal{M})^{*^{n+1}} \ominus Ker(T|\mathcal{M})^{*^n}, \quad n \in \mathbb{N}. \tag{6}$$

Then

$$[C_0]_T = [x_n \otimes x_n]_T, \quad n = 1, 2, \cdots . \tag{7}$$

Now let us put

$$\mathcal{N} = \bigvee_{k \geq 0} Ker((T|\mathcal{M})^{*k}). \tag{8}$$

Then $\mathcal{N} \in Lat(T|\mathcal{M})^*$ and $x_n \in \mathcal{N}$. Setting $A := (T|\mathcal{M})^*|\mathcal{N}$, it follows from [7, Proposition 2.8] that $A \in C_{0\cdot}$. Notice that

$$(9) \qquad T^* = \begin{pmatrix} (T|\mathcal{M})^* & 0 \\ * & * \end{pmatrix}$$

relative to a decomposition $\mathcal{M} \oplus \mathcal{M}^\perp$. Since $\mathcal{N} \in Lat(T|\mathcal{M})^*$, one has

$$(10) \qquad T = \begin{pmatrix} * & * & * \\ 0 & A^* & * \\ 0 & 0 & * \end{pmatrix}$$

relative to a decomposition $(\mathcal{H} \ominus \mathcal{M}) \oplus \mathcal{N} \oplus (\mathcal{M} \ominus \mathcal{N})$. Hence $T_{\mathcal{N}} = A^* \in C_{\cdot 0}$ and the proof is complete. □

The following is an immediate corollary of Theorem 3.

COROLLARY 4. *Suppose that $T \in \mathbb{A}_{1,\aleph_0}$. Then there exists a semi-invariant subspace $\mathcal{N}$ for T satisfying* (i) *and* (ii) *in Theorem 3.*

In general the converse implication of Corollary 4 is not always true because of Example 6. To discuss the example we need the following lemma.

LEMMA 5 [8, THEOREM 2.4]. *Suppose that $T \in C_{\cdot 0} \cap \mathbb{A}$ satisfies the defect index*

$$d_T = dim\overline{\{(I - T^*T)^{\frac{1}{2}}\mathcal{H}\}} < \infty,$$

and let n be a positive interger. Then the following are equivalent:

(i) $T \in \mathbb{A}_{n,1} \setminus \mathbb{A}_{n+1,1}$;

(ii) $T \in \mathbb{A}_{n,\aleph_0} \setminus \mathbb{A}_{n+1,1}$;

(iii) *the minimal coisometric extension B_T of T can be decomposed by $S_T^* \oplus B^{(n)}$, where S_T^* is the backward unilateral shift part of B_T and $B^{(n)}$ is the bilateral shift operator of multiplicity n.*

EXAMPLE 6. Let S be a unilateral shift operator of multiplicity one. Let $T = S^* \oplus \lambda I_{\mathcal{H}}$ with $\lambda = 0$. Then there exists a semi-invariant subspace $\mathcal{N}$ for T satisfying (i) and (ii) in Theorem 3, but T is not in the class $\mathbb{A}_{1,\aleph_0}$. Indeed, take $\mathcal{N} := (0) \oplus \mathcal{H}$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathcal{H}$. Then $\{\epsilon_n\}_{n=1}^{\infty}$ converges weakly to zero. Without loss of generality, we assume that ϵ_n belongs to $\mathcal{N}$, i.e., we consider ϵ_n as $0 \oplus \epsilon_n$. Since

$$(11) \qquad < T^k, [C_0]_T >= \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k = 1, 2, \cdots \end{cases}$$

and for any $n \in \mathbb{N}$

$$(12) \qquad < T^k, [\epsilon_n \otimes \epsilon_n]_T >= \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k = 1, 2, \cdots \end{cases}$$

we have that

$$(13) \qquad [C_0]_T = [\epsilon_n \otimes \epsilon_n]_T, \quad n = 1, 2, \cdots .$$

Let B_T be the minimal coisometric extension of T. Obviously we have $B_T = S^* \oplus S^*$. Hence by lemma 5 we have $T \notin \mathbb{A}_{1,\aleph_0}$. □

In particular, we apply the operator in Example 6 for the the membership of the classes $\mathbb{A}_{m,n}$ in the following proposition.

PROPOSITION 7. *Suppose that* $T = S^{*(n)} \oplus \lambda I_{\mathcal{H}}$ *with* $\lambda = 0$ *for some* $n \in \mathbb{N}$. *Then*

$$(14) \qquad T \in \mathbb{A}_{\aleph_0,n} \setminus (\mathbb{A}_{n+1} \cup \mathbb{A}_{1,\aleph_0})$$

PROOF. It is obvious that $T \in \mathbb{A}_{\aleph_0,n}$. Furthermore, since $T^* = S^{(n)} \oplus 0_{\mathcal{H}}$, the minimal coisometric extension B_{T^*} of T^* is $B^{(n)} \oplus S^*$, where B is the bilateral shift operator of multiplicty one. According to Lemma 5, $T^* \notin \mathbb{A}_{n+1}$. So $T \notin \mathbb{A}_{n+1}$. Moreover, since $B_T = S^{*(n)} \oplus S^*$, by Lemma 5 $T \notin \mathbb{A}_{1,\aleph_0}$. □

Finally, we discuss a structure theorem for contractions in the class $\mathbb{A}$ as following.

THEOREM 8. *Suppose that $T \in \mathbb{A}(\mathcal{H})$. Assume that there exists a sequence $\{x_k\}_{k=1}^{\infty}$ of vectors from the closed unit ball of $\mathcal{H}$ converging weakly to zero such that*

(i) $[C_0]_T = [x_k \otimes x_k]_T$, $k = 1, 2, \cdots$, *and*

(ii) *there exists a sequence $\{\mathcal{M}_k\}_{k=1}^{\infty}$ of invariant subspaces for T^* such that for each $k \in \mathbb{N}$, $x_k \in \mathcal{M}_k$ and $T_{\mathcal{M}_k} \in C_{\cdot 0}$.*

Then $T_{\mathcal{M}} \in C_{\cdot 0}$, where

(15)
$$\mathcal{M} = \bigvee_{k=1}^{\infty} \mathcal{M}_k.$$

PROOF. Since $\mathcal{M}_k \in Lat(T^*)$, we may set $A := T^*|\mathcal{M}_k$. Then we have

(16)
$$T = \begin{pmatrix} * & * \\ 0 & A^* \end{pmatrix}$$

relative to a decomposition $(\mathcal{H} \ominus \mathcal{M}_k) \oplus \mathcal{M}_k$. So $A^* = T_{\mathcal{M}_k}$. Let $x \in \mathcal{M}_k$. Then $\|A^n x\| \to 0$ and $\|T^{*n} x\| \to 0$. So

(17)
$$\mathcal{M}_k \subset \{x \in \mathcal{H} : \|T^{*n} x\| \to 0\}$$

for all positive integers k. Now let us put again

(18)
$$\mathcal{M} := \bigvee_{k=1}^{\infty} \mathcal{M}_k.$$

Then $\mathcal{M}$ is a semi-invariant subspace for T and $\{x_k\}_{k=1}^{\infty}$ is contained in the closed unit ball of $\mathcal{M}$. Furthermore, since $\{x \in \mathcal{H} : \|T^{*n} x\| \to 0\}$ is a subspace of $\mathcal{H}$, we have

(19)
$$\mathcal{M} \subset \{x \in \mathcal{H} : \|T^{*n} x\| \to 0\}.$$

Set

(20)
$$T^* = \begin{pmatrix} * & 0 \\ * & B \end{pmatrix},$$

where $B := T^*|\mathcal{M}$, relative to a decomposition $(\mathcal{H} \ominus \mathcal{M}) \oplus \mathcal{M}$. Let $x \in \mathcal{M}$. Then

(21)
$$\|{T_{\mathcal{M}}^*}^n x\| = \|B^n x\| = \|T^{*n} x\| \to 0.$$

So $T_{\mathcal{M}} \in C_{\cdot 0}$. Hence the proof is complete. □

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Department of Mathematics
Kyonggi University
Suwon 442-760, Korea